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THE RANK OF THE SUM OF TWO RECTANGULAR MATRICES

BY
IAN S. MURPHY

In what follows, the transposed complex conjugate of a complex rectangular matrix D is denoted by D^* and the rank of D by $r(D)$. Meyer [1] proved the following result using generalized inverses:

THEOREM. *Let A and B be complex $m \times n$ matrices such that $AB^*=B^*A=0$. Then $r(A+B)=r(A)+r(B)$.*

Below we prove this result by repeated use of the fact that for every complex $m \times n$ matrix D we have $r(D)=r(D^*D)=r(DD^*)$ (e.g. See [2] Theorem 5.5.4).

Proof. Note that $AB^*=B^*A=A^*B=BA^*=0$. Consider the $m \times 2n$ partitioned matrix $C=[A\ \ B]$.

$$\begin{aligned} r(C) &= r(C^*C) = r\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}[A\ \ B]\right) = r\left(\begin{bmatrix} A^*A & 0 \\ 0 & B^*B \end{bmatrix}\right) \\ &= r(A^*A)+r(B^*B) = r(A)+r(B). \end{aligned}$$

Also

$$\begin{aligned} r(C) &= r(CC^*) = r\left([A\ \ B]\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) = r(AA^*+BB^*) \\ &= r(AA^*+AB^*+BA^*+BB^*) = r((A+B)(A+B)^*) \\ &= r(A+B). \end{aligned}$$

Hence

$$r(A+B) = r(A)+r(B).$$

REFERENCES

1. C. D. Meyer, *On the rank of the sum of two rectangular matrices*, Canad. Math. Bull. **12** (1969), 508.

2. L. Mirsky, *An Introduction to linear algebra*, Oxford, 1955.

UNIVERSITY OF EDINBURGH,
EDINBURGH, SCOTLAND