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DEGENERATE CASES OF UNIFORM APPROXIMATION BY SOLUTIONS OF SYSTEMS WITH SURJECTIVE SYMBOLS

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ABSTRACT We prove that each (vector-valued) function in Sobolev space on a compact set K, which in the interior K^0 of K satisfies a system of differential equations, can be approximated by solutions in a neighbourhood of K plus sums of potentials of measures supported on the boundary of K We discuss the particular case where, for all compact sets K, one can dispense with potentials in such approximations

1. **Introduction.** Unless otherwise indicated, we let $P \in \mathrm{do}_p(E \longrightarrow F)$ be a differential operator of order p with surjective symbol on an open set X in $\mathbf{R}^n$. Here, $E = X \times \mathbf{C}^l$ and $F = X \times \mathbf{C}^k$ are (trivial) vector bundles over X whose sections of some class $\mathcal{C}$ are interpreted as columns of functions of the class $\mathcal{C}(X)$, that is $\mathcal{C}(E) = [\mathcal{C}(X)]^k$ and similarly for F. Thus, P may be represented as an $(l \times k)$-matrix of scalar differential operators of order $\leq p$ on X. We shall assume throughout that the transpose P' of the differential operator P satisfies the so-called uniqueness condition for the Cauchy problem in the small on X. Under this condition and certain other natural conditions which we assume to be satisfied (see [17]), the differential operator P has a "special" right fundamental solution $\Phi \in \mathrm{pdo}_{-p}(F \longrightarrow E)$.

In particular, for an integer $s \geq 0$ and a real number $1 \leq q \leq \infty$, we have the Sobolev space $W^{s,q}_{\mathrm{loc}}(E)$ of sections of E whose generalized derivatives up to order s together with their q-th powers are integrable on compact subsets of X.

If K is a compact subset of X, it is natural to try to define $W^{s,q}(E|_K)$ (also denoted by $[W^{s,q}(K)]^k$) as the quotient of $W^{s,q}_{\mathrm{loc}}(E)$ by the subspace of sections whose derivatives up to order s vanish on K. However, such a subspace is not clearly defined. The sections of $W^{s,q}_{\mathrm{loc}}(E)$ (and the derivatives up to order s) are basically functions of class L_q, and hence it is meaningless to talk about their values at the individual points of K. We therefore define $W^{s,q}(E|_K)$ as the quotient of $W^{s,q}_{\mathrm{loc}}(E)$ by the closure in $W^{s,q}_{\mathrm{loc}}(E)$ of $C^\infty_0(\mathbf{R}^n \setminus K)$. According to the spectral synthesis theorem of Hedberg and Wolff [10], this closure can be described in terms of function values assumed "almost everywhere" in the sense of appropriate capacities. With the quotient topology, $W^{s,q}(E|_K)$ is a Banach space.

Often, instead of $W^{s,\infty}_{\mathrm{loc}}(E)$, it is more convenient to work with the space $C^s_{\mathrm{loc}}(E)$ of s times continuously differentiable sections of E with the usual Fréchet topology. Then, we are able to introduce the Banach space $C^s(E|_K)$ or $[C^s(K)]^k$ as above. Clearly, $C^s(E|_K)$ is a closed subspace of $W^{s,\infty}(E|_K)$.

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For a (open or closed) set $\sigma \subset X$, we denote by $S(\sigma)$ the (vector) space of all sections of E which, on (a variable neighbourhood of) σ, are infinitely differentiable solutions of the system $Pu = 0$.

Setting $\tilde{W}^{s,q}(E|_K) = W^{s,q}(E|_K)$ if $q < \infty$, and $\tilde{W}^{s,q}(E|_K) = C^s(E|_K)$ if $q = \infty$, we have, for any s, a natural embedding $S(K) \subset \tilde{W}^{s,q}(E|_K)$.

The following problem has its roots in rational approximation in the plane.

PROBLEM 1.1. *Let $0 \leq s < p$ be an integer. Under which conditions on a compact set $K \subset X$ does the closure of $S(K)$ in $\tilde{W}^{s,q}(E|_K)$ coincide with the subspace of $\tilde{W}^{s,q}(E|_K)$ formed by weak solutions in K^0 of the system $Pu = 0$?*

We shall treat the cases $1 \leq q < \infty$ (approximation in Sobolev spaces) and $q = \infty$ (uniform approximation) separately.

In general, it is known that certain capacitary conditions on K are required (see, for example, the survey [17] and the bibliography therein). However, in some cases the answer to Problem 1.1 is "for all compacts sets". We call these cases of approximation the degenerate ones. Degeneracy depends on a certain correlation between the dimension n of the manifold X, the order p of the operator P, and the indexes s and q of the space $W^{s,q}(E|_K)$ in which we are approximating. This correlation, at least for nowhere dense compact sets K, is in terms of the "principal" index $d = n/q' - (p - s)$, where $1/q + 1/q' = 1$.

The present paper is devoted to a systematic study of degenerate cases of uniform approximation in parallel with approximation in Sobolev spaces. Contrary to approximation in Sobolev spaces, the degenerate cases of uniform approximation occur only if the order of the operator P is relatively high.

Sections 2, 3 and 4 are concerned with approximation in Sobolev norms. The theorems given in these sections are new, but they are very close to results of Hedberg [8, Section 6, pp. 261–262] (the restriction on p in [8] are removed in [10]); and the proofs are basically the same as those given by Hedberg, namely an application of the spectral synthesis theorem of Hedberg. For these reasons these sections might be considered as a "sharpening of results of Hedberg and Wolff".

The ensuing sections are concerned with uniform approximation and emphasize the comparison between approximation in uniform norms and approximation in Sobolev norms. The best known theorem of this type is the following, due to several authors; a convenient reference is [6, Theorem 13 and Theorem 11].

THEOREM 1.1 (STABILITY THEOREM). *Let K be a compact subset of $\mathbf{R}^n$. Then the following conditions are equivalent.*

(a) (Uniform harmonic approximation) Every continuous function on K which is harmonic on K^0 can be uniformly approximated by functions harmonic on neighbourhoods of K.

(b) (Sobolev harmonic approximation) Every function in $W^{1,2}$ of $\mathbf{R}^n$ which is harmonic on K^0 can be approximated in $W^{1,2}$ of $\mathbf{R}^n$ by functions harmonic on neighbourhoods of K.

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Moreover, if $n = 2$, then each of these is equivalent to the following condition.

(c) (L_2 holomorphic approximation) ($n = 2$) Every function in $L_2(K)$ which is holomorphic on K^0 can be approximated in $L_2(K)$ by rational functions with poles off K.

The classical theory of Keldysh [11] and Deny [4] gives many conditions equivalent to condition (a); and papers by Havin [5], Bagby [1], and Hedberg and Wolff [10] give many conditions equivalent to conditions (b) and (c). For a survey of the Stability Theorem, see [9].

We thank the referee as well as Joan Verdera for several helpful comments which we have unabashedly incorporated. We are also grateful to Lars Hedberg for sending us a copy of his very interesting survey of the Stability Theorem 1.1 which will appear soon [9].

2. Approximation in Sobolev spaces on compact sets by potentials with densities supported on the boundary.

First of all, in this section we will deal with potentials of the form $\Phi(f)$, where $f \in \mathcal{E}'(F)$ is a generalized section of F with compact support.

Since $P\Phi = 1$ on $\mathcal{E}'(F)$, we are able to conclude, as a consequence of the boundedness theorem for pseudo-differential operators in Sobolev spaces, that for $1 < q < \infty$, such a potential $\Phi(f)$ belongs to $W^{s,q}_{\mathrm{loc}}(E)$ if and only if $f \in W^{s-p,q}_{\mathrm{comp}}(F)$.

Let $M_\sigma(F)$ (or $[M_\sigma(X)]^l$) denote the space of all measure type sections of F supported on a compact set $\sigma \subset X$.

We now present the main theorem of this section which affirms that, in fact, in order to successfully solve Problem 1.1, it is sufficient to know how to approximate potentials of derivatives of measures supported on ∂K, at least for approximation in Sobolev norms. The idea of reducing matters to the case of potentials of derivatives of measures appears also in [12] and [13].

THEOREM 2.1. *Let $0 \le s < p$ and $1 < q < \infty$. Then, for each section $u \in W^{s,q}(E|_K)$ satisfying $Pu = 0$ weakly in the interior of K, and for each $\epsilon > 0$, there exists a solution $u_e \in S(K)$ and sections $m_\alpha \in M_{\partial K}(F)$ ($|\alpha| \le p - s - 1$) such that*

$$\Big\| u - \Big(u_e + \sum_{|\alpha| \le p-s-1} \Phi(D^\alpha m_\alpha) \Big) \Big\|_{W^{s,q}(E|_K)} < \epsilon. \tag{1}$$

PROOF. Let us denote by Σ the subspace of $W^{s,q}(E|_K)$ consisting of elements of the form

$$u_e + \sum_{|\alpha| \le p-s-1} \Phi(D^\alpha m_\alpha),$$

where $u_e \in S(K)$ and $m_\alpha \in M_{\partial K}(F)$ ($|\alpha| \le p - s - 1$). We would like to show that any section $u \in W^{s,q}(E|_K)$, satisfying $Pu = 0$ in the interior of K, belongs to the closure of Σ in $W^{s,q}(E|_K)$. For this, by the Hahn-Banach Theorem, it is sufficient to show that for any continuous linear functional g on $W^{s,q}(E|_K)$, which vanishes on Σ, we have $\langle g, u \rangle = 0$.

Suppose, then that g is a continuous linear functional on $W^{s,q}(E|_K)$ vanishing on Σ. First of all, according to Proposition 0.4 in [17], we can identify g with a continuous linear functional on the space $W^{s,q}_{\mathrm{loc}}(E)$ supported on K. That is, we may suppose $g \in W^{-s,q'}_K(E')$, where E' is the dual vector bundle of E. Since g vanishes on $S(K)$ in particular, we may use Lemma 5.1 of [17] to conclude that $g = P'v$, where the section $v = \Phi'(g)$ belongs to $W^{p-s,q'}_K(F')$. We now invoke the condition that g is equal to zero on the subspace of $W^{s,q}(E|_K)$ generated by elements of the form

$$\sum_{|\alpha|\leq p-s-1} \Phi(D^\alpha m_\alpha),$$

where $m_\alpha \in M_{\partial K}(F)$ ($|\alpha| \leq p-s-1$). For any such element the transposition rule gives

$$\Big\langle g, \sum_{|\alpha|\leq p-s-1} \Phi(D^\alpha m_\alpha)\Big\rangle = \Big\langle \Phi'(g), \sum_{|\alpha|\leq p-s-1} D^\alpha m_\alpha\Big\rangle = 0.$$

Hence, it is easily seen that $D^\alpha v = 0$ $(p-s-|\alpha|, q')$-a.e. outside K^0 for $|\alpha| \leq p-s-1$ (see, for example, [7]). Moreover, $v \in W^{p-s,q'}_{\mathrm{loc}}(F')$. We can now use the well-known result, proved in a number of papers of Hedberg [7] and [8] and Hedberg and Wolff [10], that each closed subset of $\mathbf{R}^n$ (in particular, the complement of K^0) admits so-called $(p-s, q')$-synthesis. This means that there exists a sequence $\{v_\nu\} \subset \mathcal{D}(F')$ such that $\operatorname{supp} v_\nu \subset K^0$ and $v_\nu \longrightarrow v$ in the topology of $W^{p-s,q'}_{\mathrm{loc}}(F')$. Finally, we have

$$\langle g, u\rangle = \langle P'v, u\rangle = \lim_{\nu\longrightarrow\infty} \langle P'v_\nu, u\rangle = 0,$$

which proves the theorem.

The preceding theorem has been proved for the Cauchy-Riemann operator in the plane successively by Bers [3] (for $q = 2$), Havin [5] (for $2 < q < \infty$), and Hedberg (see, for example, [7]) in the general case. In Hedberg's work [7] a less explicit theorem for solutions of elliptic differential equations having two-sided fundamental solutions and $s = 0$ was obtained. Namely, the measures m_α ($|\alpha| \leq p-1$) were allowed to be supported on the complement of K^0. Then clearly, u_e can be omitted. In [8] Hedberg gave a very precise result similar to ours, for polyharmonic functions, in the case where K is the closure of a bounded open subset of $\mathbf{R}^n$.

If the compact set K is assumed to be nowhere dense, we may give a more precise form of our theorem.

COROLLARY 2.1. *Let K be a compact subset of X without interior, $0 \leq s < p$ and $1 < q < \infty$. Then, for each $u \in W^{s,q}(E|_K)$ and each $\epsilon > 0$, there exists a solution $u_e \in S(K)$ and a section $m \in M_K(F)$ such that*

$$\Big\| u - \big(u_e + \Phi(m)\big)\Big\|_{W^{s,q}(E|_K)} < \epsilon. \tag{2}$$

PROOF. Let us denote by Σ the subspace of $W^{s,q}(E|_K)$ consisting of elements of the form $u_e + \Phi(m)$, where $u_e \in S(K)$ and $m \in M_K(F)$. We must show that any section $u \in W^{s,q}(E|_K)$ belongs to the closure of Σ in $W^{s,q}(E|_K)$. In order to see this, it is sufficient by the Hahn-Banach Theorem to show that any continuous linear functional g on $W^{s,q}(E|_K)$, vanishing on Σ, must be equal to zero identically

Consider some such functional g As in the proof of Theorem 2 1 one can realize g in the form $g = P'v$, where the section $v = \Phi'(g)$ belongs to $W^{p-s,q'}_{\mathrm{loc}}(F')$ and is supported on K By construction, g vanishes on the subspace of $W^{s,q}(E|_K)$ consisting of the elements $\Phi(m)$, where m are (signed) vector-valued measures in $W^{s-p,q}_K(F)$ For any such element we have by the transposition rule

$$\langle g, \Phi(m)\rangle = \langle \Phi'(g), m\rangle = 0$$

Hence we may conclude that $v = 0$ $(p-s, q')$-a e on K Thus, $v = 0$ a e on X and $g = 0$ which proves the corollary

3 **Degenerate cases of approximation in Sobolev spaces on compact sets with empty interior.** As already mentioned in the introduction, we shall make use of the principal index $d = n/q' - (p-s)$

THEOREM 3 1 *Let K be a compact subset of X with empty interior, and $0 \le s < p$ If $d < 0$, then for any section $m \in M_K(F)$, the potential $\Phi(m)$ belongs to the closure of $S(K)$ in $W^{s,q}(E|_K)$*

PROOF We first note that for $d < 0$ it follows from Sobolev's embedding theorem that $W^{p-s,q'}_{\mathrm{loc}}(F') \subset C_{\mathrm{loc}}(F')$ So, if $m \in M_K(F)$, then $m \in W^{s-p,q}_K(F)$, and we have $\Phi(m) \in W^{s,q}(E|_K)$

We wish to prove that each such potential $\Phi(m)$ lies in the closure of $S(K)$ in $W^{s,q}(E|_K)$ To this purpose, the Hahn-Banach Theorem will be used in the standard way

Let g be a continuous linear functional on $W^{s,q}(E|_K)$ vanishing on $S(K)$ Then according to Lemma 5 1 in [17], one can write $g = P'v$, where $v \in W^{p-s,q'}_{\mathrm{loc}}(F')$ Since v is continuous on X and equals zero outside K, we have $v = 0$ on the boundary of K and hence everywhere on X Thus, g vanishes identically so that $\langle g, \Phi(m)\rangle = 0$ which proves the theorem

In particular, we have the following result

COROLLARY 3 1 *Let K be a nowhere dense compact subset of X and $0 \le s < p$ Then $S(K)$ is dense in $W^{s,q}(E|_K)$ for $1 \le q < q_0$, where $q_0 = n/(n-p+s)$ if $s > p-n$ and $q_0 = \infty$ if $s \le p-n$*

PROOF This follows from Corollary 2 1 and Theorem 3 1

This fact was first proved in [17] (see also [18]) where complete references may be found

It follows from a result of Polking [14], that the range of q in Corollary 3 1 is sharp

EXAMPLE 3 1 Polking [14, Theorem 4] constructed, for any real number $1 < r < \infty$, a nowhere dense compact set $K \subset X$ of positive Lebesgue measure and a non-zero bounded function in $W^{r,n/r}_{\mathrm{loc}}(X)$ which is supported in K The compact set K was constructed as a modification of the standard "swiss cheese" or Sierpinski curve in $\mathbf{R}^2$ The

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term "swiss cheese" is traditionally applied to any compact set K obtained by removing from the closed unit disc an infinite sequence $\{B^{(\nu)}\}$ of disjoint open discs such that $\bigcup_\nu B^{(\nu)}$ is dense in the unit disc. Polking's example illustrates in a rather striking manner that the Sobolev embedding theorem is sharp. Supposing $s > p - n$, we apply this construction with $r = p - s$ and obtain a compact set $K \subset X$ of positive Lebesgue measure such that $W_K^{p-s,n/(p-s)}(X) \neq \{0\}$. Now by a theorem in [17] one can infer that $S(K)$ is not dense in $W^{s,n/(n-p+s)}(E|_K)$.

Of course, Example 3.1 and Corollary 2.1 show that if $d \geq 0$ and K is the compact set constructed in Example 3.1, then for some vector-valued measure $m \in W_K^{s-p,q}(F)$, the potential $\Phi(m)$ does not belong to the closure of $S(K)$ in $W^{s,q}(E|_K)$. But we shall prove slightly more.

We shall denote by χ_σ the characteristic function of a set $\sigma \subset X$.

THEOREM 3.2. *Let $d \geq 0$ and K be the compact subset of X constructed in Example 3.1. Then there exists a section $f \in C^\infty_{\mathrm{loc}}(F)$ such that the potential $\Phi(\chi_K f)$ does not belong to the closure of $S(K)$ in $W^{s,q}(E|_K)$.*

PROOF. First we note that if $f \in C^\infty_{\mathrm{loc}}(F)$, the potential $\Phi(\chi_K f)$ lies in $W^{p,q}_{\mathrm{loc}}(E)$ for every $q < \infty$. So the formulation of the theorem is plausible.

Since $d \geq 0$ we have $n - p + s > 0$ and $q > n/(n - p + s)$.

Let us assume that for each section $f \in C^\infty_{\mathrm{loc}}(F)$, the potential $\Phi(\chi_K f)$ lies in the closure of $S(K)$ in $W^{s,q}(E|_K)$. It follows that for any continuous linear functional g on $W^{s,q}(E|_K)$, vanishing on $S(K)$, we have $\langle g, \Phi(\chi_K f)\rangle = 0$.

But according to Lemma 5.1 in [17], if v is an arbitrary element of $W_K^{p-s,q'}(F')$ then $g = P'v$ is such a functional. Thus,

$$\langle g, \Phi(\chi_K f)\rangle = \langle P'v, \Phi(\chi_K f)\rangle = \langle v, \chi_K f\rangle = \langle v, f\rangle = 0.$$

Since this holds for every $f \in C^\infty_{\mathrm{loc}}(F)$, it is easy to see that $v = 0$. Thus, $W_K^{p-s,q'}(F') = \{0\}$ and $q' \leq n/(p - s)$, which contradicts the choice of K. This contradiction proves the theorem.

Let us say that a partial differential operator $P(D)$ in $\mathbf{R}^n$ is *L_q-degenerate-for-nowhere-dense-compacta* provided that, for every compact nowhere-dense set $K \subset \mathbf{R}^n$ of positive Lebesgue measure, every function in $L_q(K)$ can be approximated in $L_q(K)$ by solutions of $Pu = 0$ near K. From the very general theorem of Polking [15, Theorem 1.1], we conclude the following fact.

THEOREM 3.3 (POLKING). *If $P(D)$ and $Q(D)$ are elliptic operators with constant coefficients in $\mathbf{R}^n$ which have the same order, then $P(D)$ is L_q-degenerate-for-nowhere-dense-compacta if and only if $Q(D)$ is L_q-degenerate-for-nowhere-dense-compacta.*

The following result of Polking [15, Theorems 1.3 and 1.4] is also highly relevant to our investigation.

THEOREM 3.4 (POLKING). *Let $1 \leq q < \infty$. An elliptic operator $P(D)$ of order p with constant coefficients in $\mathbf{R}^n$ is L_q-degenerate-for-nowhere-dense-compacta if and only if $q < n/(n - p)$.*

4. **Degenerate cases of approximation in Sobolev spaces on arbitrary compact sets.** If the compact set K is allowed to have interior points, Corollary 3.1 becomes false for q in the same range. In this case, the range of q should be defined not by the existence of the embedding $W^{p-s,q'}_{\mathrm{loc}}(X) \subset C_{\mathrm{loc}}(X)$ but from the existence of the embedding $W^{p-s,q'}_{\mathrm{loc}}(X) \subset C^{p-s,1}_{\mathrm{loc}}(X)$. We have the following result.

THEOREM 4.1. *Let $0 \le s < p$ and $1 < q < n/(n-1)$. Then for any sections $m_\alpha \in M_{\partial K}(F)$ ($|\alpha| \le p-s-1$), the potential*

$$\Phi\Big(\sum_{|\alpha|\le p-s-1} D^\alpha m_\alpha\Big) \tag{3}$$

belongs to the closure of $S(K)$ in $W^{s,q}(E|_K)$.

PROOF. First we observe that, for $1 < q < n/(n-1)$, it follows from Sobolev's Embedding Theorem that $W^{p-s,q'}_{\mathrm{loc}}(F') \subset C^{p-s,1}_{\mathrm{loc}}(F')$. This means that, for any sections $m_\alpha \in M_{\partial K}(F)$ ($|\alpha| \le p-s-1$), the expression $\sum_{|\alpha|\le p-s-1} D^\alpha m_\alpha$ belongs to $W^{s-p,q}_{\partial K}(F)$. Then, the potential (3) lies in $W^{s,q}_{\mathrm{loc}}(E)$ so that the formulation of the theorem is plausible.

We would like to prove that each such potential (3) belongs to the closure of $S(K)$ in $W^{s,q}(E|_K)$. We shall show this using the Hahn-Banach Theorem.

Let g be a continuous linear functional on $W^{s,q}(E|_K)$ which vanishes on $S(K)$. According to Lemma 5.1 in [17], $g = P'v$, where $v \in W^{p-s,q'}_{\mathrm{loc}}(F')$ is supported on K.

After a change on a set of zero measure, the section v can be made continuous together with its derivatives up to order $(p-s-1)$ on X. Since v is supported on K, it follows that the derivatives up to order $(p-s-1)$ of v are equal to zero on the boundary of K.

Hence, we have

$$\begin{aligned}
\Big\langle g, \Phi\Big(\sum_{|\alpha|\le p-s-1} D^\alpha m_\alpha\Big)\Big\rangle &= \Big\langle P'v, \Phi\Big(\sum_{|\alpha|\le p-s-1} D^\alpha m_\alpha\Big)\Big\rangle \\
&= \Big\langle v, P\Phi\Big(\sum_{|\alpha|\le p-s-1} D^\alpha m_\alpha\Big)\Big\rangle \\
&= \Big\langle v, \sum_{|\alpha|\le p-s-1} D^\alpha m_\alpha\Big\rangle \\
&= \sum_{|\alpha|\le p-s-1} (-1)^{|\alpha|}\langle D^\alpha v, m_\alpha\rangle \\
&= 0.
\end{aligned}$$

The theorem now follows immediately from the Hahn-Banach Theorem.

Combining the results from Sections 2 and 4, we obtain the following fact.

COROLLARY 4.1. *Let K be any compact subset of X. Then for $0 \le s < p$ and $1 < q < n/(n-1)$, the space $S(K)$ is dense in the subspace of $W^{s,q}(E|_K)$ formed by weak solutions of the system $Pu = 0$ on K^0.*

PROOF. This follows from Theorems 2.1 and 4.1.

Corollary 4.1 appeared in full generality first in [17] (see also [18]). Complete references may be found in these papers.

There is a well-known example of Hedberg [7] explaining why the range for q in Corollary 4.1 is sharp. Namely, Hedberg [8, Example 6.6] constructed, for each integer $r = 1, 2, \ldots$, a compact set $K \subset X$ with non-empty interior such that $\mathcal{D}(K^0)$ is not dense in $W_K^{r,n}(X)$. If we apply this construction with $r = p - s$, we obtain an example of a compact set $K \subset X$ with non-empty interior for which $\mathcal{D}(K^0)$ is not dense in $W_K^{p-s,n}(X)$. In view of Theorem 5.4 in [17] this means that $S(K)$ is not dense in the subspace of $W^{s,n/(n-1)}(E|_K)$ formed by weak solutions of the system $Pu = 0$ on K^0. We require a slight modification of Hedberg's example.

EXAMPLE 4.1. Suppose that we are given an integer $r = 1, 2, \ldots$ and a non-negative integer δ such that $r - n < \delta \leq r - 1$. We wish to construct a compact set $K \subset \mathbf{R}^n$ and a function $\varphi \in W_{\mathrm{loc}}^{r,n/(r-\delta)}(\mathbf{R}^n)$ supported on K such that φ does not belong to the closure of $\mathcal{D}(K^0)$ in $W_K^{\delta+1,n/(r-\delta)}(\mathbf{R}^n)$. By the $\big(\delta+1, n/(r-\delta)\big)$-synthesis property of Hedberg [7, 8] and of Hedberg and Wolff [10], it is enough to construct a compact set $K \subset \mathbf{R}^n$ and a function $\varphi \in W_{\mathrm{loc}}^{r,n/(r-\delta)}(\mathbf{R}^n)$ such that for some multi-index α, with $|\alpha| = \delta$, we have $D^\alpha\varphi(x) \neq 0$ on a subset of ∂K with positive $\big(r-\delta, n/(r-\delta)\big)$-capacity. Denote the unit ball in $\mathbf{R}^n$ by B_1 and the $(n-1)$-dimensional (closed) ball $\{x \in \mathbf{R}^n : |x| \leq 1/2, x_n = 0\}$ by S. We shall choose suitable disjoint (open) balls $B^{(\nu)}$, $\nu = 1, 2, \ldots$, with centers $x_\nu \in S$ and radii r_ν, and set

$$K = B_1 \Big\backslash \bigcup_{\nu=1}^{\infty} B^{(\nu)}.$$

Let $R_\nu > r_\nu$. We can find functions $\omega_\nu \in C_{\mathrm{loc}}^\infty(0,\infty)$, $\nu = 1, 2\ldots$ such that $\omega_\nu(\xi) = 1$ for $0 < \xi \leq r_\nu$, $\omega_\nu(\xi) = 0$ for $\xi \geq R_\nu$, $0 \leq \omega_\nu \leq 1$, and moreover

$$|(d/d\xi)^j\omega_\nu(\xi)| \leq C\big(\log(R_\nu/r_\nu)\big)^{-1}\xi^{-j}, \quad j = 1, 2, \ldots, r.$$

Set $\varphi_\nu = \omega_\nu(|x - x_\nu|)$, $\nu = 1, 2, \ldots$, and choose a function $\varphi_0 \in \mathcal{D}(B_1)$ such that $\varphi_0(x) = x_n^\delta$ in a neighbourhood of S. It is easily verified that

$$\int |D^\alpha(\varphi_0\varphi_\nu)|^{n/(r-\delta)}\,dx \leq C\big(\log(R_\nu/r_\nu)\big)^{1-n/(r-\delta)},$$

for all $|\alpha| \leq r$, if R_ν is small enough. We have used the letter C to denote various positive constants that may take different values. Now choose $\{R_\nu\}$ so that

$$\sum_{\nu=1}^{\infty} R_\nu^{n-1} < (1/2)^{n-1},$$

and $\{x_\nu\}$ so that the balls $\{x \in \mathbf{R}^n : |x - x_\nu| \leq R_\nu\}$ are disjoint. Finally choose $\{r_\nu\}$ so that

$$\sum_{\nu=1}^{\infty}\big(\log(R_\nu/r_\nu)\big)^{(r-\delta)/n-1} < \infty.$$

Since $(r-\delta)/n-1<0$, this is possible Let us consider the function

$$\varphi=\varphi_0\Big(1-\sum_{\nu\ 1}^{\infty}\varphi_\nu\Big)$$

Clearly $\varphi\in W^{r\,n/(r\ \delta)}(\mathbf{R}^n)$ and supp $\varphi\subset K$ But every $x\in S$ that is not contined in one of the balls $\{x\in\mathbf{R}^n\ \ |x-x_\iota|\le R_\nu\}$ is a boundary point of K On the line perpendicular to S through such a point, we have $\varphi=\varphi_0$, and thus $(\partial/\partial x_n)^\delta\varphi(x)=\delta!$ Since the set of such points has positive $(n-1)$-dimensional Hausdorff measure, φ has the desired properties

It is interesting to note that the boundary of K in Example 4 1 has finite $(n-1)$-dimensional Hausdorff measure

Certainly, Hedberg's example [7] together with Theorem 2 1 show that already for $q=n/(n-1)$ there is a compact set $K\subset X$ and there are sections $m_\alpha\in M_{\partial K}(F)$ $(|\alpha|\le p-s-1)$ such that the potential,

$$\Phi\Big(\sum_{\alpha|<p\ \ s-1}D^\alpha m_\alpha\Big)\in W^{s\,q}_{\mathrm{loc}}(E)\cap S(K^0),$$

does not belong to the closure of $S(K)$ in $W^{s\,q}(E|_K)$ But one can prove slightly more We recall that the index $\big(n/q'-(p-s)\big)$ is denoted by d

THEOREM 4 2 *Let* $0\le s<p$, $n/(n-1)\le q<\infty$ *and let* δ *be any non-negative integer with* $-d\le\delta\le p-s-1$ *Then, if* K *is the compact subset of* X *constructed in Example 4 1 for* $r=p-s$, *there exist sections* $m_\alpha\in M_{\partial K}(F)$ $(|\alpha|\le\delta)$ *such that the potential,*

$$\Phi\Big(\sum_{|\alpha|<\delta}D^\alpha m_\alpha\Big)\in W^{p\ \delta\ \ 1\,q}_{\mathrm{loc}}(E)\cap S(K^0),$$

does not belong to the closure of $S(K)$ *in* $W^{s\,q}(E|_K)$

PROOF First we observe that, for our choice of δ, we have $p-\delta-1\ge s$, so elements of $W^{p\ \delta\ \ 1\,q}_{\mathrm{loc}}(E)$ can be considered to be elements of $W^{s\,q}(E|_K)$

According to Theorem 6 1 in Hedberg [7], [8] and results in Hedberg and Wolff [10], the subspace of $W^{\delta-1\,q}_{\partial K}(F)$ consisting of elements of the form

$$\sum_{|\alpha|<\delta}D^\alpha m_\alpha,\quad m_\alpha\in M_{\partial K}(F),\quad |\alpha|\le\delta,$$

is dense in $W^{\ \delta\ \ 1\,q}_{\partial K}(F)$ Hence it is enough to prove that there is a section $f\in W^{\ \delta\ \ 1\,q}_{\partial K}(F)$ such that the potential $\Phi(f)$ does not belong to the closure of $S(K)$ in $W^{s\,q}(E|_K)$

Suppose, on the contrary, that for every section $f\in W^{\ \delta\ \ 1\,q}_{\partial K}(F)$, the potential $\Phi(f)$ belongs to the closure of $S(K)$ in $W^{s\,q}(E|_K)$ It follows that for each continuous linear functional g on $W^{s\,q}(E|_K)$, we have $\langle g,\Phi(f)\rangle=0$

But according to Lemma 5 1 in [17], the set of such functionals is precisely the set of $g=P'v$, where v is an arbitrary element of $W^{p-s\,q}_K(F')$ Then we have

$$\langle g,\Phi(f)\rangle=\langle P'v,\Phi(f)\rangle=\langle v,P\Phi(f)\rangle=\langle v,f\rangle=0$$

Since this holds for every $f \in W_{\partial K}^{-\delta-1,q}(F)$, it is easy to see that $D^\alpha v = 0$ $(\delta+1-|\alpha|, q')$-a.e. on ∂K for $|\alpha| \leq \delta$.

In Example 4.1 we have the existence of a function $\varphi \in W_{\text{loc}}^{p-s,n/(p-s-\delta)}(X)$ supported on K such that for some multi-index α with $|\alpha| = \delta$ the derivative $D^\alpha \varphi$ is not equal to zero on a subset of ∂K of positive $(n-1)$-dimensional measure. Moreover we have chosen our range for q such that $W_{\text{loc}}^{p-s,n/(p-s-\delta)}(X) \subset W_{\text{loc}}^{p-s,q'}(X)$. Thus, our assumption contradicts the choice of the compact set K. This contradiction proves the theorem.

In particular, for $d \geq 0$, and $\delta = 0$, Theorem 4.2 is analogous to the case of nowhere dense compact sets considered in Theorem 3.2.

5. **Uniform approximation on compact sets by potentials with densities supported on the boundary.** We shall now consider Problem 1.1 for uniform approximation, that is, for $q = \infty$. It is easy to calculate that $d = n - p + s$ in this case. We do not know in general whether Theorem 2.1 is valid for $q = \infty$. This is perhaps, a difficult problem. However Corollary 2.1 carries over to $q = \infty$ without any difficulty

THEOREM 5.1. *Let K be a compact subset of X with empty interior, and $0 \leq s < p$. Then, for each $u \in C^s(E|_K)$ and each $\epsilon > 0$, there exists a solution $u_e \in S(K)$ and a section $f \in C_{\text{loc}}^\infty(F)$ such that*

$$\left\| u - \left(u_e + \Phi(\chi_K f)\right)\right\|_{C^s(E|_K)} < \epsilon. \tag{4}$$

PROOF. If K has zero n-dimensional Lebesgue measure, the section $\chi_K f$, as a generalized section of the bundle F, is equal to zero too, so that $\Phi(\chi_K f) = 0$. In this case Theorem 5.1 asserts that the subspace $S(K)$ is everywhere dense in $C^s(E|_K)$. But this follows from the Hartogs-Rosenthal type theorem for solutions of systems with surjective symbols (see Theorem 3.4 in [17]).

It remains to consider approximation on compact sets $K \subset X$ of positive n-dimensional measure. In this case, for any section $f \in C_{\text{loc}}^\infty(F)$ we have $\chi_K f \in L_{\text{comp}}^\infty(F)$ so that, by the boundedness theorem for pseudo-differential operators in Sobolev spaces, the potentials $\Phi(\chi_K f)$ belong to the space $W_{\text{loc}}^{p,q}(E)$ for each $q < \infty$. In particular, by the Sobolev Embedding Theorem, one can see that $\Phi(\chi_K f) \in C_{\text{loc}}^{p-1}(E)$.

Let us denote by Σ the subspace of $C^s(E|_K)$ consisting of elements of the form $u_e + \Phi(\chi_K f)$, where $u_e \in S(K)$ and $f \in C_{\text{loc}}^\infty(F)$. We claim that any section $u \in C^s(E|_K)$ belongs to the closure of Σ in $C^s(E|_K)$. In order to see this it is enough, by the Hahn-Banach Theorem, to show that any continuous linear functional g on $C^s(E|_K)$ vanishing on Σ must vanish identically.

Let g be some continuous linear functional on $C^s(E|_K)$ vanishing on Σ. First, according to Proposition 0.2 in [17], we can identify g with a continuous linear functional on the space $C_{\text{loc}}^s(E)$ supported on K. Since g vanishes, in particular, on $S(K)$, it follows from Lemma 12.9 in [19] that we have $g = P'v$, where the section $v = \Phi'(g)$ is supported on K. Moreover, it is easy to see from the Sobolev Embedding Theorem and the boundedness theorem for pseudo-differential operators in Sobolev spaces that $v \in W^{p\ s\ 1\ q}_{\mathrm{loc}}(F')$ for each $q' < n/(n-1)$

We now invoke the condition that g is equal to zero on the subspace of $C^s(E|_K)$ formed by potentials $\Phi(\chi_K f)$ with $f \in C^\infty_{\mathrm{loc}}(F)$ For any such potential we have, by the transpo sition rule,

$$\langle g, \Phi(\chi_K f)\rangle = \langle P'v, \Phi(\chi_K f)\rangle = \langle v, P\Phi(\chi_K f)\rangle = \langle v, \chi_K f\rangle = \langle v, f\rangle = 0$$

Since f is arbitrary, $v = 0$ a e on X Hence $g = 0$ which proves the theorem

Of course we could use the above reasoning to obtain a stronger formulation of Corol lary 2 1 However for this it is sufficient to use the above theorem and the density of $C^s(E|_K)$ in $W^{s\ q}(E|_K)$ for $q < \infty$ On the other hand, we are unable to improve Theo rem 2 1

6 Degenerate cases of uniform approximation on nowhere dense compact sets.

Extrapolating on Corollary 3 1, one can formulate the following proposition

PROPOSITION 6 1 *Let K be a compact subset of X with empty interior Then $S(K)$ is dense in $C^s(E|_K)$ for $0 \le s < p - n$*

Of course this is meaningful only if $p > n$ Proposition 6 1 was confirmed for the first time in full generality in the recent survey [17] to which we refer for references In fact, we shall verify this proposition as a consequence of the previous section and the following result

THEOREM 6 1 *Let $0 \le s < p$ be such that $d < 0$ and let δ be any non negative integer with $\delta < -d$ Then given sections $m_\alpha \in M_{\partial K}(F)$ $|\alpha| \le \delta$ the potential*

$$\Phi\Big(\sum_{|\alpha|<\delta} D^\alpha m_\alpha\Big) \tag{5}$$

belongs to the closure of $S(K)$ in $C^s(E|_K)$

PROOF Suppose we are given sections $m_\alpha \in M_{\partial K}(F)$, $|\alpha| \le \delta$ Then the order of singularity of the section $\sum_{|\alpha|<\delta} D^\alpha m_\alpha$ does not surpass δ Since all derivatives up to order $(p-n-1)$ of the kernel Φ are continuous, we have that the potential given by (5) is in $C^{p\ n\ 1\ \delta}_{\mathrm{loc}}(E)$ But from our choice of δ, it is easy to see that $p - n - 1 - \delta \ge s$ so that the potential given by (5), which is in $C^s_{\mathrm{loc}}(E)$, may be considered as an element of $C^s(E|_K)$

We wish to prove that each such potential (5) belongs to the closure of $S(K)$ in $C^s(E|_K)$ To this end, we will use the Hahn Banach Theorem in the standard way Let g be a continuous linear functional on $C^s(E|_K)$ which vanishes on $S(K)$ From Proposition 0 2 in [17], it follows that g may be realized as a continuous linear functional on $C^s_{\mathrm{loc}}(E)$ with support in K As in the proof of Theorem 5 1, one can check that $g = P'v$ where the section $v = \Phi'(g)$ is supported in K, and then show that $v \in C^{d\ 1}_{\mathrm{loc}}(F')$ Since v vanishes outside K, the derivatives up to order $-d-1$ of v are equal to zero on the boundary of K

By hypothesis, we have $\delta \leq (-d-1)$. Hence,

$$\begin{aligned}\Big\langle g, \Phi\Big(\sum_{|\alpha|\leq\delta} D^\alpha m_\alpha\Big)\Big\rangle &= \Big\langle P'v, \Phi\Big(\sum_{|\alpha|\leq\delta} D^\alpha m_\alpha\Big)\Big\rangle \\ &= \Big\langle v, P\Phi \sum_{|\alpha|\leq\delta} D^\alpha m_\alpha\Big\rangle \\ &= \sum_{|\alpha|\leq\delta} (-1)^{|\alpha|}\langle D^\alpha v, m_\alpha\rangle \\ &= 0.\end{aligned}$$

To complete the proof it suffices to invoke the Hahn-Banach Theorem.

Certainly the strongest result which can be obtained from Theorem 6.1 is for $\delta = -d-1$.

COROLLARY 6.1. *Suppose that K is a compact subset of X with empty interior, and $0 \leq s < p$ are such that $d < 0$. Then $S(K)$ is dense in $C^s(E|_K)$.*

PROOF. This follows from Theorems 5.1 and 6.1.

Proposition 6.1 has now been confirmed, and Corollary 3.1 has been extended to $q = \infty$. A question remains as to whether the range for s in Corollary 6.1 is sharp. We shall now extrapolate the result of Theorem 3.2.

THEOREM 6.2. *Let $0 \leq s < p$ be such that $d > 0$, and let K be the compact subset of X constructed in Example 3.1. Then there exists a section $f \in C^\infty_{\mathrm{loc}}(F)$ such that the potential $\Phi(\chi_K f) \in C^{p-1}_{\mathrm{loc}}(E)$ does not belong to the closure of $S(K)$ in $C^s(E|_K)$ and hence $S(K)$ is not dense in $C^s(E|_K)$.*

PROOF. In fact, it follows from Theorem 3.2 that there exists a section $f \in C^\infty_{\mathrm{loc}}(F)$ such that the potential $\Phi(\chi_K f) \in C^{p-1}_{\mathrm{loc}}(E)$ does not belong to the closure of $S(K)$ in $W^{s,n/d}(E|_K)$. A fortiori this potential does not belong to the closure of $S(K)$ in $C^s(E|_K)$ which proves the theorem.

7. **Distinguished case of uniform approximation on nowhere dense compact sets.** As far as the case $s = p-n$ (that is, $d = 0$) is concerned, there may be no simple universal answer. Whether $S(K)$ is dense in $C^s(E|_K)$ depends also on the particular differential operator P. For one differential operator P, it may be so, and for another P it may be quite the reverse. This is in contrast with the case of approximation in Sobolev spaces. We shall illustrate this by two examples in $\mathbf{R}^2$: the operators Δ and $\bar{\partial}^2$. But first, we mention a simple example in $\mathbf{R}^1$.

EXAMPLE 7.1. Consider the differentiation operator $P(D) = -\sqrt{-1}\, d/dx$ on the real axis $\mathbf{R}^1$. We claim that, for any compact set $K \subset \mathbf{R}^1$ with empty interior, the subspace $S(K)$ is dense in $C(K)$. In fact, by Tietze's Theorem, it is easy to see that our definition of the space $S(K)$ and the intrinsic definition of that space, using functions defined only on K and the induced topology of K, coincide. Let $u \in C(K)$ be any continuous function on K, and $\epsilon > 0$ be an arbitrary real number. We can extend u to a function continuous on some segment $[a, b]$ with $K \subset (a, b)$. Let us again denote this continuation by u. Since u is uniformly continuous, there exists a real $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ for $x, y \in [a, b]$ and $|x - y| < \delta$. We choose now an integer N such that $(b - a)/N < \delta/2$, and divide the segment $[a, b]$ into N parts with the help of points

$$x_j = a + \big((b - a)/N\big) \cdot j, \quad j = 0, 1, \ldots, N.$$

Since K is a nowhere dense compact set, it follows that, in each interval (x_j, x_{j+1}), we can find a point $a_{j+1} \notin K$. Having removed the points a_1 and a_N, if need be, we may assume that $K \subset (a_1, a_N)$. We note that $|a_{j+1} - a_j| < \delta$ for $j = 1, 2, \ldots, N - 1$. Let us define now a locally constant function u_ϵ in a neighbourhood of K. Namely, we set $u_\epsilon(x) = u(x_j)$ for $x \in (a_j, a_{j+1})$, where $j = 1, 2, \ldots, N - 1$. Then u_ϵ satisfies $Pu_\epsilon = 0$ in a neighbourhood of K. On the other hand, let us evaluate the difference $|u_\epsilon(x) - u(x)|$ for $x \in K$. If the point x falls in an interval (a_j, a_{j+1}), $|x_j - x| < \delta$ so that

$$|u_\epsilon(x) - u(x)| = |u(x_j) - u(x)| < \epsilon.$$

Thus we have $\|u_\epsilon - u\|_{C(K)} < \epsilon$ and so u_ϵ is the required approximation of u.

In this example, we had $n = 1, p = 1$, and $s = 0$, so $d = 0$.

EXAMPLE 7.2. On the other hand, let $P = \Delta$ be the Laplace operator in the space $\mathbf{R}^2$. We shall construct a compact set $K \subset \mathbf{R}^2$ with empty interior such that $S(K)$ is not dense in $C(K)$. For this purpose we shall use a modification of the standard construction of a "swiss cheese". We choose $r_0 = 1$ and take some decreasing sequence $\{r_j\} \subset (0, 1)$ which converges to zero. For example, one can set $r_j = 2^{-j}, j = 1, 2, \ldots$. In each ring

$$R_j = \{x \in \mathbf{R}^2 : r_j < |x| \leq r_{j-1}\},$$

where $j = 1, 2, \ldots$, we choose an everywhere dense system $\{B_j^{(\nu)}\}_{\nu=1,2,}$ of pairwise disjoint open discs. Since the harmonic capacity, denoted henceforth by cap, of each disc $B_j^{(\nu)}$ can be calculated via its radius, we can choose these radii so that

$$\sum_{\nu=1}^{\infty} \operatorname{cap} B_j^{(\nu)} < \epsilon_j.$$

The sequence $\epsilon_j, j = 1, 2, \ldots$, will be given later. We note that by the subadditivity of harmonic capacity we have

$$\operatorname{cap} \bigcup_{\nu=1}^{\infty} B_j^{(\nu)} < \epsilon_j, \quad j = 1, 2, \ldots.$$

Consider the compact set

$$K = \{|x| \leq 1\} \Big\backslash \bigcup_{j,\nu=1}^{\infty} B_j^{(\nu)}.$$

It is clear from the construction that this compact set is nowhere dense and the point $x = 0$ belongs to K. We claim that with a suitable choice of the sequence $\{\epsilon_j\}$, the complement of K is thin at the point $x = 0$. In fact, according to the well-known Wiener criterion [24] the set $\mathbf{R}^2 \setminus K$ is thin at the point $x = 0$ if and only if

$$\sum_{\nu=1}^{\infty} j \operatorname{cap}(R_j \setminus K) < \infty.$$

But

$$R_j \setminus K = \bigcup_{\nu=1}^{\infty} B_j^{(\nu)},$$

so that $\operatorname{cap}(R_j \setminus K) < \epsilon_j$. It follows that one may take, for example, $\epsilon_j = j^{-3}$. Now we can use the classical Keldysh Theorem [11] adapted to the two-dimensional situation (see, for example, 1.3.7 in [2]). Namely, in order for $S(K)$ to be dense in $C(K)$ it is necessary and sufficient that the complement of K be thick at each point of K. Thus, for a suitable choice of the sequence $\{\epsilon_j\}$, the subspace $S(K)$ is not dense in $C(K)$, which furnishes the desired example. Note that K may have positive measure.

In Example 7.2, we had $n = 2, p = 2$ and $s = 0$, so that also $d = 0$.

Recall that a partial differential operator $P(D)$ in $\mathbf{R}^n$ is L_q-degenerate-for-nowhere-dense-compacta provided that, for every compact nowhere-dense set $K \subset \mathbf{R}^n$ of positive Lebesgue measure, every function, in $L_q(K)$ can be approximated in $L_q(K)$ by solutions of $Pu = 0$ near K. It follows from Theorem 3.4 that there exist compact nowhere dense sets $K \subset \mathbf{R}^2$ for which condition (c) of the Stability Theorem 1.1 will fail, and hence the other conditions of the Stability Theorem fail. Thus, the existence of a compact set having the properties of Example 7.2 is actually known. Example 7.2 is also (independently of the present paper) included in Hedberg's survey paper [9]. For $\mathbf{R}^3$, an example similar to Example 7.2 is given in [11, Section 5].

In Theorem 3.3 we stated the fact that if $P(D)$ and $Q(D)$ are elliptic operators with constant coefficients in $\mathbf{R}^n$ which has the same order, then $P(D)$ is L_q-degenerate-for-nowhere-dense-compacta if and only if $Q(D)$ is L_q-degenerate-for-nowhere-dense-compacta. It is natural to ask whether a similar fact holds for uniform approximation. However, this is not true for operators of order 2 in $\mathbf{R}^2$; in fact, we have established (Example 7.2) that there are compact nowhere dense sets $K \subset \mathbf{R}^2$ which do *not* satisfy condition (a) of the Stability Theorem 1.1, but Trent and Wang [20, Theorem, p. 63] have proved the following theorem.

THEOREM 7.1 (TRENT AND WANG). *If K is any compact nowhere dense subset of $\mathbf{R}^2$, then the set of functions u satisfying $\bar{\partial}^2 u \equiv 0$ near K must be uniformly dense in $C(K)$.*

8. **Problem of degeneracy for uniform approximation on arbitrary compact sets.** It is natural to ask whether the above theorem of Trent and Wang can be extended to arbitrary compact sets, and we might call this the "problem of Trent and Wang".

PROBLEM 8 1 (TRENT AND WANG) *If K is any compact subset of $\mathbf{R}^2$, and $u \in C(K)$ satisfies $\bar\partial u \equiv 0$ in K^0, can we approximate u uniformly on K by functions u_e which satisfy $\bar\partial u_e \equiv 0$ near K?*

This problem is still unsolved, the most recent contribution being by Verdera [23], however, Theorem 8 2 below (with $s = 0$) shows that *the problem of Trent and Wang has a negative answer for elliptic equations (and also some elliptic systems) in dimensions $n \geq 3$*

By analogy with Theorem 4 2 it is natural to generalize the problem of Trent and Wang as follows

PROBLEM 8 2 ($n = 2$) *For any integer s with $0 \leq s < p$ and any compact set $K \subset X$, is $S(K)$ dense in the subspace of $C^s(E|_K)$ formed by weak solutions of $Pu = 0$ in K^0?*

Although the theorems of Trent and Wang as well as the work of Verdera point to an affirmative answer to the problem of Trent and Wang, we shall see from the following discussion that for $n > 2$, the answer to the generalized problem of Trent and Wang is *always* negative Of course, if $s < p - n$, it follows from Corollary 6 1 that an example of such a compact set K would have to be found among compact sets with non-empty interior

Indeed, a step towards a negative answer to Problem 8 2 was made in [16] Therein, an arbitrary homogeneous elliptic differential operator P with constant coefficients in $\mathbf{R}^n$ was considered It was shown that, if $p - n < s < p$, then, for each column $h \neq 0$ of homogeneous polynomials of degree $p - s - 1$ satisfying $P^*(D_\xi)h(\xi) = 0$, there is a compact set $K \subset \mathbf{R}^n$ with non-empty interior and a subset $S \subset \partial K$ of positive (n dimensional) Lebesgue measure such that the potential $\Phi\big(h(D)\chi_S\big)$ does not belong to the closure of $S(K)$ in $C^s(E|_K)$ Of course, it is easily verified that $\Phi\big(h(D)\chi_S\big)$ lies in $C^s_{\mathrm{loc}}(E) \cap S(K^0)$

Theorem 6 2 provides us with an analogous example for an arbitrary differential operator P and where K is a nowhere dense compact set Thus, the analog, for general n, of Problem 8 2 remains open only for $0 \leq s \leq p - n$

In view of a possible analog of Theorem 2 1 for uniform approximation, it is natural to seek an example of a non-approximable solution among potentials For this reason, we shall consider precisely such solutions

Let K be a compact subset of X, and S be a set of positive n-dimensional measure on ∂K For some fixed differential operator $\mathcal{D}$ of type $X \times \mathbf{C}^1 \longrightarrow F$ and order δ, we consider the potential $u = \Phi(\mathcal{D}\chi_S)$

Since mes $S > 0$, we have $u \in C^{p-1\ \delta}_{\mathrm{loc}}(E)$ Moreover the boundedness theorem for pseudo-differential operators in Sobolev spaces implies that $u \in W^{p\ \delta\ q}_{\mathrm{loc}}(E)$ for each $q < \infty$ In particular, if $\delta \leq p - s - 1$, then the potential u belongs to the space $C^s_{\mathrm{loc}}(E)$ Henceforth, we assume δ is within this range From the fact that $Pu = \mathcal{D}\chi_S$ it is easy to see that $u \in S(K^0)$ It follows from Theorem 6 1 that for $\delta < -d$ the potential $\Phi(\mathcal{D}\chi_S)$ belongs to the closure of $S(K)$ in $C^s(E|_K)$

THEOREM 8.1. *Let K and S be as above and suppose, for each differential operator $\mathcal{D}$ of type $X \times \mathbf{C}^1 \longrightarrow F$ and order $\delta \leq p - s - 1$, the potential $\Phi(\mathcal{D}\chi_S)$ belongs to the closure of $S(K)$ in $C^s(E|_K)$. Then, for any section $v \in W_K^{p-s,q'}(F')$, $q' > 1$, we have $D^\alpha v = 0$ a.e. on S for $|\alpha| \leq \delta$.*

PROOF. Since $\Phi(\mathcal{D}\chi_S)$ belongs to the closure of $S(K)$ in $C^s(E|_K)$, any continuous linear functional g on $C^s(E|_K)$ vanishing on $S(K)$ is zero on $\Phi(\mathcal{D}\chi_S)$.

In particular this is true for each functional $g = P'v$ where $v \in W_K^{p-s,q'}(F')$, for some $q' > 1$. Since $C^s_{\text{loc}}(E) \subset W^{s,q}_{\text{loc}}(E)$, we have $W^{-s,q'}_{\text{comp}}(E') \subset \left(C^s_{\text{loc}}(E)\right)'$. This means that $P'v \in W_K^{-s,q'}(E')$ is a continuous linear functional on $C^s_{\text{loc}}(E)$ supported on K. We can apply Proposition 0.2 of [17] to see that $P'v \in \left(C^s(E|_K)\right)'$. On the other hand it is clear that $P'v$ vanishes on $S(K)$. Thus,

$$\langle P'v, \Phi(\mathcal{D}\chi_S)\rangle = \langle v, P\Phi(\mathcal{D}\chi_S)\rangle = \langle v, \mathcal{D}\chi_S\rangle = \langle \mathcal{D}'v, \chi_S\rangle = \int_S \mathcal{D}'v = 0.$$

EXAMPLE 8.1. Suppose $\delta > -d$. Then for some $1 < q' \leq n$, namely for $q' = n/(p-s-\delta)$, we have $\delta = -(n/q' - p + s)$. It seems likely that by modifying Hedberg's Example 4.1 we can construct a compact set $K \subset X$ and a set $S \subset \partial K$ of positive n-dimensional measure such that for some section $v \in W_K^{p-s,q'}(F')$ it is not the case that $D^\alpha v = 0$ a.e. on S for all $|\alpha| \leq \delta$. In view of Thoerem 8.1, this would mean that there exists a differential operator $\mathcal{D}$ of type $X \times \mathbf{C}^1 \longrightarrow F$ and order δ such that the potential $\Phi(\mathcal{D}\chi_S)$ does not belong to the closure of $S(K)$ in $C^s(E|_K)$. We recall that

$$\Phi(\mathcal{D}\chi_S) \in C^{p-1-\delta}_{\text{loc}}(E) \cap S(K^0),$$

so that such a result would be an adequate extension of Theorem 4.2 to the case $q = \infty$. In particular, for $d > 0$ we could take $\delta = 0$. In general it is always possible to take $\delta = p - s - 1$, at least for $n > 1$. Thus we would obtain a complete negative answer to the analog of Problem 8.2 for $n > 1$.

For $\delta < -d$, the potentials $\Phi(\mathcal{D}\chi_S)$ can be approximated in $C^s(E|_K)$ by elements of $S(K)$, whereas for $\delta > -d$, there are such potentials which cannot be so approximated. As far as the case $\delta = -d$ is concerned, the solution of the question apparently depends on the choice of the particular differential operator P.

However, for $n > 2$ we are able to obtain a complete negative answer to the analog of Problem 8.2 directly from Theorem 4.2.

THEOREM 8.2. *If $n > 2$, the for any integer s with $0 \leq s < p$, there is a compact set $K \subset X$ such that $S(K)$ is not dense in $C^s(E|_K) \cap S(K^0)$.*

PROOF. For $s > p - n$ such a compact set K has already been constructed in Theorem 6.2, so we may assume that $p \geq n$ and $s \leq p - n$.

Let K be the compact subset of X constructed in Example 4.1 for $r = p - s$. We apply Theorem 4.2 with $\delta = -(n - p + s) + 1$ and any fixed $q > n$. Since $n > 1$, it is easy to see that all the conditions for the theorem are satisfied.

According to Theorem 4.2 there exist sections $m_\alpha \in M_{\partial K}(F)$, $|\alpha| \leq \delta$, such that the potential given by (5), which is in $W^{p-\delta-1,q}_{\text{loc}}(E) \cap S(K^0)$, does not belong to the closure of $S(K)$ in $W^{s,q}(E|_K)$.

Now we observe that by the Sobolev Embedding Theorem,

$$W^{p-\delta-1,q}_{\text{loc}}(E) \subset C^{p-\delta-2}_{\text{loc}}(E), \quad \text{for } q > n.$$

Since $p - \delta - 2 \geq s$, at least for $n > 2$, it follows that the potential given by (5) belongs to $C^s_{\text{loc}}(E) \cap S(K^0)$.

However this potential cannot be approximated in $C^s(E|_K)$ by elements of $S(K)$ because otherwise it would be approximated also in $W^{s,q}(E|_K)$ by elements of $S(K)$. This completes the proof of the theorem.

We now formulate a problem which appears to be simple.

CONJECTURE 8.1. *Let $n = 1$ and $0 \leq s < p$. Then for any compact set $K \subset X$ the space $S(K)$ is dense in the subspace of $C^s(E|_K)$ formed by weak solutions of $Pu = 0$ in the interior of K.*

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