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ON THE PLETHYSM OF S-FUNCTIONS

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1. Introduction. Many authors have studied the theory and calculation of the plethysms of S-functions. The significance of S-functions lies in their relationship [**9**] to the characters of the continuous groups, and plethysms play a crucial role in the determination of branching rules associated with the decomposition of a continuous group into its subgroups [**2**; **14**; **16**]. Tables have been published for the plethysm $\{\lambda\} \otimes \{\mu\}$, where (λ) and (μ) are any partitions of l and m, respectively, with $lm \leqq 18$. These tables have been drawn up both with [**1**] and without [**5**] the aid of computers and some results are also known for $lm > 18$ [**3**; **4**; **7**].

The method given here deals with the notion of q-quotients and is based on a theorem of Littlewood's relating these to plethysms of S-functions with symmetric power sums. Use is made of some results concerning modular congruences between the symmetric power sums. A general rule is obtained for $\{l\} \otimes \{\mu\}$, where $\{l\}$ is a symmetric S-function and (μ) is any partition of 3. In addition, the method has been used for the computation of $\{l\} \otimes \{\mu\}$ beyond the range currently available.

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2. S-functions and plethysm. S-functions, or Schür functions, $\{\lambda\}$, are defined [**9**] in terms of symmetric power sums S_l of independent variables $\alpha_1, \alpha_2, \ldots \alpha_n$ given by

$$S_l = \sum_{i=1}^{n} \alpha_i^{\,l}. \tag{2.1}$$

For any partition $\rho = (1^a 2^b 3^c \ldots)$, the product S_ρ is defined by

$$S_\rho = S_1^{\,a} S_2^{\,b} S_3^{\,c} \ldots, \tag{2.2}$$

and the Schür function $\{\lambda\}$ corresponding to the partition $(\lambda_1, \lambda_2, \ldots)$ of l may then be expressed in the form

$$\{\lambda\} = \frac{1}{l!} \sum_{\rho} h_\rho \chi_\rho^{(\lambda)} S_\rho, \tag{2.3}$$

where $\chi_\rho^{(\lambda)}$ is the character of the class ρ of size h_ρ in the irreducible representa-

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tion of the symmetric group specified by (λ). The inverse of (2.3) is the relationship

(2.4) $$S_\rho = \sum_\lambda \chi_\rho^{(\lambda)}\{\lambda\}.$$

The outer product of two S-functions, $\{\lambda\}\{\mu\}$, may be evaluated by means of the well known Littlewood-Richardson rule [**10**]. Powers of S-functions may be split into parts corresponding to some degree of symmetry between the factors. Thus,

$$\{\lambda\}^2 = \{\lambda\} \otimes \{2\} + \{\lambda\} \otimes \{1^2\},$$

where the square is divided into its symmetrised and anti-symmetrised parts; and

$$\{\lambda\}^3 = \{\lambda\} \otimes \{3\} + 2\{\lambda\} \otimes \{21\} + \{\lambda\} \otimes \{1^3\},$$

etc. In general [**13**],

(2.5) $$\{\lambda\}^m = \sum_\mu f^\mu\{\lambda\} \otimes \{\mu\},$$

where (μ) is a partition of m for which the symmetric group representation is of degree f^μ, and $\{\lambda\} \otimes \{\mu\}$ defines the operation of plethysm. This operation was introduced by Littlewood [**6**] who also established its algebra, which is such that

(2.6) $$\{\lambda\} \otimes (\{\mu\} + \{\nu\}) = \{\lambda\} \otimes \{\mu\} + \{\lambda\} \otimes \{\nu\},$$

and

(2.7) $$\{\lambda\} \otimes (\{\mu\}\{\nu\}) = (\{\lambda\} \otimes \{\mu\})(\{\lambda\} \otimes \{\nu\}).$$

3. q-residues and q-quotients. The notions of q-residue, q-sign, and q-quotient were introduced by Robinson [**11**; **12**; **13**] and developed by Littlewood [**8**]. With every partition $(\lambda) = (\lambda_1, \lambda_2, \ldots \lambda_i)$ of l into i parts, there is associated a q-quotient, which is a sum of partitions of s, with an associated sign, and a q-residue or q-core, which is a partition of r, where s and r are such that $l = sq + r$. The definitions of these quantities are best illustrated by an example. Consider the partition (9542^21) of 23, and let $q = 3$. The numerical working consists of a series of lines:

A	9	5	4	2	2	1
B	5	4	3	2	1	0
C	14	9	7	4	3	1
D	2	3	7	4	0	1
E	7	4	3	2	1	0
F	2	0	0	0	0	0

A is the partition, B the numbers $i - 1, i - 2, \ldots, 1, 0$, and C the sum of A and B. D is obtained from C by reducing each number (mod 3) to the smallest non-negative integer so far unused, working from the right. E contains the numbers in D rearranged in descending order, and F is the difference between E and B.

The partition in F, i.e., (2), is the 3-residue. The sign of the permutation by which E is obtained from D, here positive, is the 3-sign. To obtain the 3-quotient, consider the decrease between C and D, in multiples of 3, of terms congruent to 0 (mod 3):

$$(9, 3) \to (3, 0) : (2, 1),$$

of terms congruent to 1:

$$(7, 4, 1) \to (7, 4, 1) : (0),$$

and of terms congruent to 2:

$$(14) \to (2) : (4).$$

The outer product of S-functions corresponding to these three partitions is found:

$$\{21\}\{4\}\{0\} = \{61\} + \{52\} + \{511\} + \{421\}, \tag{3.1}$$

and the 3-quotient is the corresponding set of partitions with the 3-sign appended:

$$+ (61) + (52) + (511) + (421).$$

The q-quotient is a sum of partitions of, say, n which is obtained from outer products of S-functions. The S-functions $\{n\}$ and $\{1^n\}$ can occur only with coefficient ± 1 (or 0) in such a product. For example, if $n = 4$ all possible quotients correspond to the S-functions:

$\{4\}; \{31\}; \{2^2\}; \{21^2\}; \{1^4\};$

$\{3\}\{1\} = \{4\} + \{31\}; \{21\}\{1\} = \{31\} + \{2^2\} + \{21^2\}; \{1^3\}\{1\} = \{21^2\} + \{1^4\};$

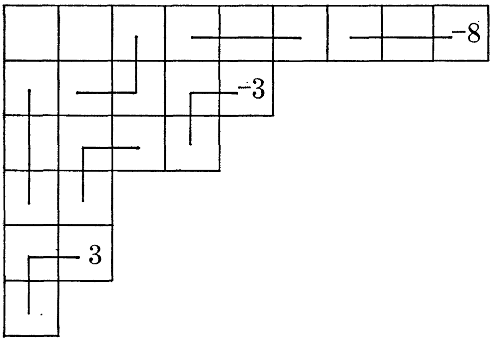
$\{2\}\{2\} = \{4\} + \{31\} + \{2^2\}; \{2\}\{1^2\} = \{31\} + \{21^2\}; \{1^2\}\{1^2\} = \{2^2\} + \{21^2\} + \{1^4\};$

$\{2\}\{1\}\{1\} = \{4\} + 2\{31\} + \{2^2\} + \{21^2\}; \{1^2\}\{1\}\{1\} = \{31\} + \{2^2\} + 2\{21^2\} + \{1^4\};$

$\{1\}\{1\}\{1\}\{1\} = \{4\} + 3\{31\} + 2\{2^2\} + 3\{21^2\} + \{1^4\}.$

So the partitions (n) and (1^n) can occur in a q-quotient only with coefficient ± 1 or 0.

The q-residue, q-sign, and q-quotient may also be obtained in a graphical manner. From the tableau for the partition (λ), hooks are removed whose length is a multiple of q. This multiple is denoted by n_j for a hook starting on the jth row, and each n_j is made as large as possible subject to three conditions. Each hook must (i) start from the right hand end of a row, each row being tried in turn starting at the bottom, (ii) move only to the left and down, and (iii) leave a regular tableau. Figure 1 illustrates this process for the tableau for (9542^21). The q-residue is the partition of the tableau which remains. If m_j is the number of rows covered by the hook starting at the end of the jth row, the q-sign is $\prod_j(-1)^{m_j+1}$. To find the q-quotient, the quantity $j - \lambda_j$ is found for each hook. If, for hooks starting on the rows $j_1, j_2, j_3 \ldots$, this quantity is congruent (mod q), then the S-function $\{n_{j_1}n_{j_2}n_{j_3} \ldots\}$ is constructed. The outer product of these S-functions, one for each congruence class, is found as before, giving the q-quotient. In Figure 1, the first square of each hook is marked with the value of $j - \lambda_j$. Since $-8 \equiv 1 \pmod 3$, $-3 \equiv 3 \equiv 0 \pmod 3$, and $n_1 = 4$, $n_2 = 2$, $n_5 = 1$, the 3-quotient is $\{4\}\{21\}\{0\}$, in agreement with (3.1).



The removal of hooks of length 3, 6 and 12 from the tableau for (9542^21) leaving the tableau for (2).

FIGURE 1

4. Application to the calculation of plethysms. Littlewood [8] proves the theorem that if the q-residue of (ν) is null and the q-quotient is $\sum k_{\lambda\nu}(\lambda)$, then

$$\{\lambda\} \otimes S_q = \sum k_{\lambda\nu}\{\nu\}.$$

This result can be used to calculate plethysms of the form $\{\lambda\} \otimes \{\mu\}$. Littlewood has two methods to suggest, but both involve fairly lengthy calculations and the establishing of tables of prior results. One method uses the symmetric function identity

$$\{m\} = \frac{1}{m}\sum_{r=0}^{m-1} S_{m-r}\{r\}$$

to obtain

$$\{\lambda\} \otimes \{m\} = \frac{1}{m}\sum_{r=0}^{m-1} (\{\lambda\} \otimes S_{m-r})(\{\lambda\} \otimes \{r\}),$$

by means of (2.6) and (2.7). The evaluation of this expression involves the finding of $\{\lambda\} \otimes S_r$, for $2 \leqq r \leqq m$, and $\{\lambda\} \otimes \{r\}$, for $2 \leqq r < m$. Then further calculations are necessary to find $\{\lambda\} \otimes \{\mu\}$.

The other method uses (2.3) in conjunction with (2.6) and (2.7) to obtain

$$\text{(4.1)} \qquad \{\lambda\} \otimes \{\mu\} = \frac{1}{m!}\sum_{\rho} h_\rho \chi_\rho^{(\mu)} \{\lambda\} \otimes S_\rho$$

$$= \frac{1}{m!}\sum_{\rho} h_\rho \chi_\rho^{(\mu)} (\{\lambda\} \otimes S_1)^a (\{\lambda\} \otimes S_2)^b (\{\lambda\} \otimes S_3)^c \ldots .$$

Here, again, $\{\lambda\} \otimes S_r$ for $2 \leqq r \leqq m$ must be known, and also $(\{\lambda\} \otimes S_1)^r$, i.e. $\{\lambda\}^r$, for $2 \leqq r \leqq m$. This second method can be greatly simplified by observing a relationship between these products.

For p prime,

(4.2)
$$\begin{aligned} S_a^{p^b} &= (\alpha_1^{\ a} + \alpha_2^{\ a} + \ldots + \alpha_n^{\ a})^{p^b} \\ &\equiv \alpha_1^{\ ap^b} + \alpha_2^{\ ap^b} + \ldots + \alpha_n^{\ ap^b} \pmod p \\ &= S_{ap^b}. \end{aligned}$$

Special cases of this result are particularly useful. For $a = b = 1$,

$$S_1^{\ p} \equiv S_p,$$

for $a = 1$,

$$S_1^{\ p^b} \equiv S_{p^b},$$

and for $b = 1$,

$$S_a^{\ p} \equiv S_{ap}.$$

Thus,

(4.3) $$\{\lambda\} \otimes S_p \equiv \{\lambda\} \otimes S_1^{\ p} = \{\lambda\}^p,$$

(4.4) $$\{\lambda\} \otimes S_{p^b} \equiv \{\lambda\} \otimes S_1^{\ p^b} = \{\lambda\}^{p^b},$$

(4.5) $$\{\lambda\} \otimes S_{ap} \equiv \{\lambda\} \otimes S_a^{\ p} = (\{\lambda\} \otimes S_a)^p.$$

So we have

$$\begin{aligned} \{\lambda\} \otimes S_2 &\equiv \{\lambda\}^2 \pmod 2, \\ \{\lambda\} \otimes S_3 &\equiv \{\lambda\}^3 \pmod 3, \\ \{\lambda\} \otimes S_4 &\equiv \{\lambda\}^4 \pmod 2, \\ \{\lambda\} \otimes S_5 &\equiv \{\lambda\}^5 \pmod 5, \\ \{\lambda\} \otimes S_6 &\equiv (\{\lambda\} \otimes S_3)^2 \pmod 2, \end{aligned}$$

etc.

These congruences are not in themselves sufficient to obtain $\{\lambda\} \otimes S_r$ from $\{\lambda\}^r$, but in certain cases the result can be determined. Rewriting Littlewood's theorem: if

$$\{\lambda\} \otimes S_r = \sum k_{\lambda\nu}\{\nu\},$$

then the r-quotient of (ν) contains $k_{\lambda\nu}(\lambda)$. But we have shown that an r-quotient can contain (l) or (1^l) only with coefficient ± 1 or 0. So $k_{l\nu}$ and $k_{1^l\nu}$ are ± 1 or 0. Therefore, the coefficients of the S-functions appearing in $\{l\} \otimes S_r$ and $\{1^l\} \otimes S_r$ are simply the r-signs of the corresponding partitions. Thus, the modular congruences give the coefficients $k_{l\nu}$ and $k_{1^l\nu}$ unambiguously except for congruences (mod 2), for which $+1 \equiv -1$. But in these cases the r-sign is easily determined.

The method for finding $\{l\} \otimes \{\mu\}$, for all partitions (μ) of m, is as follows. First, $\{l\}^m$ is calculated, noting the $\{l\}^r$, $2 \leqq r < m$, on the way. From these, the $\{l\} \otimes S_r$ can easily be found as shown above. Then the character-class-size products are used to complete (4.1). It is important to emphasize that the characters involved are only those for $\sum_m$ and not for the much larger group $\sum_{lm}$.

This method has been used for the machine calculation of $\{l\} \otimes \{\mu\}$ on the University of London's CDC 6600 computer. With $m = 4$, the values of l range up to 10; and for $m = 5$, up to 6. Table 3 shows a typical set of plethysms.

5. Symmetrized squares of S-functions. As a simple illustration, the result for $\{l\} \otimes \{2\}$ and $\{l\} \otimes \{1^2\}$ can easily be established. First of all

(5.1) $$\{l\}^2 = \{2l\} + \{2l - 1, 1\} + \{2l - 2, 2\} + \{2l - 3, 3\} + \dots .$$

In order to find $\{l\} \otimes S_2$, we must know the 2-sign of each partition. It is clear diagramatically that for partitions into even parts, hooks of length 2 can be removed from the two rows separately giving a positive 2-sign, while for partitions into two odd parts, one 2-hook must cover the two rows giving a negative 2-sign (see Figure 2). So we have

(5.2) $$\{l\} \otimes S_2 = \{2l\} - \{2l - 1, 1\} + \{2l - 2, 2\} - \{2l - 3, 3\} + \dots ,$$

and also

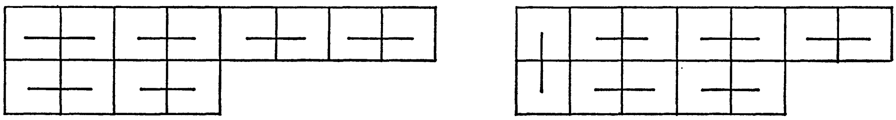
(5.3) $$\{l\} \otimes S_1^2 = \{l\}^2 = \{2l\} + \{2l - 1, 1\} + \{2l - 2, 2\} + \{2l - 3, 3\} + \dots .$$

Hence, the well-known results [**7**]:

(5.4) $$\{l\} \otimes \{2\} = \{l\} \otimes [\tfrac{1}{2}(S_1^2 + S_2)] = \{2l\} + \{2l - 2, 2\} + \dots ,$$

and

(5.5) $$\{l\} \otimes \{1^2\} = \{l\} \otimes [\tfrac{1}{2}(S_1^2 - S_2)] = \{2l - 1, 1\} + \{2l - 3, 3\} + \dots .$$



(a) (b)

FIGURE 2

Removal of 2-hooks from two-rowed tableaux with (a) rows of even numbers of boxes (b) rows of odd number of boxes.

6. Symmetrized cubes of S-functions. Thrall [**15**] produced a simple rule for writing down the plethysm $\{l\} \otimes \{3\}$. We can re-derive this result and also produce similar rules for immediately obtaining $\{l\} \otimes \{21\}$ and $\{l\} \otimes \{1^3\}$.

Again, an illustration makes the method clearest. We take $l = 4$ and find

(6.1) $$\{4\}^2 = \{8\} + \{71\} + \{62\} + \{53\} + \{44\},$$

and

(6.2) $$\{4\}^3 = \{12\} + 2\{11.1\} + 3\{10.2\} + 4\{93\} + 5\{84\} + 3\{75\} + \{66\} + \{10.1.1\} + 2\{921\} + 3\{831\} + 4\{741\} + 2\{651\} + \{822\} + 2\{732\} + 3\{642\} + \{552\} + \{633\} + \{543\} + \{444\}.$$

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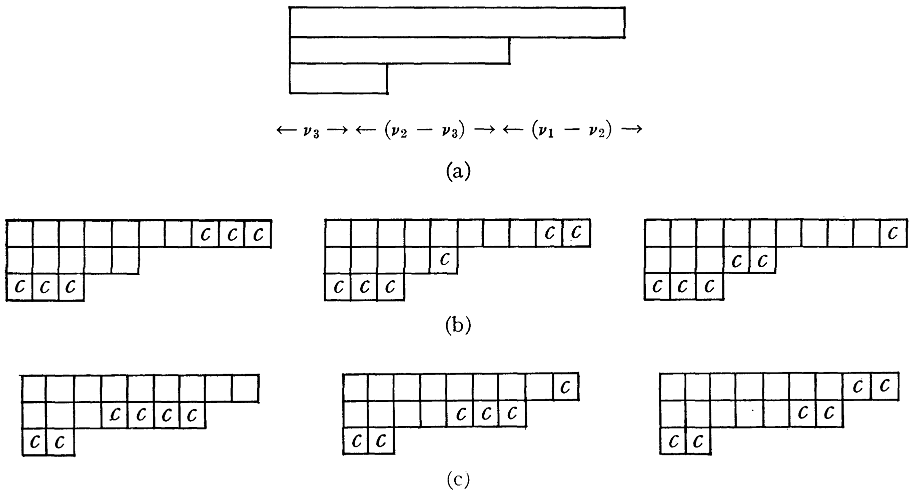
The result has been set out so that the pattern of the coefficients in $\{4\}^3$ is clear. It will be observed that the coefficient of $\{\nu\} = \{\nu_1\nu_2\nu_3\}$ is

$$M_\nu = 1 + \min(\nu_1 - \nu_2, \nu_2 - \nu_3).$$

This can be shown in general as follows. The coefficient of $\{\nu\}$ is the number of ways $\{\nu\}$ can be obtained from terms in $\{\lambda\}^2$ by multiplication with $\{\lambda\}$. This is equal to the number of ways in which l c's can be placed in the tableau for $\{\nu\}$ following the usual rules and completely filling the third row. This leaves n c's to be distributed between the first and second rows, where

$$3n = 2(\nu_2 - \nu_3) + (\nu_1 - \nu_2). \tag{6.3}$$

(See Figure 3(a).) From this, it is clear that the greater of $(\nu_2 - \nu_3)$ and $(\nu_1 - \nu_2)$ cannot be less than n, so the number of ways of distributing the c's is one more than the lesser of $(\nu_2 - \nu_3)$ and $(\nu_1 - \nu_2)$. See Figure 3(b), (c).



The number of ways of placing six c's in the tableaux (10 5 3) and (9 7 2). In both cases, the result is three, which equals $1 + \min(\nu_1 - \nu_2, \nu_2 - \nu_3)$.

FIGURE 3

Continuing with the calculation,

$$\{4\} \otimes S_2 = \{8\} - \{71\} + \{62\} - \{53\} + \{44\}, \tag{6.4}$$

so

$$\begin{aligned}\{4\} \otimes S_2S_1 = (\{4\} \otimes S_2)\{4\} = {} & \{12\} + \{10.2\} + \{84\} - \{75\} + \{66\} \\ & - \{10.1.1\} - \{831\} \\ & + \{822\} + \{642\} - \{552\} \\ & - \{633\} \\ & + \{444\}.\end{aligned} \tag{6.5}$$

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Each term in $\{4\} \otimes S_2S_1$ is obtained from a series of successive terms in $\{4\} \otimes S_2$, just as those in $\{4\}^3$ came from $\{4\}^2$. The alternation in signs in $\{4\} \otimes S_2$ means that a coefficient in $\{4\} \otimes S_2S_1$ must be 0 if an even number of terms contribute, and ± 1 if an odd number. In the latter case, the sign will be that of the first (or last) of the series of contributing terms in $\{4\} \otimes S_2$. If $(\nu_2 - \nu_3)$ is less than (or equal to) $(\nu_1 - \nu_2)$, this sign will be positive (negative) if ν_2 is even (odd). If $(\nu_1 - \nu_2)$ is less than $(\nu_2 - \nu_3)$, the sign will be similarly determined by ν_1. But $(\nu_1 - \nu_2)$ must be even in order to give an odd number of terms, so $\nu_1 \equiv \nu_2 \pmod 2$. See Figure 3. So the terms which occur in $\{\lambda\} \otimes S_2S_1$ have coefficient ± 1, according as ν_2 is even or odd.

$\{4\} \otimes S_3$ is obtained by reducing the coefficients in $\{4\}^3$ (mod 3) to ± 1 or 0:

$$(6.6) \qquad \begin{aligned} \{4\} \otimes S_3 = {} & \{12\} - \{11.1\} + \{93\} - \{84\} + \{66\} \\ & + \{10.1.1\} - \{921\} + \{741\} - \{651\} \\ & + \{822\} - \{732\} + \{552\} \\ & + \{633\} - \{543\} \\ & + \{444\}. \end{aligned}$$

Now,

$$(6.7) \qquad \{4\} \otimes \{3\} = \tfrac{1}{6}[\{4\} \otimes S_1^3 + 2.\{4\} \otimes S_3 + 3.\{4\} \otimes S_2S_1].$$

The coefficient of each S-function in the sum in square brackets must be divisible by 6. Since $\{4\} \otimes S_2S_1$ can only contribute coefficients ± 1 or 0, and this entry is multiplied by 3, the coefficients obtained from the sum of the first two terms must be divisible by 3 and, further, if even, will receive no contribution from the third term but, if odd, will receive ± 3 as ν_2 is even or odd. The coefficients in $\{4\} \otimes S_3$ are also ± 1 or 0, so the contribution from the second term will be ± 2 or 0. The coefficients in the first term are the M_ν. So we have Thrall's rule: $\{l\} \otimes \{3\} = \sum k_\nu \{\nu\}$, summed over all partitions of $3l$ with 3 or fewer parts, where k_ν is obtained by adding ± 2 or 0 to M_ν to give a result divisible by 3, then if even, dividing by 6, but if odd, first adding (subtracting) 3 if ν_2 is even (odd) and then dividing by 6.

Similarly for $\{l\} \otimes \{1^3\}$. We have

$$(6.8) \qquad \{l\} \otimes \{1^3\} = \tfrac{1}{6}[\{l\} \otimes S_1^3 + 2\{l\} \otimes S_3 - 3\{l\} \otimes S_2S_1],$$

so the only alteration in the above rule is the interchanging of "adding" and "subtracting".

Also,

$$(6.9) \qquad \{l\} \otimes \{21\} = \tfrac{1}{3}[\{l\} \otimes S_1^3 - \{l\} \otimes S_3].$$

So $\{l\} \otimes \{21\} = \sum k_\nu \{\nu\}$, where k_ν is obtained by adding ± 1 or 0 to M_ν to obtain a multiple of 3, and then dividing by 3.

Thus, $M_\nu = 1 + \min(\nu_1 - \nu_2, \nu_2 - \nu_3)$ and the "parity" of ν_2 determine the coefficient of $\{\nu\}$ in $\{l\} \otimes \{\mu\}$, (μ) a partition of 3. These coefficients are given in Table 1 for $M_\nu \leqq 11$ which suffices for $l \leqq 10$. Table 2 lists the partitions of 12 into not more than three parts with their M_ν and ν_2 "parity", and tabulates the plethysms $\{4\{ \otimes \{3\}$, $\{4\} \otimes \{21\}$, $\{4\} \otimes \{1^3\}$.

TABLE 1

M_ν	$\{l\} \otimes \{3\}$ ν_2 even	$\{l\} \otimes \{3\}$ ν_2 odd	$\{l\} \otimes \{21\}$	$\{l\} \otimes \{1^3\}$ ν_2 even	$\{l\} \otimes \{1^3\}$ ν_2 odd
1	1	0	0	0	1
2	0	0	1	0	0
3	1	0	1	0	1
4	1	1	1	1	1
5	1	0	2	0	1
6	1	1	2	1	1
7	2	1	2	1	2
8	1	1	3	1	1
9	2	1	3	1	2
10	2	2	3	2	2
11	2	1	4	1	2

$M_\nu = 1 + \min(\nu_1 - \nu_2, \nu_2 - \nu_3)$ determines the coefficient of $\{\nu\}$ in the three plethysms $\{l\} \otimes \{3\}$, $\{l\} \otimes \{21\}$, $\{l\} \otimes \{1^3\}$ except that if M_ν is odd it is necessary to know also the "parity" of ν_2 for $\{l\} \otimes \{3\}$ and $\{l\} \otimes \{1^3\}$.

TABLE 2

$\{\nu\}$	M_ν	$\nu_2 (\text{mod } 2)$	$\{4\} \otimes \{3\}$	$\{4\} \otimes \{21\}$	$\{4\} \otimes \{1^3\}$
$\{12\}$	1	0	1	0	0
$\{11\ 1\}$	2	1	0	1	0
$\{10\ 2\}$	3	0	1	1	0
$\{93\}$	4	1	1	1	1
$\{84\}$	5	0	1	2	0
$\{75\}$	3	1	0	1	1
$\{66\}$	1	0	1	0	0
$\{10\ 1\ 1\}$	1	1	0	0	1
$\{921\}$	2	0	0	1	0
$\{831\}$	3	1	0	1	1
$\{741\}$	4	0	1	1	1
$\{651\}$	2	1	0	1	0
$\{822\}$	1	0	1	0	0
$\{732\}$	2	1	0	1	0
$\{642\}$	3	0	1	1	0
$\{552\}$	1	1	0	0	1
$\{633\}$	1	1	0	0	1
$\{543\}$	2	0	0	1	0
$\{444\}$	1	0	1	0	0

The plethysms $\{4\} \otimes \{3\}$, $\{4\} \otimes \{21\}$ and $\{4\} \otimes \{1^3\}$ calculated from Table 1.

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TABLE 3

	$\{5\} \otimes \{4\}$	$\{5\} \otimes \{31\}$	$\{5\} \otimes \{2^2\}$	$\{5\} \otimes \{21^2\}$	$\{5\} \otimes \{1^4\}$
{20}	1	0	0	0	0
{19 1}	0	1	0	0	0
{18 2}	1	1	1	0	0
{18 1 1}	0	0	0	1	0
{17 3}	1	2	0	1	0
{17 2 1}	0	1	1	1	0
{17 1 1 1}	0	0	0	0	1
{16 4}	2	2	2	1	0
{16 3 1}	0	2	1	2	1
{16 2 2}	1	1	1	0	0
{16 2 1 1}	0	0	0	1	0
{15 5}	1	4	1	2	0
{15 4 1}	1	3	2	3	1
{15 3 2}	1	2	1	2	0
{15 3 1 1}	0	0	1	1	1
{15 2 2 1}	0	1	0	0	0
{14 6}	2	3	3	2	1
{14 5 1}	1	4	3	5	1
{14 4 2}	2	4	3	2	1
{14 4 1 1}	0	1	0	2	1
{14 3 3}	0	1	0	2	1
{14 3 2 1}	0	1	1	1	0
{14 2 2 2}	1	0	0	0	0
{13 7}	1	4	1	3	0
{13 6 1}	2	5	3	5	2
{13 5 2}	2	6	3	5	1
{13 5 1 1}	0	1	2	2	2
{13 4 3}	1	3	2	3	1
{13 4 2 1}	1	2	1	2	0
{13 3 3 1}	0	0	1	1	1
{13 3 2 2}	0	1	0	0	0
{12 8}	2	2	3	2	1
{12 7 1}	1	5	3	5	2
{12 6 2}	3	6	5	5	2
{12 6 1 1}	0	2	1	3	0
{12 5 3}	1	5	3	6	2
{12 5 2 1}	1	3	2	3	1
{12 4 4}	2	2	2	1	0
{12 4 3 1}	0	2	1	2	1
{12 4 2 2}	1	1	1	0	0
{12 3 3 2}	0	0	0	1	0
{11 9}	0	3	0	2	0
{11 8 1}	1	4	3	4	1
{11 7 2}	2	6	3	6	1
{11 7 1 1}	1	1	2	2	2
{11 6 3}	2	6	4	6	2
{11 6 2 1}	1	4	2	3	1
{11 5 4}	1	5	3	4	1

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TABLE 3 (*continued*)

	{5} ⊗ {4}	{5} ⊗ {31}	{5} ⊗ $\{2^2\}$	{5} ⊗ $\{21^2\}$	{5} ⊗ $1^4\}$
{11 5 3 1}	1	2	3	4	2
{11 5 2 2}	1	2	0	1	0
{11 4 4 1}	1	2	0	1	0
{11 4 3 2}	0	1	1	1	0
{11 3 3 3}	0	0	0	0	1
{10 10}	1	0	2	0	1
{10 9 1}	1	2	1	3	1
{10 8 2}	2	4	4	3	2
{10 8 1 1}	0	2	0	2	0
{10 7 3}	1	5	3	6	2
{10 7 2 1}	1	3	2	3	1
{10 6 4}	3	5	4	4	2
{10 6 3 1}	1	4	2	4	1
{10 6 2 2}	1	1	2	1	0
{10 5 5}	0	2	1	4	1
{10 5 4 1}	1	3	2	3	1
{10 5 3 2}	0	2	1	2	1
{10 4 4 2}	1	1	1	0	0
{10 4 3 3}	0	0	0	1	0
{ 9 9 2}	0	2	0	2	0
{ 9 9 1 1}	0	0	2	0	1
{ 9 8 3}	1	3	2	3	1
{ 9 8 2 1}	1	2	1	2	0
{ 9 7 4}	1	4	2	4	1
{ 9 7 3 1}	1	2	3	3	2
{ 9 7 2 2}	0	2	0	1	0
{ 9 6 5}	1	3	2	3	1
{ 9 6 4 1}	1	4	2	3	0
{ 9 6 3 2}	1	2	1	2	1
{ 9 5 5 1}	0	1	2	2	2
{ 9 5 4 2}	1	2	1	2	0
{ 9 5 3 3}	0	0	1	1	1
{ 9 4 4 3}	0	1	0	0	0
{ 8 8 4}	1	1	2	1	0
{ 8 8 3 1}	0	2	0	1	0
{ 8 8 2 2}	1	0	1	0	1
{ 8 7 5}	0	2	1	2	1
{ 8 7 4 1}	1	2	1	2	1
{ 8 7 3 2}	0	1	1	2	0
{ 8 6 6}	1	1	1	0	0
{ 8 6 5 1}	1	2	1	2	0
{ 8 6 4 2}	1	2	2	1	1
{ 8 6 3 3}	0	1	0	1	0
{ 8 5 5 2}	0	1	0	2	1
{ 8 5 4 3}	0	1	1	1	0
{ 8 4 4 4}	1	0	0	0	0
{ 7 7 6}	0	0	0	1	0
{ 7 7 5 1}	0	0	1	1	1

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TABLE 3 (*concluded*)

	$\{5\}\otimes\{4\}$	$\{5\}\otimes\{31\}$	$\{5\}\otimes\{2^2\}$	$\{5\}\otimes\{21^2\}$	$\{5\}\otimes\{1^4\}$
{7 7 4 2}	0	1	0	1	0
{7 7 3 3}	0	0	1	0	1
{7 6 6 1}	0	1	0	0	0
{7 6 5 2}	0	1	1	1	0
{7 6 4 3}	1	1	0	1	0
{7 5 5 3}	0	0	1	1	1
{7 5 4 4}	0	1	0	0	0
{6 6 6 2}	1	0	0	0	0
{6 6 5 3}	0	1	0	0	0
{6 6 4 4}	0	0	1	0	0
{6 5 5 4}	0	0	0	1	0
{5 5 5 5}	0	0	0	0	1

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