Multiple Periodic Variable Stars
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NON-RADIAL OSCILLATIONS OF DEGENERATE DWARFS: A THEORETICAL REVIEW

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Abstract. The current status of the theory of non-radial oscillations of white dwarfs and hydrogen shell-burning degenerate stars, in the linear, quasi-adiabatic approximation, is reviewed. Relevant aspects of the thermal structure of such stars are summarized, and several problem areas requiring further investigation are identified. Among the most interesting and important of these are the question of the existence of g^- modes in white dwarfs; the nature of the excitation mechanisms for non-radial oscillations; the possibility of tidal excitation; the problem of mode-coupling between pulsations and convection; and the question of the effects of a solid, crystallizing core.

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1. Introduction

Recent observations, as reviewed by Warner (1975) at this conference, have shown the existence of two classes of degenerate stars exhibiting multiperiodic variability. One class is comprised of Z Cam stars, which are a subgroup of the dwarf novae. These stars display optical periodicities in the 20-30 second range, but only near the peak of an outburst. The oscillation periods are too long for radial pulsations of stars of $\sim 1.2M_\odot$, which is typical of the dwarf novae (Warner 1973), unless the Z Cam stars are very close indeed to the limiting mass for their internal chemical composition (cf. Warner 1974). The fundamental mode radial pulsation periods for degenerate dwarfs are $\Pi_o \sim 2$ to 5 seconds for stars in the mass range ~ 1.0 to $1.4M_\odot$, depending upon the composition (Wheeler, Hansen, and Cox 1968; Faulkner and Gribbin 1968). For this reason, and because of the existence of multiple periodicities with small frequency separations (uncharacteristic of radial modes, but understandable in terms of rotation-splitting of the non-radial modes); because of "mode-hopping" in these stars; and because the oscillation period is sometimes observed to change significantly during the course of an outburst, Warner and Robinson (1972) and Chanmugam (1972) have suggested that the high-frequency variability in these stars is associated with non-radial "g-mode" oscillations of the degenerate dwarf.

The second class of degenerate variable stars consists of the variable white dwarfs. This group is much less homo-

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geneous than the Z Cam stars and includes i) suspected or known binaries, ii) a variety of spectral types, and iii) effective temperatures ranging from ~ 9000°K to more than 20,000°K. (There is, however, evidence for some concentration of variable white dwarfs with colors near B - V ≈ +0.25, U - B ≈ -0.55: Lasker and Hesser 1971, Richer and Ulrych 1974). The observed periods in this group, most of which exhibit multiple periodicities, range from 29.08 and 30.15 seconds in CD-42° 14462 (Hesser, Lasker, and Osmer 1974) to 1311 seconds in EG65 (Richer and Ulrych 1974). Since the absolute magnitude of EG65 is known, a photometric estimate of the mass is possible, giving M ~ $1.0M_{\odot}$ (Richer and Ulrych 1974). The observed periods are thus again much too long to be associated with radial pulsations, and it has been suggested that these are also g-mode oscillations (Chanmugam 1972; Warner and Robinson 1972; Hesser, Lasker, and Osmer 1974).

It is accordingly the purpose of this review to summarize the current state of our theoretical understanding of non-radial oscillations in degenerate dwarfs. An excellent recent discussion of this problem is contained in the review paper of Ledoux (1974), and I shall mainly attempt to supplement his summary and bring it up to date. To place the discussion in context, those aspects of white dwarf structure which are relevant to the non-radial oscillation problem are first summarized briefly in Section 2. In Section 3 the linear quasi-adiabatic theory of non-radial oscillations is briefly reviewed in order to

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establish notation and to emphasize the role of the non-adiabatic terms. Specific calculations are reviewed in Section 4 and we conclude in Section 5 with a summary of the current status of the theoretical situation and an identification of certain aspects of the problem that merit immediate attention.

2. The Physical Structure of Degenerate Dwarfs

White dwarfs are well-known to be stars which are supported against their self-gravitational attraction primarily by the pressure of highly degenerate electrons (cf. Chandrasekhar 1935, 1957; Hamada and Salpeter 1961). White dwarf masses are typically in the range 0.5 to $1.0M_\odot$, and radii are $R \sim 10^{-2}R_\odot$. Fundamental mode radial oscillation periods are thus expected to be of order $\Pi_o \sim (G\bar{\rho})^{-1/2} \sim 10$ seconds. Because of the high degeneracy, electron conduction is extremely efficient, and the degenerate core of a white dwarf is very nearly isothermal. The luminosity is supplied from the thermal content of the interior (Mestel 1952):

$$T\frac{ds}{dt} \approx C_v\frac{dT}{dt} \neq 0. \qquad (1)$$

In such circumstances a star is said to be in "thermal imbalance" (radiative losses not balanced by nuclear energy production), and this has a bearing on the vibrational stability of the star.

The core of a white dwarf probably consists primarily of a mixture of carbon and oxygen, and at sufficiently low temperatures the ions may crystallize onto a regular lattice structure. The best current estimate for the crystallization temperature is given by:

$$\Gamma = 2.28 \frac{Z^2}{A^{1/3}} \frac{(\rho/10^6 g\ cm^{-3})^{1/3}}{(T/10^7 °K)} \approx 160 \qquad (2)$$

where the ratio of Coulomb to thermal energies Γ is $\approx$ 160 at the point of crystallization of the plasma (Hansen 1973, Lamb 1974). Crystallization temperatures are high enough to occur at the observed luminosities of the white dwarfs ($L \sim 10^{-2}$ to 10^{-4} $L_\odot$); the cooler and more massive white dwarfs thus have a solid core within the fluid star.

Because the white dwarf core is nearly isothermal, most of the temperature drop occurs in a very thin non-degenerate surface layer. If this region is hydrogen-rich, the layer is radiative for surface temperatures $T_e \gtrsim$ 16,000°K, and the luminosity L is related to the core temperature T_c by (cf. Schwarzschild 1958, Van Horn 1971)

$$\frac{L}{L_\odot} \sim 2 \times 10^{-3} \frac{M}{M_\odot} \left(\frac{T_c}{10^7 °K}\right)^{3.5} . \qquad (3)$$

Cooler white dwarfs and degenerate stars with He- or C-rich envelopes all have thin subsurface convection zones, however (Böhm 1968, 1969, 1970; Böhm and Cassinelli 1971a, b; Böhm and Grenfell 1972; Fontaine 1973; Baglin and Vauclair

1973; D'Antona and Mazitelli 1974, 1975; Fontaine *et al.* 1975). Since the existence of a convection zone is a necessary precondition for the existence of dynamically unstable g^- modes of non-radial oscillation, we therefore expect such modes to occur in sufficiently cool white dwarfs. Convection may also mix surface (or accreted) hydrogen deep enough for nuclear burning (which may affect the stability of white dwarf oscillations: cf. Richer and Ulrych 1974), and convective timescales are short enough so that interactions between convective motions and the non-radial oscillation modes may be important.

Some of the properties of the surface convection zones of $0.612M_\odot$, He-rich envelope white dwarfs, taken from the calculations of Fontaine (1973; see also Fontaine *et al.* 1974) are listed in Table 1.

Table 1

log Te	v_c (cm s^{-1})	z (cm)	$\tau=z/v_c$	$\log\Delta M_c$	log T_b	ε_b/X_H^2	log T_c	Γ_c (^{12}C)
4.280	2.091 +5	2.920 +5	1.40	——	5.341	——	7.284	28.6
4.217	1.146 +5	1.544 +6	13.5	-8.037	6.072	2.543 -8	7.188	35.7
4.155	7.006 +4	3.663 +6	52.3	-6.352	6.411	4.757 -4	7.090	44.7
4.092	5.066 +4	4.933 +6	97.4	-5.622	6.527	1.272 -2	6.995	55.6
4.030	3.647 +4	4.806 +6	131.8	-5.539	6.496	9.598 -3	6.893	70.3
3.967	2.569 +4	4.199 +6	163.4	-5.607	6.427	2.954 -3	6.788	89.6
3.905	1.738 +4	3.254 +6	187.2	-5.768	6.334	4.716 -4	6.672	117.
3.842	1.222 +4	2.571 +6	210.4	-5.976	6.276	4.535 -5	6.566	153.
3.780	8.815 +3	2.112 +6	239.6	-6.217	6.108	2.728 -6	6.427	206.
3.717	6.043 +3	1.563 +6	258.6	-6.376	5.960	——	6.274	293.

The notation 1.00 + n ≡ 1.00 x 10^n is employed, and all units are c.g.s. except for the mass of the convection zone, ΔM_c, which is given in solar masses.

As the effective temperature Te decreases, the central temperature T_c drops monotonically, and the value of Γ at the center ($\propto \rho_c^{1/3}/T_c$), computed assuming a carbon core, increases. Crystallization begins at the center for log $L/L_\odot \approx -3.55$. As the star cools, the extent of the partial ionization region increases inward until, at log Te ≈ 4.1, the base of the convection becomes degenerate, and the increasing efficiency of electron conduction pushes the zone back out toward the surface. The depth z, convection zone mass ΔM_c and temperature T_b at the base of the zone thus first increase with decreasing L and subsequently decrease. The nuclear burning rate ε_b at the convection zone base, due to the reaction $3\,{}^1H \rightarrow {}^3He + e^+ + \nu$ (Clayton 1969, p. 376) and computed assuming $X_H << 1$, thus increases sharply but peaks at relatively high Te (log Te ~ 4.1). Since both ΔMc and the maximum T_b increase somewhat on going to lower mass stars, the conditions for the excitation of white dwarf oscillations by nuclear burning of accreted hydrogen appears more favorable for hot, low mass white dwarfs than for any others.

The characteristic timescale of convection, as measured by $\tau = z/v_c$, where v_c is the maximum convective velocity, increases monotonically with decreasing temperature in these models. For the hotter models the convective velocity is high [thus requiring a relatively large superadiabatic gradient in the outermost layers: $(\nabla-\nabla ad)_{max} \gtrsim 0.4$], and the convective timescales are comparable to the radial pulsation periods. For models cooler than ~ 13,000°K, however, τ is comparable to the non-radial g-mode oscillation periods, as we shall see.

3. Linear, Quasi-Adiabatic Theory of Non-Radial Oscillations

The linearized, Eulerian equations of fluid dynamics are (cf. Ledoux and Walraven 1958, Ledoux 1974, Cox 1974)

$$\frac{\partial \rho'}{\partial t} + \mathrm{div}\,(\rho_o \underset{\sim}{v}') + \mathrm{div}\,(\rho' \underset{\sim}{v}_o) = 0 \tag{4a}$$

$$\frac{\partial \underset{\sim}{v}'}{\partial t} + \frac{1}{\rho_o}\nabla p' - \frac{\rho'}{\rho_o^2}\nabla p_o + \nabla\Phi'$$
$$+\left[\frac{\partial \underset{\sim}{v}_o}{\partial t} + (\underset{\sim}{v}_o\cdot\nabla)\underset{\sim}{v}_o + (\underset{\sim}{v}_o\cdot\nabla)\underset{\sim}{v}' + (\underset{\sim}{v}'\cdot\nabla)\underset{\sim}{v}_o - \underset{\sim}{f}\right] = 0 \tag{4b}$$

$$\nabla^2\Phi' = 4\pi G\rho' \tag{4c}$$

$$\frac{\partial p'}{\partial t} + \underset{\sim}{v}'\cdot\nabla p_o = \frac{\Gamma_1 p_o}{\rho_o}\left(\frac{\partial \rho'}{\partial t} + \underset{\sim}{v}'\cdot\nabla\rho_o\right)$$
$$+\left[\left(-\underset{\sim}{v}_o\cdot\nabla p' + \frac{\Gamma_1 p_o}{\rho_o}\,\underset{\sim}{v}_o\cdot\nabla\rho'\right) + \rho_o(\Gamma_3-1)T_o\left(\frac{\partial s'}{\partial t} + \underset{\sim}{v}'\cdot\nabla s_o + \underset{\sim}{v}_o\cdot\nabla s'\right)\right.$$
$$\left. + \left(\frac{\rho'}{\rho_o} + \frac{\Gamma_3'}{\Gamma_3-1} + \frac{T'}{T_o}\right)\rho_o(\Gamma_3-1)T_o\frac{ds_o}{dt} + \left(\frac{\Gamma_1'}{\Gamma_1} + \frac{p'}{p_o} - \frac{\rho'}{\rho_o}\right)\frac{\Gamma_1 p_o}{\rho_o}\frac{d\rho_o}{dt}\right], \tag{4d}$$

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$$\frac{\partial s'}{\partial t}+\underset{\sim}{v}_o\cdot\nabla s'+\underset{\sim}{v}'\cdot\nabla s_o=-\frac{T'}{T_o}\frac{ds_o}{dt}+\frac{1}{T_o}(\varepsilon'-\frac{1}{\rho_o}\mathrm{div}\underset{\sim}{F}'+\frac{\rho'}{\rho_o^2}\mathrm{div}\underset{\sim}{F}_o), \quad (4e)$$

$$\underset{\sim}{F}' = -\frac{4ac}{3}\frac{T_o^3}{\kappa_o\rho_o}[(3\frac{T'}{T_o}-\frac{\kappa'}{\kappa_o}-\frac{\rho'}{\rho_o})\nabla T_o + \nabla T']. \quad (4f)$$

All symbols have their usual meanings, primes denote Eulerian perturbations, $d/dt = \partial/\partial t + \underset{\sim}{v}'\cdot\nabla$, and we have assumed the flux $\underset{\sim}{F}$ to be non-convective in writing equation (4f). In addition, the body force per unit mass $\underset{\sim}{f}$ in equation (4b) is intended as the sum of all other body forces per unit mass (due, e.g., to viscosity, magnetic forces, tidal interactions, etc.). Because we wish to consider perturbations about a spherical state, we have also included the terms $\partial\underset{\sim}{v}_o/\partial t$ and $(\underset{\sim}{v}_o\cdot\nabla)\underset{\sim}{v}_o$ in this equation, even though they do not involve perturbed quantities, since they are neglected in the construction of spherical models.

Note that we have also retained all perturbation terms involving $\underset{\sim}{v}_o$ in equations (4). The order of magnitude of these terms can be seen from equation (4a). Here the first two terms are $\propto \Pi^{-1}$, where Π is the characteristic oscillation period, while the third term involving $\underset{\sim}{v}_o$ is clearly $\propto \tau^{-1}$, where τ is the timescale associated with the unperturbed velocity field. For pure gravitational contraction, $\tau \sim \tau_{KH}$, the Kelvin-Helmholtz timescale, while for rotation $\tau \sim \Omega^{-1}$(the rotation period) and τ is the convective timescale if $\underset{\sim}{v}_o$ represents convective motions.

Neglect of the $\underset{\sim}{v}_o$-terms is clearly satisfactory in the first case, but is certainly not in the others.

In equation (4d), in addition to the obvious $\underset{\sim}{v}_o$-terms, there also occur terms involving the time derivatives of the unperturbed entropy and density: ds_o/dt and $d\rho_o/dt$. The $d\rho_o/dt$-term is almost certainly negligible for degenerate dwarfs, but the ds_o/dt-term is not [cf. equation (1)]. These are the so-called "thermal imbalance" terms. Finally, the terms involving the entropy perturbation s' are the usual non-adiabatic terms which, in the quasi-linear approximation, determine the secular stability of the oscillations.

In the linear, adiabatic approximation, all terms involving $\underset{\sim}{v}_o$, $\underset{\sim}{f}$, the entropies, and the time derivatives of unperturbed quantities are neglected. The time-dependence of perturbed quantities is assumed to be of the form $e^{i\sigma t}$ and the angular dependences are assumed given by the spherical harmonic $Y_{\ell m}(\theta,\phi)$. Equations (4) then reduce to the fourth-order system (cf. Ledoux and Walraven 1958, Ledoux 1974, Cox 1974)

$$\sigma^2\zeta = \frac{\partial\chi}{\partial r} - A\,\frac{\Gamma_1 p_o}{\rho_o}\,\alpha \quad , \qquad (5a)$$

$$-\frac{\Gamma_1 p_o}{\rho_o}\,\alpha = \chi - \Phi' + \zeta\frac{1}{\rho_o}\,\frac{\partial p_o}{\partial r} \quad , \qquad (5b)$$

$$\alpha = \text{div}\,\underset{\sim}{\zeta} = \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2\zeta) - \frac{\ell(\ell+1)}{\sigma^2 r^2}\,\chi \quad , \qquad (5c)$$

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$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi'}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}\Phi' = -4\pi G\left(\zeta\frac{\partial\rho_o}{\partial r} + \alpha\rho_o\right). \qquad (5d)$$

Here $\underset{\sim}{v}' \equiv i\sigma\underset{\sim}{\zeta}e^{i\sigma t}$, $\zeta \equiv \underset{\sim}{\zeta}\cdot\underset{\sim}{1}_r$, $\chi' \equiv \Phi' + p'/\rho_o$, and

$$A \equiv \frac{1}{\rho_o}\frac{\partial\rho_o}{\partial r} - \frac{1}{\Gamma_1 p_o}\frac{\partial p_o}{\partial r} = \frac{1}{H_p}\frac{\chi_T}{\chi_\rho}(\nabla - \nabla ad), \qquad (6)$$

where H_p is the pressure scale height, $\chi_T \equiv (\partial\ln p/\partial\ln T)_\rho$, and $\chi_\rho \equiv (\partial\ln p/\partial\ln\rho)_T$.

The solution of the system (5) is well-known to yield three different types of modes (cf. Cowling 1941, Ledoux and Walraven 1958, Cox 1974), termed the p-, g-, and f-modes. With the aid of Cowling's approximation ($\Phi'=0$) and Castor's approximation (cf. Cox 1974), the frequencies of the p-mode and g-mode oscillations are approximately given by

$$\sigma_p^2 \approx \left[k^2 + \frac{\ell(\ell+1)}{r^2}\right]\frac{\Gamma_1 p_o}{\rho_o}, \qquad (7a)$$

$$\sigma_g^2 \approx N^2/\left[1 + \frac{k^2 r^2}{\ell(\ell+1)}\right], \qquad (7b)$$

while the f-mode frequencies are intermediate between these two. Here k^{-1} is the radial length scale of the oscillations, $(\Gamma_1\ p_o/\rho_o)$ is the square of the sound speed, and $N^2 = -Ag$ is the square of the Brunt-Väisälä frequency, where g is the gravitational acceleration. Thus p-mode frequencies are ~ $(GM/R^3)^{1/2}$ ~ $(10 \text{ seconds})^{-1}$ and are insensitive to temperature, while g-mode frequencies depend upon temperature through both $(\nabla-\nabla ad)$ and $\chi_T \propto P_{ion}/P_{el} <<1$. Note that $A<0$ (stability against convection) implies $\sigma_g^2 > 0$: these are the dynamically stable "g^+-modes." Conversely for $A>0$ (convective instability) dynamically unstable "g^--modes" occur. For a given ℓ-value we follow the practice of denoting those modes with precisely n nodes in the radial eigenfunction as p_n- or $g_n^{\pm}$ - modes.

For the oscillation mode specified by $(n,\ell,m) \equiv j$, the effects of the neglected terms in equations (4) can be assessed by perturbation methods. Thus the non-adiabatic (s') terms lead to an exponential time-dependence $e^{-\kappa t}$ of the oscillations, where (cf. Cox 1974)

$$\kappa_j = - \frac{\int (\Gamma_3-1)(\delta\rho/\rho)_j^* \delta(\varepsilon - \text{div } \underset{\sim}{F}/\rho)_j \rho_o \, d^3r}{2\sigma_j^2 \, J_j} \tag{8}$$

with

$$J_j \equiv \int \underset{\sim}{\zeta}_j \cdot {}^* \underset{\sim}{\zeta}_j \rho_o \, d^3r, \tag{9}$$

and δ denotes a Lagrangian variation. In a similar manner, uniform rotation with angular velocity $\underset{\sim}{\Omega} \equiv \Omega \underset{\sim}{1}_z$ is found to lead simply to a shift in the oscillation frequency (as seen by an observer at rest) from σ_j to $\sigma_j + m\Omega(1-C_j)$, $m=-\ell, -\ell+1, ---, +\ell$, where (Ledoux 1951)

$$m\Omega C_j = \frac{i\int (\underset{\sim}{\Omega} \times \underset{\sim}{\zeta}_j) \cdot \underset{\sim}{\zeta}_j^* \rho_o \, d^3r}{J_j} \quad , \tag{10}$$

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(See Smeyers and Denis 1971 for a self-consistent second-order calculation of rotational effects). The effects of the "thermal imbalance" terms, tidal perturbations, and convective interactions are more complex, and most of these terms have yet to be studied in the context of degenerate dwarfs. Thus we merely note here that recent discussions of these three effects can be found in the papers of Aizenman and Cox (1975), Denis (1972), and Gabriel et al. (1975), respectively.

4. Calculations for Specific Models

The first investigation of g-mode oscillations in white dwarfs appears to have been that of Baglin and Schatzman (1969). Motivated by the relatively long (71 second) period of DQ Her, they obtained rough estimates for the temperature dependence of the g_1 mode periods. The full fourth-order system was first solved by Harper and Rose (1970) for two $0.75M_{\odot}$ hydrogen shell-burning models. They obtained oscillation periods for the p_o, p_1, and p_2 modes belonging to $\ell=0$ and for the g_2, g_1, f, and p_o modes belonging to $\ell=2$. For the g_1 modes the periods were $\Pi_{g_1}=30.4$ seconds at $L=58L_{\odot}$ and $\Pi_{g_1} = 42.2$ seconds at $L = 377L_{\odot}$. These are comparable to the periods observed in the Z Cam stars. More recently Böhm, Ledoux, and Robe (quoted in Ledoux 1974) have obtained the periods of the

g_6 through g_1, f, and p_1 through p_4 modes of a $0.6M_\odot$, 12,000°K white dwarf model with a relatively extensive He-rich surface convection zone. The periods they obtained for the g_1 and g_2 modes, $\Pi_{g_1} = 200$ seconds, $\Pi_{g_2} = 250$ seconds are very reminiscent of the two periods observed in R548 (213 seconds and 273 seconds: Lasker and Hesser 1971).

A systematic investigation of the dependence of the non-radial oscillation periods upon the thermal structure of the unperturbed degenerate model has quite recently been carried out by Brickhill (1975). He employed Cowling's (1941) approximate second-order system of equations ($\Phi' = 0$), which are expected to be quite satisfactory for degenerate dwarfs; utilized moderately realistic static (non-evolutionary) models for the unperturbed star; and studied two groups of models: i) "white dwarfs" with masses ranging from 0.2 to 0.8 $M_\odot$ and effective temperatures mostly clustered around $T_e \sim 8350$°K, and ii) hydrogen shell-burning models with masses of 1.0 and $1.2M_\odot$ and $T_e \gtrsim 20{,}000$°K ($10^{-2}L_\odot \lesssim L \lesssim 10\ L_\odot$). For the former case he has computed the g-mode periods belonging primarily to $\ell=1$ and $\ell=2$ (up to $\ell=4$ in one case), and he has given the "excitation energy" E[$\propto$ Jj: cf. equation (9)] for the g_1 modes. His results for the g_1 and g_2 mode periods again suggest an identification of the oscillations of R548 with these modes. He also finds that the presence of the surface convection zone does not significantly affect the oscillation periods, in agreement with the conclusions of Böhm, Ledoux, and Robe. The eigen-functions of the lowest $\ell=2$ g-modes, however, are strongly concentrated toward the surface. This both reduces the "excitation energy" and enhances the possibility of excitation of the oscillations by mechanisms located in the convection zone.

Interestingly, none of the oscillation periods found by Brickhill are long enough to be identified with, e.g., the 750 second period of HL Tau 76. The $\ell=1$ g-modes, however, are significantly longer than the $\ell=2$ modes - for example, in a $0.8M_{\odot}$, 8340°K model, Πg_1 = 236 seconds and Πg_2 = 299 seconds for $\ell=2$, while Πg_1 = 408 seconds and Πg_2 = 511 seconds for $\ell=1$. While it is not easy to understand how the $\ell=1$ modes could be excited in an isolated star, it seems not improbable that they could be excited by tidal interactions in a close binary system. In this regard Fitch's (1973) interpretation of the oscillation spectrum of HL Tau 76 is particularly interesting. He finds three periodicities, at 494.22 seconds, 746.16 seconds, and ~ 3.24 hours, the last of which he suggests may be the orbital period of a faint companion. If this is correct, and if tidal forces can excite the $\ell=1$ mode, Brickhill's results suggest a larger mass than $0.8M_{\odot}$ for this star.

None of the papers discussed above have considered the problem of the damping or excitation of non-radial oscillations, however, and only one (Harper and Rose 1970) has been based upon evolutionary models (so that thermal imbalance effects can be investigated). Both of these

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shortcomings were alleviated in an important paper by Osaki and Hansen (1973). Their calculations were carried out for several models selected from the pre-white-dwarf evolutionary sequences of ^{56}Fe stars computed by Savedoff, Van Horn, and Vila (1969), and they calculated g_2, g_1, f, p_1, and p_2 modes belonging to $\ell=2$ as well as the radial eigenfunctions and damping integrals for these models. They showed that the g_1 mode periods for the white dwarfs obeyed the simple period-luminosity relations

$$\log \Pi_{g_1} \text{(seconds)} = 1.587-0.178\log L/L_\odot,\ M=0.398M_\odot \quad (11a)$$

$$= 1.331-0.171\log L/L_\odot,\ M=1.0M_\odot, \quad (11b)$$

which can be readily understood on the basis of equations (3) and (6), together with the relation $\chi_T \propto T$.

Perhaps the most significant aspect of their work, however, was their discussion of the damping mechanisms for non-radial oscillations (there are no driving mechanisms in the models they studied). In addition to the radiative and neutrino loss damping processes familiar from analyses of radial pulsations, they recognized the potential of non-radial oscillations for the generation of gravitational waves, and they investigated this energy loss mechanism as well. In fact, they found gravitational radiation to be the dominant damping mechanism for the p- and f- modes. For the $1.0M_\odot$, ^{56}Fe model with $\log L/L_\odot = -1.934$ (model 9N), their results are summarized in Table 2, where all units are c.g.s. except for the damping times, κ^{-1}, which are in years. Following Osaki and Hansen we have written the components of equations (8) and (9) in the forms, e.g.,

$$E \equiv \frac{1}{2}\sigma^2 \int |\underset{\sim}{\zeta}|^2 \rho_o d^3r \ , \ L_\nu \equiv -\frac{1}{2}\int (\delta T/T)^* \delta\varepsilon_\nu \rho_o d^3r, \text{ etc.,} \quad (12)$$

with $\zeta(r) = 1$ at the surface. It is important to note that both of these quantities are proportional to the square of the oscillation amplitude.

Table 2

Mode	Π(sec)	E	L_{ph}	L_ν	L_{GW}	κ^{-1}(ph+ν)	κ^{-1}(total)
p_2	1.009	1.0 +49	1.6 +34	1.5 +32	6.1 +39	4.0 +7	104.
p_1	1.493	2.8 +49	5.7 +33	3.6 +32	3.3 +40	3.0 +8	54.
f	2.703	1.1 +50	1.2 +33	1.5 +32	1.7 +41	4.9 +9	40.
g_1	45.98	1.3 +47	2.5 +36	1.1 +31	4.0 +25	3.3 +3	3.3 +3
g_2	62.96	3.8 +46	8.0 +36	1.9 +31	5.7 +24	3.0 +2	3.0 +2

Quite recently, Hansen, Lamb, and Van Horn (1975) have carried out this same type of calculation for the pure ^{12}C evolutionary models constructed by Lamb (1974: see also Lamb and Van Horn 1975). These models are of interest

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because the treatment of the physical state of the interior is believed to be the most accurate currently available for homogeneous stars in these phases of evolution. A fully temperature-dependent treatment of the Coulomb interactions is used, which includes the effects of crystallization and Debye cooling, and self-consistent carbon convective envelope models are employed. The results for a model with log $L/L_{\odot} = -1.928$ are briefly summarized in Table 3 for comparison with the calculations of Osaki and Hansen.

Table 3

Mode	Π	$\kappa^{-1}(ph+\nu)$	$\kappa^{-1}(tot)$
p_2	1.496	4.7 +7	530.
p_1	2.217	2.7 +8	190.
f	4.204	6.8 +9	120.
g_1	116.60	1.2 +3	1.2 +3
g_2	122.82	3.6 +1	3.6 +1

The dominance of gravitational radiation damping for the f- and p- modes is again clear, and the damping rates are not grossly different in spite of the rather larger oscillation periods of the Lamb models. In addition, we have investigated the effects of thermal imbalance, following Aizenman and Cox (1975), and have found them

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to be completely negligible for these modes, at typical white dwarf luminosities. We have also found a hint of secular instability due to strong driving in the partial ionization zone in the g_1^+ and g_2^+ modes of one model with log $L/L_\odot$ = -0.167 (log Te = 4.765). Whether this result is real, however, remains to be established. Evidently further studies of the secular stability of models with realistic surface convection zones and various masses and compositions are sorely needed.

5. Summary and Conclusions

The dependence of the non-radial mode oscillation periods upon the thermal properties of the unperturbed degenerate star models are beginning to be understood. The magnitudes of the $\ell=2$ g-mode periods in 1.0 to 1.2$M_\odot$ hydrogen shell-burning stars are of the correct magnitude to account for the periodicities in the Z Cam stars, and rotation splitting of these modes, as suggested by Warner and Robinson (1972) to account for the observed fine structure, seems plausible. The case of the variable white dwarfs is in less satisfactory shape, however. Although the $\sim 10^2$ second periodicities seem understandable as low-order $\ell=2$ g-modes, as in the case of R548, the origin of the $\sim 10^3$ second oscillations is still mysterious. Tidally-driven $\ell=1$ g-modes seem promising, but much work remains to be done.

Probably the most important current theoretical problems concerning non-radial oscillations of degenerate dwarfs are the following.

i) The question of the existence of $\bar{g}$-modes. These are expected to be present on the basis of calculations for other types of stellar models, but none have yet been discovered in any of the degenerate dwarf models. This may simply be due to the difficulty of locating these modes in models with convection zones as thin as those in white dwarfs, but it is important that this be resolved, since the presence of $\bar{g}$-modes may have important consequences for the excitation mechanisms of non-radial oscillations.

ii) The nature of the excitation mechanisms for non-radial modes. The strong gravitational radiation damping of the p- and f- modes in white dwarfs makes these modes difficult to excite. In addition, the fact that the eigenfunctions of the low-order g-modes are largest near the surface (where the potential excitation mechanisms are concentrated in white dwarfs) enhances the possibility of excitation of these modes. Aside from the single tentative case of secular instability (due to the usual κ- and γ-mechanisms: cf. Cox and Giuli 1969) recently found by Hansen, Lamb, and Van Horn, however, there have been no investigations of de-stabilitzing influences. Such studies are sorely needed.

In addition, the possibility of nuclear excitation of the oscillations needs to be investigated both for the shell-burning stars and for the DA (hydrogen spectra) white dwarfs, which may have relatively deep hydrogen surface layers. For white dwarfs of other spectral types, however, nuclear excitation due to hydrogen-burning at the base of the convection zone appears unlikely to be significant. If we set Osaki and Hansen's (amplitude-dependent) L_{ph}, which is the dominant damping mechanism for g-modes, equal to the product $\varepsilon_b \Delta M_c$ from Table 1, which provides a generous upper limit to the maximum thermonuclear luminosity, we find relative oscillation amplitudes $\lesssim 10^{-5}$.

iii) The possibility of tidal excitation in a close binary system. The fact that the $\ell=1$ modes, which probably can be excited only in a binary system, have substantially longer periods than the modes belonging to higher ℓ-values indicates the potential interest of these modes. The fact that many of the variable white dwarfs are now suspected or known close binaries makes it important to investigate this problem in more detail. If $\ell=1$ modes are the explanation of the $\sim 10^3$ second periodicities, observational tests of this mechanism may also be possible.

iv) Mode-coupling. If dynamically unstable $\bar{g}$-modes do exist in white dwarfs, mode coupling may permit the excitation of otherwise stable modes, as recently suggested by Osaki (1974) for the β Ceph stars. Coupling with convective motions (if these are different from $\bar{g}$-modes) also seems likely to be important because of the near com-

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mensurability of the g^+ mode periods and convective time-scales. These processes may also have a bearing on the low-frequency "flickering" found in some of the variable white dwarfs (e.g., CD - 42° 14462: Hesser, Lasker, and Osmer 1974).

v) Finally, the fact that crystallization is expected to occur in some of the more massive and fainter stars suggests the interesting possibility of whole new classes of stellar oscillations and instabilities. The solid core can sustain shear modes, and the solid-liquid core interface may cause significant changes in the oscillation spectrum as the core grows in the cooling star. The possibility of erratic excitation of oscillations triggered by "star quakes" in the cooling and contracting solid core also immediately suggests itself.

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Discussion to the paper of VAN HORN

BATH: The periodicities observed in dwarf novae exhibit large changes (∿10%) in period on very short timescales (∿10^3 - 10^4 sec). If interpreted (as you have implied) as due to varying g-mode periods associated with changes of luminosity of equilibrium models, then the structural changes within the dwarf are enormous. They require internal energy changes of ∿10^{45} erg - much larger than the outburst energy itself. Brickhill agrees with this in the paper you refer to.

If one assumes that runaway nuclear burning occurring in the surface layers can change the period in the way observed, that is a different problem. But in that case, none of the work on g-mode pulsations presently published is relevant to your argument.

VAN HORN: I agree that if you consider equilibrium white dwarf models, then the observed period changes do indeed imply unacceptably large changes in the internal energy. However, in the case of degenerate, hydrogen-shell-burning stars, it does not seem impossible to me that the rapidly changing thermal structure of the envelope - especially the distension of the hydrogen-rich surface layers - may be able to produce the observed period changes. Brickhill's calculations are indicative of this, and they are certainly relevant to this point, but calculations must clearly be extended to rapidly evolving, hydrogen-shell-burning, degenerate stars before these indications can be regarded as being either confirmed or rejected.

BATH: Since I shall not have time to discuss the periodicity problem of dwarf novae in my talk, it should be pointed out here that the periods so far observed all fall in the range 16-35 sec. The period with which matter orbits the dwarf surface at the inner boundary of the accretion disc is in the range 14-55 sec for white dwarf masses $\sim$1.4-0.1 $M_\odot$. Clearly, the regular eclipse by the dwarf of inhomogeneities (or luminosity fluctuations) in the central regions of the disc could account for the observed periodicities. The changing periods then only imply small changes in the orbital radius of the inhomogeneity. Accurate determination of the lifetime of such a disc fluctuation is obviously a difficult problem.

VAN HORN: I agree that the periodicity of matter at the inner edge of an accretion disk around a white dwarf is of the correct order of magnitude to account for the observed periods. Whether the systematic properties of the observed periodicities can be satisfactorily explained by this hypothesis, however, is another matter. To mention only two points that bother me: (1) It is not obvious to me that something as fragile as I expect an accretion disk to be, can sustain fluctuations long enough to account for the observed coherency of the oscillations, and (2), I would expect the fluctuations to be carried inward through the disk, and this will cause the period to decrease rather than increase as observed.

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