Canad. Math. Bull. Vol. 19 (2), 1976

THE GIBBS PHENOMENON FOR $[S, \alpha_n]$ MEANS AND $[T, \alpha_n]$ MEANS

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The Gibbs phenomenon of the Fourier series $\sum_{n=1}^{\infty} \sin nx/n$ for different summability methods has been investigated by various authors. In this note, we study the same for the $[S, \alpha_n]$ method of summability introduced by Meir and Sharma [3]. The corresponding result for the $[T, \alpha_n]$ method of summability due to Powell [5] can be worked out in exactly the same way.

The elements c_{nk} of the $[S, \alpha_n]$ matrix are defined by the relations:

$$(1)\qquad \prod_{j=0}^{n} \frac{1-\alpha_j}{1-\alpha_j z} = \sum_{k=0}^{\infty} c_{nk} z^k \qquad (n=0, 1, \ldots),$$

where $0<\alpha_n<1$. The $[S, \alpha_n]$ matrix is regular if and only if $\sum_{j=0}^{\infty} \alpha_j = +\infty$.

The elements a_{nk} of the $[T, \alpha_n]$ matrix are given by the relations:

$$a_{nk} = 0 \qquad k<n$$

$$\prod_{j=1}^{n+1} \frac{(1-\alpha_j)z}{1-\alpha_j z} = \sum_{k=n}^{\infty} a_{nk} z^{k+1} \qquad (n=0, 1, \ldots),$$

where $0<\alpha_n<1$.

Let $\sigma_n(x)$ denote the T-transform of the sequence of partial sums of the Fourier series for the function

$$(2)\qquad \phi(x) = \begin{cases} -\pi/2 & -\pi<x<0 \\ +\pi/2 & 0<x<\pi, \end{cases}$$

$\phi(-\pi)=\phi(0)=\phi(\pi)=0$, and $\phi(x)=\phi(x+2\pi)$. In order to show that a regular summability method T preserves the Gibbs phenomenon, Miracle [4] has proved that it sufficies to show that if δ is in $[-\pi, +\pi]$, there exists a sequence $\{t_n\}$ such that $t_n \to 0$ and

$$\lim_{n\to\infty} \sigma_n(t_n) = \int_0^{\delta} \frac{\sin y}{y}\, dy.$$

THEOREM I. *Suppose that* $0<\alpha_j\le q<1$ $(j=0, 1, \ldots)$. *Then the* $[S, \alpha_n]$ *transform completely preserves the Gibbs phenomenon for Fourier series.*

THEOREM II. *Suppose that* $0<\alpha_j\le q<1$ $(j=1, 2, \ldots)$. *Then the* $[T, \alpha_n]$ *transform completely preserves the Gibbs phenomenon for Fourier series.*

Received by the editors August 8, 1973 and, in revised form, June 19, 1975.

https://doi.org/10.4153/CMB-1976-032-8 Published online by Cambridge University Press

Proof of Theorem I. The Fourier series expansion for the function $\phi(x)$ of (2) is

$$2\sum_{\nu=1}^{\infty}\frac{\sin(2\nu-1)x}{2\nu-1}.$$

Let $\{s_n(x)\}$ denote the sequence of partial sums of this series. Then

$$s_n(x)=\int_0^x\frac{\sin 2nt}{\sin t}\,dt \qquad (n=0,1,\ldots).$$

In the sequel, we consider only values of x in $[0,\pi/4]$. The $[S,\alpha_n]$ transform $\{\sigma_n(x)\}$ of the sequence $\{s_n(x)\}$ is given by

$$(3)\qquad \sigma_n(x)=\sum_{k=0}^{\infty}c_{nk}s_k(x)=\sum_{k=0}^{\infty}c_{nk}\int_0^x\frac{\sin 2kt}{\sin t}\,dt.$$

We have by an easy calculation from (1) that

$$\left|\sum_{k=0}^{\infty}c_{nk}\frac{\sin 2kt}{\sin t}\right|\le\frac{\pi}{\sqrt2}\sum_{j=0}^{n}\frac{\alpha_j}{1-\alpha_j}\le\frac{1}{\sqrt2}\frac{\pi q}{1-q}(n+1).$$

Hence the series (3) is uniformly convergent for $0\le t\le\pi/4$ and we may therefore interchange the order of integration and summation in (3). We now have by (1)

$$\sigma_n(x)=\int_0^x\frac{1}{\sin t}\operatorname{Im}\left\{\prod_{j=0}^{n}\left(\frac{1-\alpha_j}{1-\alpha_j\exp(i2t)}\right)\right\}dt.$$

Define ρ_j and θ_j $(j=0,1,\ldots)$ by

$$\rho_j\exp(-i\theta_j)=1-\alpha_j\exp(i2t)$$

Then, after some calculations, we obtain for $\sigma_n(x)$ the representation

$$\sigma_n(x)=\int_0^x\operatorname{cosec} t\sin\left(\sum_{j=0}^{n}\theta_j\right)dt-V_n\int_0^x\lambda t^2\operatorname{cosec} t\sin\left(\sum_{j=0}^{n}\theta_j\right)dt,$$

where V_n is defined as $\sum_{j=0}^{n}\alpha_j/(1-\alpha_j)^2$, and λ is a function of n, α_j, and t. Write

$$u_n=\sum_{j=0}^{n}\frac{\alpha_j}{1-\alpha_j},\qquad W_n=\sum_{j=0}^{n}\frac{\alpha_j}{(1-\alpha_j)^3}.$$

Careful estimations then lead to

$$\left|\sigma_n(x)-\int_0^x t^{-1}\sin 2u_nt\,dt\right|<\pi V_nx^2+9\pi W_nx^3+x^2+243\pi W_n^2x^6.$$

Consequently, given $\varepsilon > 0$, there exists an integer N such that for $n \geq N$,

$$\left| \sigma_n(t_n) - \int_0^\delta \frac{\sin y}{y} \, dy \right| < \varepsilon.$$

Hence the theorem.

REMARK. In the statement of Theorem I, the condition that $q < 1$ cannot be relaxed; that is to say, there exist $[S, \alpha_n]$ transformations, for which there is no number q such that $0 < \alpha_j \leq q < 1$ $(j = 0, 1, \ldots)$ and which do not preserve the Gibbs phenomenon.

This can be illustrated by choosing

$$\alpha_j = 1 - \exp(-10^{-3}(j+1)) \qquad (j = 0, 1, \ldots).$$

COROLLARY I. *If* $\alpha_j = \alpha$, $(j = 0, 1, \ldots)$, *Theorem I reduces to the known result of Ishiguro* [2] *for the classical* S_α*-transform of Meyer-König.*

COROLLARY II. *If* $\alpha_j = \alpha$, $(j = 1, 2, \ldots)$, *Theorem II reduces to the known result of Ishiguro* [1] *for the classical Taylor transform.*

The author is grateful to Dr. Y. Sitaraman for his guidance and help in the preparation of this note. The author is also grateful to the referee for his suggestions.

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