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GROUPS SATISFYING THE DOUBLE CHAIN CONDITION ON ABELIAN SUBGROUPS

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Abstract

If θ is a subgroup property, a group G is said to satisfy the double chain condition on θ-subgroups if it admits no infinite double sequences

$$\cdots < X_{-n} < \cdots < X_{-1} < X_0 < X_1 < \cdots < X_n < \cdots$$

consisting of θ-subgroups. We describe the structure of generalised radical groups satisfying the double chain condition on abelian subgroups.

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1. Introduction

Maximal and minimal conditions on systems of subgroups are among the most important finiteness conditions in the theory of groups. The structure of (generalised) soluble groups satisfying the maximal or minimal condition on the set of all subgroups has been completely described, and there are also many results for groups with similar restrictions imposed only on certain relevant systems of subgroups, such as abelian, normal or subnormal subgroups (see, for instance, [7, Chs. 3 and 5]). In particular, Schmidt [8, Theorem 9] showed that for soluble groups the minimal condition is equivalent to the minimal condition on abelian subgroups, and a corresponding result for the maximal condition was obtained by Mal'cev [6, Theorem 8].

The imposition of weaker forms of the classical chain conditions produces some remarkable effects. In particular, Zaicev [11] and Shores [9] independently proved that if G is a generalised soluble group admitting no chains of subgroups with the same order type of the set of integers, then G is soluble-by-finite and it satisfies either the minimal or the maximal condition on subgroups. Conditions of this type can be

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considered for special subgroup systems: if θ is a subgroup property, we shall say that a group G satisfies the *double chain condition* on θ-subgroups if for each *double chain*

$$\cdots \leq X_{-n} \leq \cdots \leq X_{-1} \leq X_0 \leq X_1 \leq \cdots \leq X_n \leq \cdots$$

of θ-subgroups of G there exists an integer k such that either $X_n = X_k$ for all $n \leq k$ or $X_n = X_k$ for all $n \geq k$. Obviously, both the minimal and the maximal conditions on θ-subgroups imply the double chain condition on θ-subgroups.

More recently, the double chain condition on various relevant systems of subgroups has been considered (see [2–4]), and the strict relation between the pure maximal and minimal conditions on θ-subgroups and the respective double chain conditions was pointed out.

The aim of this article is to give a further contribution to this theory by studying groups satisfying the double chain condition on abelian subgroups. In particular, a complete description of generalised radical groups with this property will be obtained. Recall here that a group is said to be *generalised radical* if it contains an ascending (normal) series all of whose factors are either locally nilpotent or locally finite. Note that there exist soluble groups satisfying the double chain condition on abelian subgroups which satisfy neither the maximal nor the minimal condition on subgroups. Finally, we will prove that a generalised radical group satisfying the double chain condition on abelian subgroups contains no infinite double chain consisting of subnormal subgroups.

Most of our notation is standard and can be found in [7].

2. Results and proofs

It is straightforward to see that a group G satisfies the double chain condition on abelian subgroups if and only if every abelian subgroup of G satisfies either the maximal or the minimal condition on subgroups.

The class of groups satisfying the double chain condition on abelian subgroups is obviously subgroup closed, while we will prove that it is closed under homomorphic images in the universe of generalised radical groups. Finally, it is clear that this class is not extension closed, as shown by considering the direct product of an infinite cyclic group and a group of type p^∞ for some prime p.

Recall that a group is said to be *minimax* if it has a series of finite length each of whose factors satisfies either the maximal or the minimal condition on subgroups. An important theorem due to Baer [1, Hauptsatz 4.2] states that every *radical* group all of whose abelian subgroups are minimax is a soluble minimax group (recall here that a group is *radical* if it has an ascending series with locally nilpotent factors). The reader can refer to [7, Theorem 10.33] for the complete description of the structure of a soluble minimax group.

Our first lemma generalises a result of Shores [9] and deals with the case of Baer groups, that is, groups which are generated by their abelian subnormal subgroups. Clearly, Baer groups are locally nilpotent.

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Lemma 2.1. *Let G be a Baer group satisfying the double chain condition on abelian subgroups. Then G is nilpotent and satisfies either the maximal or the minimal condition on subgroups.*

Proof. Let T be the torsion subgroup of G. Clearly, every abelian subgroup of T satisfies the minimal condition on subgroups. It follows that T is a Černikov group and hence it contains a subgroup of finite index which is contained in the centre of G (see [7, Lemma 3.13]). If $Z(T)$ is infinite, then G satisfies the minimal condition on subgroups by hypothesis. Thus, we may assume that T is finite. Therefore, every abelian subgroup of G is finitely generated and hence G satisfies the maximal condition on subgroups. □

Note that Lemma 2.1 cannot be extended to arbitrary locally nilpotent groups. In fact, let P be a group of type p^∞ for a prime p, and let x be an automorphism of P of infinite order. Clearly, there exists a positive integer m such that x^m acts trivially over the subgroup of order p of P. It is easy to show that the semidirect product $G = \langle x^m \rangle \ltimes P$ is a metabelian hypercentral group satisfying the double chain condition on its abelian subgroups. On the other hand, G satisfies neither the maximal nor the minimal condition on subgroups.

In order to investigate the structure of generalised radical groups satisfying the double chain condition on abelian subgroups, we need to prove that this latter property is closed under homomorphic images in the universe of generalised radical groups. This result will be obtained through a series of lemmas.

Lemma 2.2. *Let G be a group satisfying the double chain condition on abelian subgroups. If H is a finite-by-nilpotent subgroup of G, then H satisfies either the maximal or the minimal condition on subgroups.*

Proof. It follows from a well-known result of P. Hall that H contains a normal nilpotent subgroup N such that H/N is finite (see [7, Theorem 4.25]). Thus, the statement is a direct consequence of Lemma 2.1. □

Lemma 2.3. *Let G be a group satisfying the double chain condition on abelian subgroups and let A be a soluble-by-finite normal subgroup of G. Then G/A satisfies the double chain condition on abelian subgroups.*

Proof. As A contains a G-invariant soluble subgroup of finite index, we may suppose by Lemma 2.2 that A is soluble and infinite. First assume that A is abelian. It follows by hypothesis that A is either finitely generated or a Černikov group. Let B/A be an abelian subgroup of G/A. Clearly, the subgroup $C = C_B(A)$ of B is nilpotent. If A is not a Černikov group, then it is polycyclic and so is B/C (see [7, Theorem 3.27]). On the other hand, C is finitely generated by Lemma 2.1 and hence so is B. Therefore, B/A satisfies the maximal condition on subgroups. Now let A be a Černikov group. By Lemma 2.2, we may suppose that A is radicable. For a contradiction, choose the pair (G, A) with A radicable of smallest total rank subject to the existence of an abelian subgroup B/A of G/A which satisfies neither the maximal nor the minimal condition on

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subgroups. Clearly, A is not central in B. Moreover, if A contains a proper B-invariant infinite subgroup H, then the pair $(B/H, A/H)$ contradicts the choice of (G, A). Thus, every proper B-invariant subgroup of A is finite and hence there exists a subgroup X of B such that $B = XA$ and $X \cap A$ is finite (see [5, 6.1.7]). It follows that X is finite-by-abelian and hence by Lemma 2.2 it satisfies either the maximal or the minimal condition on subgroups. This last contradiction proves the statement when A is abelian.

Finally, let A be soluble with derived length $d > 1$. Then, by the first part of the proof, $G/A^{(d-1)}$ satisfies the double chain condition on abelian subgroups and hence the same holds for G/A by induction. □

Proposition 2.4. *Let G be a generalised radical group satisfying the double chain condition on abelian subgroups. Then G is a soluble-by-finite minimax group.*

Proof. By Baer's theorem quoted above, the maximal normal radical subgroup R of G is a soluble minimax group. By Lemma 2.3, G/R satisfies the double chain condition on abelian subgroups, so we can assume that G has no nontrivial subnormal subgroups which are locally nilpotent. Assume for a contradiction that G is infinite. By a result of Šunkov [10], every locally finite subgroup of G is a Černikov group and hence finite if normal in G. Let N be a finite normal subgroup of G. If F is a subnormal nonabelian simple subgroup of N, then $E_1 = F_1 = F^G$ is the direct product of finitely many nonabelian simple subgroups of G (see [7, Theorem 5.45]). Now consider $C_1 = C_G(F_1)$. Clearly, C_1 is nontrivial and $F_1 \cap C_1 = \{1\}$ since F_1 has trivial centre. As C_1 contains neither infinite locally finite nor locally nilpotent subnormal subgroups, arguing as above, there exists a subgroup E_2 of C_1 which is normal in G and which is the direct product of finitely many nonabelian finite simple groups. Clearly, $F_2 = E_1E_2$ is still a normal subgroup of G which is the direct product of finitely many nonabelian finite simple groups and has trivial intersection with $C_2 = C_G(F_2)$. Hence, by induction we construct the normal subgroup $L = \bigcup F_i$ which is infinite and locally finite. This contradiction proves the statement. □

Corollary 2.5. *Let G be a group satisfying the double chain condition on abelian subgroups and let A be a generalised radical normal subgroup of G. Then G/A satisfies the double chain condition on abelian subgroups.*

We are now in a position to characterise groups satisfying the double chain condition on abelian subgroups, at least within the universe of generalised radical groups.

Theorem 2.6. *Let G be a generalised radical group. Then G satisfies the double chain condition on abelian subgroups if and only if one of the following conditions holds:*

(a) *G satisfies the maximal condition on subgroups;*
(b) *G satisfies the minimal condition on subgroups;*
(c) *$G = HJ$, where H is a polycyclic-by-finite subgroup, J is a radicable abelian Černikov normal subgroup and $C_H(P)$ is finite for each subgroup P of type p^∞ of J for some prime p.*

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Proof. First, assume that G is a generalised radical group satisfying the double chain condition on abelian subgroups. By Proposition 2.4, G is a soluble-by-finite minimax group. Suppose that G satisfies neither the maximal nor the minimal condition on subgroups. Let J be the finite residual of G. Then J is the direct product of finitely many Prüfer groups and G/J contains a normal nilpotent subgroup F/J such that G/F is polycyclic-by-finite (see [7, Theorem 10.33]). By Lemma 2.3, G/J satisfies the double chain condition on abelian subgroups and hence it is polycyclic-by-finite by Lemma 2.1. Take H to be the subgroup of G generated by a transversal to J in G. Then $H/H \cap J$ is polycyclic-by-finite and hence $H \cap J = K^H$ for some finite subgroup K of $H \cap J$. But every finite subgroup of $H \cap J$ has finite normal closure in H, so that $H \cap J$ is finite and hence H is polycyclic-by-finite. Finally, let P be a subgroup of J of type p^∞ for a prime p. Clearly, $C_H(P)$ is finite since otherwise we would find a nonperiodic element a of H centralising an abelian group not satisfying the maximal condition, against the hypothesis.

Conversely, assume that G has the structure described in (c) and let A be an abelian subgroup of G which is not a Černikov group. Then A contains an element x of infinite order. Suppose that x belongs to $C_A(P)$ for some Prüfer subgroup P of J. By the factorisation of G, it follows that $xj' = hj$ for some $j' \in P$, $h \in H$ and $j \in J$. Since $\langle x, J\rangle$ has the torsion subgroup, $h = xj'j^{-1}$ is aperiodic and centralises P. This contradiction shows that x does not belong to $C_A(P)$ and hence no Prüfer subgroup of G can lie inside A. It follows that A is finitely generated. The proof is complete. □

Note that in the situation described in case (c) the group G need not split over its finite residual. This is shown by the example constructed in the final part of [2]. We report it here for the convenience of the reader. Let L be a nonsplit central extension of a group $C = \langle x\rangle$ of prime order $p > 2$ by a free abelian group Q of rank two. If P is a group of type p^∞, then Q is isomorphic to a group of automorphisms of P and so we may construct the semidirect product $K = L \ltimes P$ with $C_L(P) = C$. Let $\langle a\rangle$ be the unique subgroup of order p of P. Since C is contained in the centre of K, the subgroup $M = \langle x^{-1}a\rangle$ is normal in K. Put $G = K/M$. Then $J = PM/M$ is the finite residual of G, and $G = HJ$, where $H = LM/M$. Moreover, $H \cap J$ has order p, and G cannot split over J because L does not split over C. Finally, as K satisfies the double chain condition on abelian subgroups, the same holds for G by Lemma 2.2.

Now let G be a metabelian group satisfying the double chain condition on abelian subgroups and suppose that G is not finitely generated. If the derived subgroup G' of G is nonperiodic, then G' is finitely generated, and hence $G/C_G(G')$ is polycyclic. It follows that the finite residual J of G is contained in $C_G(G')$. Thus $G'J$ is an abelian subgroup of G which satisfies neither the maximal nor the minimal condition on subgroups. This contradiction shows that G' is a Černikov group.

On the other hand, there exist soluble groups satisfying the double chain condition on abelian subgroups which are not finitely generated and whose derived subgroup is not periodic.

In fact, let J be a radicable abelian p-group (p a prime) of rank two. It is well known that Aut J is isomorphic to $GL(2, R_p)$, where R_p is the ring of p-adic integers. Consider

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in R_p an element x of infinite multiplicative order (that is, x is an automorphism of infinite order of a group P of type p^∞) such that x does not act trivially on the unique subgroup of P of order p. If $H = \langle a, b\rangle$, where

$$a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we may construct the semidirect product $G = H \ltimes J$. Clearly, H is isomorphic to the infinite dihedral group and an easy application of Theorem 2.6 shows that G satisfies the double chain condition on abelian subgroups.

Finally, we remark that in statement (c) of Theorem 2.6, the imposition of the property on the centralisers cannot be weakened. In fact, take $J = A \times B$, where A and B are groups of type p^∞ for the same prime $p > 2$, and consider an element x of the ring of the p-adic units of infinite multiplicative order. Clearly,

$$a = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

is an automorphism of J centralising B but not A, which means that the semidirect product $G = \langle a\rangle \ltimes J$ does not satisfy the double chain condition on abelian subgroups. However, it is easy to show that a does not centralise any subgroup of type p^∞ of J other than B.

In the final part of this paper, we analyse the relation between the double chain condition on abelian subgroups and that on subnormal subgroups considered in [2]. Clearly, the finitary permutation group on $\mathbb{N}$ satisfies the double chain condition on subnormal subgroups and contains an abelian subgroup which satisfies neither the maximal nor the minimal condition on subgroups.

Theorem 2.7. *Let G be a generalised radical group satisfying the double chain condition on abelian subgroups. Then G satisfies the double chain condition on subnormal subgroups.*

Proof. By Theorem 2.6, G is a soluble-by-finite minimax group and decomposes as the product HJ, where J is its finite residual, H is polycyclic-by-finite and $C_H(P)$ is finite for each subgroup P of type p^∞ of J. In particular, $C_H(J)$ is finite.

Assume first that G is soluble. By [2, Theorem 2], it will suffice to show that every subnormal subgroup of G either strictly contains J or is a Černikov group. So, let K be a subnormal subgroup of G which does not properly contain J. We will argue by induction on the subnormal defect of K in order to prove that K is a Černikov group. Firstly, suppose that K is normal in G. By Lemma 2.4, the factor group $G/K \cap J$ satisfies the double chain condition on abelian subgroups and hence, replacing G by $G/K \cap J$, we may assume that $[K, J] = \{1\}$. Since J is not properly contained in K, the latter has to be periodic and hence it is a Černikov group.

Suppose now that K has subnormal defect $n > 1$ and let L be the $(n-1)$th term of the series of normal closures of K. Clearly, if L satisfies the minimal condition on subgroups we are done. Therefore, by induction we may assume that $J < L$. But K is normal in L and hence it is a Černikov group.

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Finally, let G be a finite extension of a soluble group S. For a contradiction, let

$$\cdots < X_{-n} < \cdots < X_{-1} < X_0 < X_1 < \cdots < X_n < \cdots$$

be an infinite double chain of subnormal subgroups of G. Clearly, X_0 cannot satisfy the minimal condition on subgroups and hence the same holds for $X_0 \cap S$. By the first part of the proof, J is properly contained in $X_0 \cap S$ and so in X_0. Therefore, G/J has an infinite ascending chain of subgroups, which is a contradiction since G/J satisfies the maximal condition on subgroups. □

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