BULL. AUSTRAL. MATH. SOC. 22E15
VOL. 64 (2001) [415–421]

MAXIMAL COMPACT SUBGROUPS AND THE CENTRE IN AN ANALYTIC GROUP

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In this paper we study the centre of a simply connected analytic group. We present two structure theorems on the centres of such groups. We also study maximal tori and maximal compact subgroups of an analytic group, and as a result, we extend a theorem by Goto.

1. INTRODUCTION

The purpose of this paper is two-fold. In Section 2 we study the centre of a simply connected analytic group. In Section 3 we generalise a theorem by Goto.

Let G be a simply connected analytic group, R the radical of G, S a semisimple Levi factor of G, and $Z(G)$ the centre of G. Let $d = rs$, where $r \in R$ and $s \in S$. Then we obtain necessary and sufficient conditions on r and s so that $d \in Z(G)$ (Proposition 2.2). We also present two structure theorems on $Z(G)$. In particular, we show that if R is nilpotent, then $Z(G) = \big(Z(G) \cap R\big) \times \big(Z(G) \cap S\big)$ (Theorem 2.3).

In [**2**], Goto proved the following theorem.

THEOREM A. *Let G be an analytic group and Z the centre of G. Let α denote the natural homomorphism $G \to G' = G/Z$. Let H be an analytic subgroup of G containing Z. Then H contains a maximal torus of G if and only if $\alpha(H)$ contains a maximal torus of G'.*

We generalise Theorem A as follows. Let G and G' be analytic groups and let α be a continuous homomorphism from G onto G'. Let H be an analytic subgroup of G containing the kernel of α. Then H contains a maximal torus of G if and only if $\alpha(H)$ contains a maximal torus of G' (Theorem 3.6). We also obtain the following result. H contains a maximal compact subgroup of G if and only if $\alpha(H)$ contains a maximal compact subgroup of G' (Theorem 3.7).

NOTATION. Let G be a locally compact group. We denote the connected component of G that contains the identity element by G°. Let $\mathbf{Z}, \mathbf{R}^+, \mathbf{R}$, and $\mathbf{C}$ denote the sets of integers, positive real numbers, real numbers, and complex numbers, respectively.

Received 19th March, 2001

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https://doi.org/10.1017/S0004972700019882 Published online by Cambridge University Press

2. The centre of a simply connected analytic group

This section will be dedicated to the portion of the paper studying the centre of a simply connected analytic group. Throughout this section, G will denote a simply connected analytic group, R the radical of G, S a semisimple Levi factor of G, and $Z(G)$ the centre of G. Recall that $G = R \cdot S$, both R and S are simply connected, and S is closed.

Theorem 2.1. *$G = E \cdot (A \times S)$, where E and A are analytic subgroups of G and E is a normal subgroup of G. In particular, $Z(G) = \big(Z(G) \cap E\big) \times \big(Z(G) \cap (A \times S)\big)$.*

Proof: Let $Z = Z(G)$ denote the centre of G. Then G/Z is a faithfully representable analytic group. Therefore, $G/Z = E' \cdot F'$, where E' is a simply connected solvable analytic group normal in G/Z, and F' is a reductive analytic group. Now $F' = T'S'$, where S' is a semisimple Levi factor of F' and T' is a cental torus of F'.

Let π denote the natural homomorphism $G \to G/Z$. Let $E = \pi^{-1}(E')^{\circ}$. Then $\pi^{-1}(E') = EZ$ and $E \supseteq Z^{\circ}$. Since $\pi(EZ) = E'$ is simply connected, $E \cap Z = Z^{\circ}$ and E is a simply connected normal solvable analytic subgroup of G.

Let $F = \pi^{-1}(S')^{\circ}$. Then $F = Z^{\circ} \times S$, where S is a semisimple Levi factor of G. Let S act on R by conjugation. Let A be the direct complement of E in R under the action of S (or A may be obtained as $\pi^{-1}(T')^{\circ} = Z^{\circ} \times A$). Thus $G = E \cdot (A \times S)$.

Now $Z = Z^{\circ} \times Z'$, where Z' is a finitely generated discrete group. We have that $Z^{\circ} \subseteq E$ and $Z' \subseteq A \times S$. Thus $Z(G) = \big(Z(G) \cap E\big) \times \big(Z(G) \cap (A \times S)\big)$. ∎

Let $d = rs \in G$, where $r \in R$ and $s \in S$. Then r and s are unique. The following proposition gives necessary and sufficient conditions on r and s so that $d \in Z(G)$.

Proposition 2.2. *Suppose $r \in R$ and $s \in S$. Then $rs \in Z(G)$ if and only if the following conditions are satisfied:*

1. *$r^{-1}r'r = sr's^{-1}$ for all $r' \in R$.*
2. *$s \in Z(S)$.*
3. *r centralises S.*

Proof: Suppose $rs \in Z(G)$. For any $s' \in S$, we have $(rs)s' = s'(rs) = (s'rs'^{-1})(s's)$, so $rs = (s'rs'^{-1})(s'ss'^{-1})$. By the uniqueness of the decomposition, we have $r = s'rs'^{-1}$ and $s = s'ss'^{-1}$ for all $s' \in S$. Thus $s \in Z(S)$ and r is in the centraliser of S. Also for any $r' \in R$, we have $r'rs = rsr' = rsr's^{-1}s$, so $r'r = rsr's^{-1}$, and therefore $r^{-1}r'r = sr's^{-1}$ for all $r' \in R$.

Suppose $r \in R$ and $s \in S$ satisfy conditions 1–3. Then $(r's')(rs) = r'rs's = r'rss' = rsr's^{-1}ss' = (rs)(r's')$, for all $r' \in R$ and $s' \in S$. Hence $rs \in Z(G)$. ∎

Now we consider the situation when R is nilpotent. If $d = rs \in Z(G)$, then $r^{-1}r'r = sr's^{-1}$ for all $r' \in R$. Since $s \in Z(S)$, $s^m \in Z(G)$ for some integer m.

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To see this, view S as a group of automorphisms acting on R by conjugation; then it has finite centre. Thus we have that $s^m r' s^{-m} = r'$ for all $r' \in R$, which implies that $r^{-m} r' r^m = r'$ for all $r' \in R$. Hence $r^m \in Z(R)$. Since R is nilpotent, $Z(R)$ is connected: since R is also simply connected, $Z(R)$ is a vector group. Thus $r \in Z(R)$, which implies that $r \in Z(G)$. Thus we have the following theorem.

THEOREM 2.3. *If R is nilpotent, then $Z(G) = \big(Z(G) \cap R\big) \times \big(Z(G) \cap S\big)$.*

We conclude this section with an example of a simply connected analytic group $\widetilde{G}$ such that $Z(\widetilde{G}) \cap \widetilde{R}$ is not a direct factor of $Z(\widetilde{G})$, where $\widetilde{R}$ denotes the radical of $\widetilde{G}$.

EXAMPLE. Let

$$G = \left\{ \sigma : \sigma \in GL(3, \mathbf{C}),\ \sigma = \begin{pmatrix} \alpha & \beta & x \\ \gamma & \delta & y \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let

$$N = \left\{ \sigma \in G :\ \sigma = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad D = \left\{ \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} :\ c \in \mathbf{C} \backslash \{0\} \right\},$$

and

$$S = \left\{ \sigma \in G :\ \sigma = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1 \right\}.$$

Then $G = N \cdot (D \times S)$, the radical of G is $N \cdot D$, and S is a semisimple Levi factor of G. Note that $S \cong SL(2, \mathbf{C})$ is simply connected. Let $\widetilde{G}$ be the universal covering group of G. Let $\widetilde{D} = \mathbf{R}^+ \times \mathbf{R}$ be the universal covering group of D, and let $\pi : \widetilde{D} \to D$ be the covering homomorphism given by

$$\pi(\rho, r) = \begin{pmatrix} \rho e^{2\pi i r} & 0 & 0 \\ 0 & \rho e^{2\pi i r} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\widetilde{G} = N \cdot \big(\widetilde{D} \times S\big)$. The centre of $\widetilde{G}$ is a subgroup of $\widetilde{D} \times S$: it is the infinite cyclic group with generator c,

$$c = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (1, 1/2), \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \in \widetilde{D} \times S.$$

Then

$$c^2 = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (1, 1), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \in \widetilde{D} \subseteq \widetilde{R}.$$

Thus, the subgroup $Z(\widetilde{G}) \cap \widetilde{R}$ is not a direct factor of $Z(\widetilde{G})$.

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3. Maximal tori and maximal compact subgroups

This section is mainly devoted to extending Theorem A. We begin by recalling the following result of Goto [**1**, (7.1)].

Theorem B. *Let G and G' be analytic groups, and let β be a continuous homomorphism from G onto G'. Let T' be a maximal torus of G', and H' a closed connected subgroup of G' containing T'. Then $\beta^{-1}(H')$ is a closed connected subgroup of G.*

Remark. $\beta^{-1}(H')$ contains the kernel of β. Also, $\beta^{-1}(H')$ contains a maximal torus of G. To see this, let T be a maximal torus of $\beta^{-1}(H')$ and suppose $T \subseteq T^*$, where T^* is maximal torus of G. Then $\beta(T^*)$ is a torus in G', so there exists $g' \in G'$ with $g'\beta(T^*)g'^{-1} \subseteq T'$. Let $g \in G$ with $\beta(g) = g'$. Then $\beta(g)\beta(T^*)\beta(g)^{-1} = \beta(gT^*g^{-1}) \subseteq T'$, which implies that $gT^*g^{-1} \subseteq \beta^{-1}(T') \subseteq \beta^{-1}(H')$. Thus there exists $\widetilde{g} \in G$ such that $\widetilde{g}T^*\widetilde{g}^{-1} \subseteq T$, which implies that $\dim T = \dim T^*$. Since $T \subseteq T^*$, $T = T^*$. Thus T is a maximal torus of G.

Corollary 3.1. *Let G and G' be analytic groups, and let β be a continuous homomorphism from G onto G'. Let K' be a maximal compact subgroup of G', and H' a closed connected subgroup of G' containing K'. Then $\beta^{-1}(H')$ is a closed connected subgroup of G containing a maximal compact subgroup of G.*

Proof: Since K' is a maximal compact subgroup of G', K' is connected and contains a maximal torus T' of G'. Thus $\beta^{-1}(K')$ is a closed connected subgroup of G. Let K be a maximal compact subgroup of $\beta^{-1}(K')$. Then $K \subseteq \beta^{-1}(H')$ and K is a maximal compact subgroup of G. This follows by using an argument similar to the one in the above remark. ∎

Let G be an analytic group and let Z be the centre of G. Let α denote the natural homomorphism $G \to G/Z = G'$. Let T' be a maximal torus of G'. Then $A = \alpha^{-1}(T')$ is a closed connected Abelian group. Let T be the maximal torus of A. Then T is also a maximal torus of G. Thus we have the following proposition.

Proposition 3.2. *Let Z be the centre of an analytic group G. Then Z is contained in an Abelian analytic subgroup A of G, which also contains a maximal torus of G.*

Remark. The original statement in [**3**, Chapter XVI, Theorem 1.2] is that $Z \subseteq A$ without mentioning that A also contains a maximal torus of G.

Now $Z \cong V \times T_c \times F \times \mathbf{Z}^m$, where V is a vector group, T_c is a central torus, and F is a finite group. Since $A/Z \cong T'$, there exists torus $T_1 \subseteq A$ with $F \subseteq T_1$, and $T_1/F \cong T_1'$. Also there exists a vector group $V_2 \subseteq A$ such that $V_2/\mathbf{Z}^m \cong T_2'$. Let $T = T_c \times T_1 \times T_3$. Then $A/Z \cong T' \cong T_1' \times T_2' \times T_3$.

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The following lemmas will be used in proving Theorems 3.6 and 3.7.

LEMMMA 3.3. *Let G be an analytic group and let D be a discrete central subgroup of G. Let α denote the natural homomorphism $G \to G/D = G'$. Let H be an analytic subgroup of G which contains D and also a maximal torus of G. Then $\alpha(H)$ contains a maximal torus of G'.*

PROOF: Let T' be a maximal torus of G'. Let $B = \alpha^{-1}(T')$. Then $B/D \cong T'$ and B is a closed connected Abelian group. Let T be the maximal torus of B. Then T is also a maximal torus of G. Let $D = F \times \mathbf{Z}^m$, where F is a finite group. Then $T' = \alpha(T) \times T''$, where T'' is a torus of dimension m.

Since D is contained in the centre of H, by the above discussion, there exists a vector subgroup $V \subseteq H$ such that $V \supseteq \mathbf{Z}^m$. Thus $\alpha(H)$ contains a torus T^* such that $\dim(T^*) = \dim(\alpha(T)) + m$. Therefore, $\alpha(H)$ contains a maximal torus of G'. ∎

LEMMA 3.4. *Let G be an analytic group and let D be a discrete central subgroup of G. Let α denote the natural homomorphism $G \to G/D = G'$. Let H be an analytic subgroup of G which contains D and a maximal compact subgroup of G. Then $\alpha(H)$ contains a maximal compact subgroup of G'.*

PROOF: Let K be a maximal compact subgroup of G contained in H. Since K is connected, K is a compact analytic group. Thus $K = T^*S^*$, where T^* is a central torus of K and S^* is a compact semisimple normal subgroup of K. Let $\widetilde{T}$ be a maximal torus of S^*. Let $T = T^*\widetilde{T}$. Then $K = TS^*$ where T is a maximal torus of K, hence a maximal torus of G. Note that the dimension of S^* is equal to the dimension of the compact part of a semisimple Levi factor of G.

Since $T \subseteq H$, by Lemma 3.3, $\alpha(H)$ contains a maximal torus of G'. Let K' be a maximal compact subgroup of G'. Since K' is a compact analytic group, $K' = T'^*S'^*$, where T'^* is a central torus of K' and S'^* is a compact semisimple normal subgroup of K'. Note that the dimension of S'^* is equal to the dimension of the compact part of a semisimple Levi factor of G'. Let $B = \alpha^{-1}(S'^*)^\circ$. Then $\alpha|_B : B \to S'^*$ is a covering. Thus B is a compact semisimple analytic group, and $\dim S'^* = \dim B \leqslant \dim S^* = \dim \alpha(S^*)$. Hence $\alpha(H)$ contains a compact semisimple subgroup, $\alpha(S^*)$, whose dimension is equal to the dimension of the compact part of a Levi factor of G'. Since $\alpha(H)$ also contains a maximal torus of G', say T', $\alpha(H)$ contains a maximal compact subgroup of G', namely $T'\alpha(S^*)$. ∎

Let G be an analytic group and let N be a normal analytic subgroup of G. Assume G/N is compact. Let K be a maximal compact subgroup of G. Then $G = NK$ ([**4**, Lemma 3.13]). In particular, if G/N is a torus, there exists a torus $T \subseteq G$ such that $G = NT$.

LEMMA 3.5. *Let G be an analytic group and let N be a closed normal analytic*

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subgroup of G. Let α denote the natural homomorphism $G \to G/N = G'$. Let H be an analytic subgroup of G which contains N and a maximal torus T of G. Then $\alpha(T)$ is a maximal torus of G'.

PROOF: Let T' be a maximal torus of G'. Let $F = \alpha^{-1}(T')$. Then F is a closed connected subgroup of G such that $F/N \cong T'$. Thus there exists a torus $T_1 \subset F$ such that $NT_1 = F$. Let $T \subseteq H$ be a maximal torus of G. Then there exists $g \in G$ such that $gT_1g^{-1} \subseteq T$. Thus $NgT_1g^{-1} \subseteq NT$, which implies that $gNT_1g^{-1} \subseteq NT$. Thus $NT_1 \subseteq g^{-1}NTg$, which implies that $NT_1/N \subseteq g^{-1}NTg/N$. Hence $\dim T' \leqslant \dim \alpha(T)$, so $\alpha(T)$ must be a maximal torus of G'. ∎

Now we are ready to prove Theorems 3.6 and 3.7.

THEOREM 3.6. *Let G and G' be analytic groups, and let α be a continuous homomorphism from G onto G'. Let M be the kernel of α. Let H be an analytic subgroup of G containing M. Then H contains a maximal torus of G if and only if $\alpha(H)$ contains a maximal torus of G'.*

PROOF: We identify G/M with G'. Let β denote the natural homomorphism $G \to G/M^\circ$, and let γ denote the induced homomorphism $G/M^\circ \to G/M$. Let T be a maximal torus of G contained in H. Then by Lemma 3.5, $\beta(T)$ is a maximal torus of G/M°. Since $\beta(H) \supseteq \beta(T)$, by Lemma 3.3, $\alpha(H) = \gamma \circ \beta(H)$ contains a maximal torus of G/M.

The converse follows from the remarks made after Theorem B. ∎

THEOREM 3.7. *Let G and G' be analytic groups, and let α be a continuous homomorphism from G onto G'. Let M be the kernel of α. Let H be an analytic subgroup of G containing M. Then H contains a maximal compact subgroup of G if and only if $\alpha(H)$ contains a maximal compact subgroup of G'.*

PROOF: We identify G/M with G'. Let β denote the natural homomorphism $G \to G/M^\circ$, and let γ denote the induced homomorphism $G/M^\circ \to G/M$. Let K be a maximal compact subgroup of G contained in H. Then, using the same argument as in Lemma 3.5, $\beta(K)$ is a maximal compact subgroup of G/M°. Since $\beta(H) \supseteq \beta(K)$, by Lemma 3.4, $\alpha(H) = \gamma \circ \beta(H)$ contains a maximal compact subgroup of G/M.

The converse follows from Corollary 3.1. ∎

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