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BEST DIOPHANTINE APPROXIMATIONS TO A SET OF LINEAR FORMS

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Abstract

We define the notion of a best Diophantine approximation vector to a set of linear forms. This generalizes definitions of a best approximation vector to a single linear form and of a best simultaneous Diophantine approximation vector. We derive necessary and sufficient conditions for the existence of an infinite set of best Diophantine approximation vectors. Finally, we prove that such approximation vectors are spaced far apart in an appropriate sense.

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1. Introduction

This paper defines the notion of a best Diophantine approximation vector to a set of linear forms. This definition generalizes the notions of a best approximation to a single linear form in [1], [2], [4] and of a best simultaneous Diophantine approximation given in [3], [5], [6], [7]. We derive necessary and sufficient conditions for an infinite set of best Diophantine approximation vectors to exist. Finally we prove that best approximation vectors are spaced far apart, in an appropriate sense.

Let $\{l_i\}_{i=1}^n$ be a set of n linear forms in m variables and let L: $\mathbf{R}^m \to \mathbf{R}^n$ be the linear transformation defined by

$$L((x_1,\ldots,x_m)) = (l_1, l_2,\ldots,l_n) \tag{1.1}$$

where .

$$l_i = \alpha_1^{(i)}x_1 + \cdots + \alpha_n^{(i)}x_m \tag{1.2}$$

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for $1 \leqslant i \leqslant n$. A *seminorm* $\|\cdot\|$ on $\mathbf{R}^k$ is a map $\mathbf{R}^k \to \mathbf{R}$ satisfying the axioms

(i) $\|\mathbf{x}\| \geqslant 0$,

(ii) $\|a\mathbf{x}\| = |a| \, \|\mathbf{x}\|$ for $a \in \mathbf{R}$ and $\mathbf{x} \in \mathbf{R}^k$,

(iii) $\|\mathbf{x}_1 + \mathbf{x}_2\| \leqslant \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^k$.

The basic definition is as follows.

DEFINITION. A nonzero integer vector $\mathbf{v} \in \mathbf{Z}^m$ is a *best (Diophantine) approximation vector* to L *with respect to the seminorms* $\|\cdot\|_1$ on $\mathbf{R}^m$ and $\|\cdot\|_2$ on $\mathbf{R}^n$ provided that for all $\mathbf{w} \in \mathbf{Z}^n$ the following implications hold.

(1.4) $$(1)\ \|\mathbf{w}\|_1 \leqslant \|\mathbf{v}\|_1 \Rightarrow \|L(\mathbf{w})\|_2 \geqslant \|L(\mathbf{v})\|_2.$$

(1.5) $$(2)\ \|L(\mathbf{w})\|_2 \leqslant \|L(\mathbf{v})\|_2 \Rightarrow \|\mathbf{w}\|_1 \geqslant \|\mathbf{v}\|_1.$$

(Note that (1) and (2) are equivalent except for cases of equality.)

A best approximation vector $\mathbf{v}$ is *nontrivial* if $\|\mathbf{v}\|_1 \neq 0$ and $\|L(\mathbf{v})\|_2 \neq 0$. If for any $\varepsilon > 0$ it is possible to find $\mathbf{v} \in \mathbf{Z}^m$ (depending on ε) such that

$$0 < \|\mathbf{v}\|_1 < \varepsilon,$$
$$0 < \|L(\mathbf{v})\|_2 < \varepsilon,$$

then no nontrivial best approximation vectors can exist. We will show that this possibility is excluded if $\|\cdot\|_1$, $\|\cdot\|_2$ and L satisfy the transversality condition defined below. Any seminorm $\|\cdot\|$ on $\mathbf{R}^k$ has a *zero set* given by

$$V = V(\|\cdot\|) = \{\mathbf{x} \colon \|\mathbf{x}\| = 0\},$$

which is a subspace of $\mathbf{R}^k$. Let V_1, V_2 denote the zero sets of $\|\cdot\|_1$, $\|\cdot\|_2$ respectively.

DEFINITION. L is *transverse* with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ if

(1.6) $$V_1 \cap L^{-1}(V_2) = \{\mathbf{0}\}.$$

We also make the following definitions which are relevant in determining the order type of the set of best approximation vectors.

DEFINITION. $\|\cdot\|_1$ is *collapsed* if there is a nonzero $\mathbf{x} \in \mathbf{Z}^m$ with $\|\mathbf{x}\|_1 = 0$.

DEFINITION. L is *degenerate* with respect to $\|\cdot\|_2$ if there exists a nonzero $\mathbf{v} \in \mathbf{Z}^m$ such that $\|L(\mathbf{v})\|_2 = 0$. Otherwise L is *nondegenerate*.

The following result characterizes the existence of best approximation vectors.

THEOREM 1.1. *If L is not transverse with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ then there are no nontrivial best approximation vectors. If L is transverse with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ then L has a set B of nontrivial best approximation vectors which can be numbered $\{\mathbf{v}_j\}$ for $n_0 < j < n_1$ such that for all $n_0 < j < k < n_1$,*

(1.8) $$\|\mathbf{v}_j\|_1 \leqslant \|\mathbf{v}_k\|_1 \quad \text{and} \quad \|L(\mathbf{v}_j)\|_2 \geqslant \|L(\mathbf{v}_k)\|_2.$$

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In addition:

(1) $n_0 = -\infty$ *if and only if* $\|\cdot\|_1$ *is not collapsed and* $\|\cdot\|_1$ *is not a norm.*

(2) $n_1 = +\infty$ *if and only if L is nondegenerate and either* $\|\cdot\|_2$ *is not a norm or* $\|\cdot\|_2$ *is a norm and L does not have full row rank.*

If $n_1 = +\infty$ *then* $\|\mathbf{v}_n\|_1 \to \infty$ *and* $\|L(\mathbf{v}_n)\|_2 \to 0$ *as* $n \to \infty$.

The following result concerns the spacing of best approximation vectors.

THEOREM 1.2. *Let L be transverse with respect to* $\|\cdot\|_1$ *and* $\|\cdot\|_2$. *Let* $\{\mathbf{v}_j\}$ *be the set of best approximation vectors. If* $D = 7^m$ *then*

$$\|\mathbf{v}_{k+D}\|_1 \geqslant 2\|\mathbf{v}_k\|_1 \tag{1.9}$$

holds for all k for which $\mathbf{v}_{k+D}$, $\mathbf{v}_k$ *exist.*

Theorems 1.1 and 1.2 are proved in Sections 2 and 3 respectively. Before giving the proofs we indicate how the notions of best approximation in [1] – [7] are special cases of the definition given here.

EXAMPLE 1 (Approximations to a Single Linear Form). Let

$$L: \mathbf{R}^{n+1} \to \mathbf{R} \quad \text{via} \quad l_i = \alpha_1 x_1 + \cdots + \alpha_n x_n - x_0.$$

Given a norm $\|\cdot\|$ on $\mathbf{R}^n$, let $\|\cdot\|$ be the seminorm on $\mathbf{R}^{n+1}$ defined by $\|(x_0, x_1, \ldots, x_n)\|_1 = \|(x_1, \ldots, x_n)\|$. $\|\cdot\|_2$ is taken to be the sup norm on $\mathbf{R}^1$. Note that $\|\cdot\|_1$ is collapsed. L is always transverse and is nondegenerate if and only if

$$\dim_{\mathbf{Q}}[1, \alpha_1, \ldots, \alpha_n] = n + 1.$$

EXAMPLE 2 (Best simultaneous Diophantine approximations). Let

$$L: \mathbf{R}^{n+1} \to \mathbf{R}^n \quad \text{via} \quad l_i = \alpha_i x_0 - x_i$$

for $1 \leqslant i \leqslant n$. Here $\|\cdot\|_1$ is the seminorm

$$\|\mathbf{x}\|_1 = |x_0|$$

and $\|\cdot\|_2 = \|\cdot\|$ is an arbitrary norm on $\mathbf{R}^n$. Note $\|\cdot\|_1$ is collapsed and L is always transverse. L is nondegenerate exactly when some $\alpha_i \notin \mathbf{Q}$.

In both examples, Theorem 1.1 guarantees that if L is nondegenerate then the set B of best approximations can be written $\{\mathbf{v}_k\}$ for $1 \leqslant k < \infty$ such that (1.8) holds.

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2. Existence of best approximation vectors

PROOF OF THEOREM 1.1. Suppose L is not transverse, so that $V_1 \cap L^{-1}(V_2) = V_3 \neq \mathbf{0}$. Let $\|\cdot\|_e$ denote the Euclidean norm on $\mathbf{R}^m$ and π orthogonal projection onto V_3. For any $\varepsilon > 0$ we can find a nonzero $\mathbf{v} \in \mathbf{Z}^m$ with .

(2.1) $$\|\mathbf{v} - \pi(\mathbf{v})\|_e < \varepsilon.$$

If $A = \max|\alpha_j^{(i)}|$, where the $\alpha_j^{(i)}$ are given by (1.2), then

(2.2) $$\|L(\mathbf{v}) - L(\pi(\mathbf{v}))\|_e < A\varepsilon.$$

Now there exist positive constants c_1, c_2 such that

$$c_1\|\mathbf{x}\|_e \geqslant \|\mathbf{x}\|_1, \qquad c_2\|\mathbf{y}\|_e \geqslant \|\mathbf{y}\|_2,$$

for all $\mathbf{x} \in \mathbf{R}^m$, $\mathbf{y} \in \mathbf{R}^n$, where

$$c_i = \sup\{\|\mathbf{x}\|_i : \|\mathbf{x}\|_e = 1\} \quad \text{for } i = 1, 2.$$

Hence

$$\|\mathbf{v}\|_1 = \|\mathbf{v} - \pi(\mathbf{v})\|_1 < c_1\varepsilon,$$
$$\|L(\mathbf{v})\|_2 = \|L(\mathbf{v}) - L(\pi(\mathbf{v}))\|_2 < c_2 A\varepsilon.$$

Since ε can be chosen arbitrarily small, there are no best approximation vectors with $\|\mathbf{x}\|_1 \neq 0$, $\|L(\mathbf{x})\|_2 \neq 0$.

Now suppose L is transverse. Consider the m-dimensional subspace

$$W = \{(\mathbf{x}, L(\mathbf{x})) : \mathbf{x} \in \mathbf{R}^m\}$$

of $\mathbf{R}^{m+n}$. For $\mathbf{x} \in \mathbf{R}^m$ we set $\mathbf{x}^* = (\mathbf{x}, L(\mathbf{x})) \in W$. The map $\mathbf{x} \to \mathbf{x}^*$ is a linear vector space isomorphism. Define a seminorm $\|\cdot\|$ on W by

$$\|\mathbf{x}^*\| = \|\mathbf{x}\|_1 + \|L(\mathbf{x})\|_2.$$

Then $\|\cdot\|$ is a norm by the transversality condition. Now set

$$\Omega(c_1, c_2) = \{\mathbf{x}^* \in W : \|\mathbf{x}\|_1 \leqslant c_1 \text{ and } \|L(\mathbf{x})\|_2 \leqslant c_2\}.$$

This is a centrally symmetric convex body. We define a volume on W arising from m-dimensional Lebesgue measure on $\mathbf{R}^m$ via the map $\mathbf{x} \to \mathbf{x}^*$. Then

$$\Lambda = \{(\mathbf{x}, L(\mathbf{x})) : \mathbf{x} \in \mathbf{Z}^m\}$$

is a lattice in W whose unit cell has volume 1. Let

$$B(c) = \{\mathbf{x}^* \in W : \|\mathbf{x}^*\| \leqslant c\},$$

and observe that

(2.3) $$B(c) \subseteq \Omega(c, c) \subseteq B(2c).$$

For sufficiently small c, $B(2c)$ and hence $\Omega(c, c)$ contains only the point $\mathbf{0}$ of Λ since $\|\cdot\|$ is a norm. By Minkowski's theorem, $\Omega(c, c)$ contains a nonzero lattice point if its volume exceeds 2^m, which must occur for some c by (2.3). Now take the smallest c for which $\Omega(c, c)$ has points of Λ on its boundary but none except $\mathbf{0}$ inside. The finite set B_0^* of lattice points $\mathbf{x}^*$ on the boundary can be divided into three classes, those with

(1) $\|\mathbf{x}\|_1 < c$, $\|L(\mathbf{x})\|_2 = c$,

(2) $\|\mathbf{x}\|_1 = c$, $\|L(\mathbf{x})\|_2 < c$,

(3) $\|\mathbf{x}\|_1 = \|L(\mathbf{x})\|_2 = c$.

If there are vectors $\mathbf{x}$ satisfying (1), those $\mathbf{x}$ with the smallest value of $\|\mathbf{x}\|_1$ are best approximation vectors. Call this set of vectors B_{-1}. If there are vectors $\mathbf{x}$ satisfying (2), those $\mathbf{x}$ with the smallest value of $\|L(\mathbf{x})\|_2$ are best approximation vectors. Call this set of vectors B_1. All vectors satisfying (3) are best approximation vectors if there are no $\mathbf{x}$ satisfying either (1) or (2), otherwise none are. Call this set of vectors B_0. Note that at least one of B_{-1}, B_0, B_1 is empty. Let (d_{-1}, r_{-1}), (d_0, r_0) and (d_1, r_1) denote the values of $(\|\mathbf{x}\|_1, \|L(\mathbf{x})\|_2)$ in each of the sets B_{-1}, B_0, B_1, respectively, which are nonempty, and set them equal to (c, c) otherwise.

Now start with $\Omega(c, r_1)$ with $c = d_1$ and increase c until the first value d_2 is encountered at which an element of Λ with $\|L(\mathbf{x})\|_2 < r_1$ appears on the boundary of $\Omega(d_2, r_1)$. Let B_2 be the set of lattice points on this boundary with minimal $\|L(\mathbf{x})\|_2 = r_2$. The elements of B_2 are all best approximation vectors. Now repeat this process starting with $\Omega(c, r_2)$ with $c = d_2$, and continue it to construct $B_3, B_4, \ldots$. If for some $\Omega(c, r_j)$ we can let $c \to \infty$ with no nonzero element of Λ ever appearing on the boundary having $\|L(\mathbf{x})\|_2 < r_j$, we say this process *terminates on the right* at B_j.

Now start with $\Omega(d_{-1}, c)$ with $c = r_{-1}$ and increase c until the first value r_{-2} is encountered at which $\|\mathbf{x}\|_1 < d_{-1}$. Let B_{-2} be the set of vectors on the boundary of $\Omega(d_{-2}, r_{-2})$ with minimal $\|\mathbf{x}\|_1 = d_{-2}$. Now repeat this process starting with $\Omega(d_{-2}, c)$ with $c = r_{-2}$ and continue it to construct $B_{-2}, B_{-3}, \ldots$. If for some $\Omega(d_{-j}, c)$ we can let $c \to \infty$ with no nonzero element of Λ ever appearing with $\|\mathbf{x}\|_1 < d_{-j}$, we say this process *terminates on the left* at B_{-j}.

We obtain in this way a sequence $\{(d_j, r_j)\colon -n_1 < j < n_2\}$ in which $d_{j-1} < d_j$ (except for $j = 0, 1$ where equality may occur) and $r_{j-1} > r_j$ (except for $j = 0, 1$ where equality may occur). Note that this construction shows there are no best approximation vectors with $d_{j-1} < \|\mathbf{x}\|_1 < d_j$ and produces all those with $\|\mathbf{x}\| = d_j$; also that there are none with $r_{j-1} > \|L(\mathbf{x})\|_2 > r_j$ and it produces all those with $\|L(\mathbf{x})\|_2 = r_j$. If truncation on the left occurs at (d_{-j}, r_{-j}), this construction guarantees there are no best approximation vectors with $0 < \|\mathbf{x}\|_1 < d_{-j}$ or with $\|L(\mathbf{x})\|_2 > r_j$. If truncation on the right occurs, it guarantees there are no best approximation vectors with $0 \leqslant \|L(\mathbf{x})\|_2 < r_{-j}$, or with $\|\mathbf{x}\|_1 > d_j$. Thus the construction produces all best approximation vectors $\mathbf{x}$ with

$$\lim_{k \to \infty} d_{-k} < \|\mathbf{x}\|_1 < \lim_{k \to \infty} d_k$$

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and

$$\lim_{k\to\infty} r_{-k} > \|L(\mathbf{x})\|_2 > \lim_{k\to\infty} r_k,$$

where we use the convention that if termination on the left occurs then $\lim_{k\to\infty} d_{-k} = 0$, $\lim_{k\to\infty} r_{-k} = \infty$, and if termination on the right occurs, then $\lim_{k\to\infty} d_k = \infty$, $\lim_{k\to\infty} r_k = 0$.

This process of obtaining the best approximations is geometrically similar to that of Voronoi's algorithm for finding units in a non-totally real cubic field (for example, see [8]).

We next check that the set of elements in the union of the B_j's exhausts the set B of best approximation vectors. Suppose not, so that there is such a best approximation vector $\mathbf{x}$. By the remarks above, it must be either that termination on the right doesn't occur and that

$$\|\mathbf{x}\|_1 \geqslant \lim_{j\to\infty} d_j, \qquad \|L(\mathbf{x})\|_2 \leqslant \lim_{j\to\infty} r_j,$$

or else that termination on the left doesn't occur and

$$\|\mathbf{x}\|_1 \leqslant \lim_{j\to\infty} d_{-j}, \qquad \|L(\mathbf{x})\|_2 \geqslant \lim_{j\to\infty} r_{-j}.$$

We will rule out both these possibilities by showing that if termination on the right doesn't occur, then $d_j \to \infty$ and $r_j \to 0$ as $j \to \infty$, and that if termination on the left doesn't occur, then $d_{-j} \to 0$, $r_{-j} \to \infty$ as $j \to \infty$.

We treat the case that termination on the right doesn't occur. Pick an element $\mathbf{x}_j \in B_j$ for all $j \geqslant 2$. Then $\|L(\mathbf{x}_j)\|_2 = r_j$ and $r_2 > r_3 > r_4 > \cdots$. Now $\|(\mathbf{x}_j, L(\mathbf{x}_j))\| = r_j + d_j \to \infty$ as $j \to \infty$ since $\|\cdot\|$ is a norm and $\mathbf{x}_j^* \in \Lambda$. Since r_j is bounded, $d_j \to \infty$ as $j \to \infty$. It remains to show $r_j \to 0$ as $j \to \infty$. To do this it suffices to prove the following fact.

FACT. *If termination on the right doesn't occur, then for any* $\varepsilon > 0$ *there exists a nonzero* $\mathbf{x} \in \mathbf{Z}^m$ *such that* $\|L(\mathbf{x})\|_2 < \varepsilon$.

If the Fact is true, we can find some $\mathbf{x}_j \in B_j$ for which $\|\mathbf{x}_j\|_1 > \|\mathbf{x}\|_1$, in which case by the definition of best approximation vector $r_j = \|L(\mathbf{x}_j)\|_2 \leqslant \|L(\mathbf{x})\|_2 < \varepsilon$. Hence $r_j \to 0$ as required.

To prove the Fact, we observe that its conclusion is equivalent to the statement that for any positive ε the region $\Omega(c, \varepsilon)$ contains a nonzero point of Λ, for some sufficiently large $c = c(\varepsilon)$. This occurs if:

1. $\|\cdot\|_2$ is not a norm. In this case $L^{-1}(\mathbf{0})$ is a non-trivial subspace of $\mathbf{R}^m$. Then the set $\{\mathbf{x}\colon \|L(\mathbf{x})\|_2 < \varepsilon\}$ contains a small open ball around the origin translated by all elements of $L^{-1}(\mathbf{0})$, hence has infinite volume in $\mathbf{R}^m$. Consequently the

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volume of $\Omega(c, \varepsilon)$ is unbounded as $c \to \infty$, and so contains nonzero lattice points of Λ for sufficiently large c by Minkowski's theorem.

2. L does not have full rank m. Then $L(\mathbf{Z}^m)$ projects onto a k-dimensional subspace $\mathbf{S}$ of $\mathbf{R}^n$, where $k \leqslant m - 1$. A standard pigeonhole principle argument then proves for any ε there exist nonzero $\mathbf{x} \in \mathbf{Z}^m$ with $\| L(\mathbf{x})\|_2 < \varepsilon$.

The exceptional case is where $\| \cdot \|_2$ is a norm and L has full rank m. In this case $L(\mathbf{Z}^m)$ is a lattice, and for any c_0 there are only a finite number of $\mathbf{x} \in \mathbf{Z}^m$ such that $\| L(\mathbf{x})\|_2 < c_0$, so termination on the right occurs in this case. This proves the fact.

One shows that $d_j \to 0$ and $r_j \to \infty$ as $j \to \infty$ if termination on the left does not occur by similar arguments.

We obtain a numbering $\{\mathbf{v}_j\}$ of the elements of B that satisfies (1.8) by ordering the elements of each B_j in a way that satisfies (1.8), concatenating the sets B_j using the obvious ordering, and picking $\mathbf{v}_0 \in B_0$.

It remains to specify the conditions under which termination on the left or right occurs.

If $\| \cdot \|_1$ is not a norm, then $\Omega(c_1, c_2)$ has unbounded volume as $c_2 \to \infty$. Minkowski's theorem then asserts that for large enough c_2 there must exist nonzero lattice points with $\|\mathbf{x}\|_1 < c_1$, $\| L(\mathbf{x})\|_2 < c_2$. Thus termination on the left can occur only if there is a nonzero $\mathbf{x} \in \mathbf{Z}^m$ with $\|\mathbf{x}\|_1 = 0$, that is, $\| \cdot \|_1$ is collapsed. If $\| \cdot \|_1$ is a norm there exists c_0 so that $\|\mathbf{x}\|_1 < c_0$ contains only $\mathbf{0} \in \mathbf{Z}^m$ and termination on the left occurs.

If $\| \cdot \|_2$ is not a norm then $\Omega(c_1, c_2)$ has unbounded volume as $c_1 \to \infty$. As above, lattice points with $\|\mathbf{x}\|_1 < c_1$, $\| L(\mathbf{x})\|_2 < c_2$ must occur. In this case termination on the right can occur only if there is an $\mathbf{x}$ with $\| L(\mathbf{x})\|_2 = 0$, that is, L is degenerate with respect to $\| \cdot \|_2$. If $\| \cdot \|_2$ is a norm and L has rank m, then it was shown earlier that termination on the right occurs. If L has rank $< m$, then $\| L(\mathbf{x})\|_2 < \varepsilon$ has an infinite number of solutions, and termination on the right can occur only if L is degenerate with respect to $\| \cdot \|_2$.

3. Spacing of best approximation vectors

PROOF OF THEOREM 1.2. Let $D = 7^m$, and set $\|\mathbf{v}_k\|_1 = r_0$, $\| L(\mathbf{v}_k)\|_2 = r_1$. Suppose that

$$\|\mathbf{v}_{k+d}\|_1 \leqslant 2\|\mathbf{v}_k\|_1. \tag{3.1}$$

By definition of a best approximation vector, we know that for $\mathbf{w} \neq 0$ in $\mathbf{Z}^m$,

$$\|\mathbf{w}\|_1 \leqslant r_0 \Rightarrow \| L(\mathbf{w}_1)\|_2 \geqslant r_1. \tag{3.2}$$

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Now consider the seminorm $\|\cdot\|_c$ defined on W by

$$\|(\mathbf{x}, L(\mathbf{x}))\|_c = \|\mathbf{x}\|_1 + c\|L(\mathbf{x})\|_2$$

where $c > 0$. Then $\|\cdot\|_c$ is actually a norm on W by the transversality condition. For $\mathbf{x} \in \mathbf{R}^m$ let $\mathbf{x}^* = (\mathbf{x}, L(\mathbf{x})) \in W$. We choose $c = r_0/r_1$ so that

$$\|\mathbf{v}_k^*\|_c = 2r_0.$$

Since $\|L(\mathbf{v}_{k+i})\|_2 \leqslant \|L(\mathbf{v}_k)\|_2$, (3.1) yields, for $0 \leqslant i \leqslant D$,

$$\|\mathbf{v}_{k+i}^*\|_c \leqslant 3r_0. \tag{3.3}$$

Now for $0 \leqslant i < j \leqslant D$,

$$\mathbf{w} = \mathbf{v}_{k+j} - \mathbf{v}_{k+i} \neq \mathbf{0}.$$

Using (3.2) we obtain that either $\|\mathbf{w}\|_1 > r_0$ or $\|L(\mathbf{w})\|_2 \geqslant r_1$, so that in either case

$$\|\mathbf{w}^*\|_c = \|\mathbf{v}_i^* - \mathbf{v}_j^*\|_c \geqslant r_0. \tag{3.4}$$

For $\mathbf{x}^* \in \mathbf{W}$ let

$$B(\mathbf{x}^*, \lambda) = \{\mathbf{y}^* \in W: \|\mathbf{x}^* - \mathbf{y}^*\|_c < \lambda\}.$$

Now consider the $7^m + 1$ balls $B(\mathbf{v}_{k+i}^*, \frac{1}{2}r_0)$ for $0 \leqslant i \leqslant 7^m$. The triangle inequality and (3.4) show these are disjoint sets. Hence they occupy a volume $(7^m + 1)V$ where V is the volume of $B(0, \frac{1}{2}r_0)$. But by (3.3) all these balls sit inside the ball $B(0, \frac{7}{2}r_0)$, which has volume 7^mV. This contradiction proves Theorem 1.2.

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