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A NOTE ON DUALIZING GOLDIE DIMENSION

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1. **Introduction and definitions.** The purpose of this note is to offer a dualization of the concept of Goldie dimension and to prove a structure theorem (Theorem 3.1) for modules satisfying the conditions of this dualization. In this paper, all rings considered are associative with unit and all modules are unital. If M is a left R-module, then a submodule, A, of M is termed small if $A+H=M$ implies $H=M$ for any other submodule, H, of M. It should be noted that if $M\supseteq X\supseteq Y$ is a sequence of submodules of M, then if Y is small in X, it is small in M.

We recall that a module is said to have finite Goldie dimension if it does not contain an infinite direct sum of submodules. This is equivalent to saying that, for any increasing sequence of submodules of M, $U_0\subseteq U_1\subseteq U_2\subseteq \cdots$ there is i and U_i is essential in U_j for $j\geq i$. There are several possible ways to dualize this and we choose the following form which is sufficient for our purposes.

DEFINITION 1.1. If M is a left R-module, we say M has finite spanning dimension if for every strictly decreasing sequence of submodules $U_0\supseteq U_1\supseteq \ldots$, there is i and U_j is small in M for every $j\geq i$.

For example, any Artinian module will satisfy this definition as will any local module in the sense of [4]. Further, if R is a left-semiprimary ring (i.e., it has dcc module its Jacobson radical) and $M=R$, then M also satisfies this definition.

We use the work spanning since codimension has other meanings in other contexts. Furthermore, there is an analogy between the above definition and the definition of a basis in a vector space. One can define a basis as either a maximal set of linearly independent vectors or as a minimal set of vectors which span the space. The former, when generalized to modules, becomes the concept of Goldie dimension. The latter, as we shall see in theorem 3.1, is the analog of definition 1.1 for a finite dimensional vector space.

2. **Elementary properties.** Of fundamental importance in the study of Goldie dimension are uniform modules (those modules which are essential extensions of all submodules) and complements of submodules (modules which are maximal with respect to the property of having zero intersection with a given submodule). We give the dualization of these notions next.

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DEFINITION 2.1. (a) Let M be a left R-module. We call M hollow if every submodule of M is small in M. (b) If U is a submodule of M, we say the submodule X is a supplement of U in M if $U+X=M$ but $U+Y\neq M$ for any proper submodule Y of X.

Finding hollow submodules of a module with finite spanning dimension is easy. If M has finite spanning dimension, then if all of its submodules are small, M is itself hollow and we are done. If not, then there is M_1 properly contained in M and there is an X_1 in M with $M_1+X_1=M$ but $X_1\neq M$. If M_1 has a submodule which is not small in M, say M_2, then there is X_2 and $M_2+X_2=M$ but $X_2\neq M$. Continuing in this way we obtain a strictly decreasing sequence $M\supseteq M_1\supseteq M_2\supseteq\ldots$. By definition, there must be an M_i which contains no non-small submodules of M yet M_i is non-small itself. Now suppose M_i is not hollow. That is, we can find A and B contained in M_i such that neither A nor B equals M_i yet $A+B=M_i$. Since M_i is not small, there is X_i and $M_i+X_i=M$, but $X_i\neq M$. Thus $A+B+X_i=M$. Since A is small in M, we get $B+X_2=M$. Since B is small, we have $X_i=M$ which is a contradiction. We thus have the following.

LEMMA 2.2. *If M has finite spanning dimension and X is a submodule of M which is not small, then X contains a hollow submodule.*

Now let us consider any module M. If X is a submodule of M, it is always possible, using Zorn's lemma, to find a complement for X. Finding a supplement for X is not so easy. The next lemma shows that finite spanning dimension is just what we need for supplements.

LEMMA 2.3. *If M has finite spanning dimension, then every submodule of M has a supplement.*

Proof. We shall actually prove a little more. I.e., if N is a submodule of M and $N+X=M$, then X contains a supplement of M. If N is small, it is clear that the supplement of N is M itself since the only submodule which we could add to N to obtain M is M itself. If, on the other hand, N is not small and $N\neq M$, we can find an X which is not M and $N+X=M$. If X is a supplement of N, we are done. If not, there is $X_1\subseteq X$ and $N+X_1=M$. If X_1 is a supplement, we are done; otherwise, we can obtain an X_2. We proceed this way obtaining the chain $X\supseteq X_1\supseteq X_2\ldots$. After a certain point, this sequence must terminate because every infinite sequence must contain only a finite number of non-small members and all the members here are certainly non-small. Thus X must contain a supplement of N. Finally if $N=M$, then the supplement is the zero submodule.

3. **The main theorem.** We are now in a position to state and prove our main theorem. Before we do that, however, we make a trivial observation which will help in understanding the proof. If N is a small submodule of N' and N' is a submodule of M, then N is a small submodule of M. In particular, if N' is hollow and $N'\subseteq M$, then all proper submodules of N' are small in M.

THEOREM 3.1. *Let M have finite spanning dimension. Then there is an integer p and $M=N_1+\cdots+N_p$ where each N_i is hollow for $i=1,\ldots,p$. Furthermore $N_1+\cdots+\hat{N}_i+\cdots+N_p\neq M$. Finally, if $M=N'_1+\cdots+N'_q$ and this summation satisfies our first two conditions, then $p=q$.*

Proof. To begin, we pick a hollow submodule as we did preceding the statement of lemma 2.2. Let us call that submodule N_1. If $N_1=M$, we are done. If not, we employ the following process. Since N_1 is not small, it has a supplement X_1. So $N_1+X_1=M$ but $N_1+Y\neq M$ for any proper submodule Y of X_1. Now if all the submodules of X_1 are small in M, then it is easy to show that X_1 is hollow as we did preceding lemma 2.2. We then would have M as the sum of two hollow submodules, neither of which can be deleted from the summation. If X_1 has a submodule which is not small in M, say N_2, we pick a supplement for N_2 in M, say X'_2. Then $N_2+X'_2=M$. Intersecting both sides of the above equation with X_1 and using the modular property of the submodule lattice of M, we see $N_2+X'_2\cap X_1=X_1$. Now it is possible to pick a supplement for N_2 in X_1, say X_2. We note, from the fact that $X'_2\cap X_1\neq X_1$ (otherwise N_2 is small) that X_2 is properly contained in X_1. We now have $M=N_1+N_2+X_2$ and if we delete any of the terms we have a proper submodule of M. We continue in this way obtaining $N_3, N_4,\ldots$ et cetera. We note that the process must eventually stop since we have a strictly decreasing sequence $X_1\supseteq X_2\supseteq\cdots$ and after a certain point, the submodules in this sequence must be small. Thus we obtain $M=N_1+\cdots+N_p$ where each N_i is hollow and we cannot delete any of them.

Now suppose $M=N'_1+\cdots+N'_q$ where each N'_i is hollow and none of the submodules in the summation may be deleted. Without loss of generality, we may assume $q>p$. Consider $N_2+\cdots+N_p$. This is a proper submodule of M by construction. We are going to show that for some i, $N'_i+N_2+\cdots+N_p=M$ and none of the terms in the sum can be deleted. First, if $N'_1+N_2+\cdots+N_p\neq M$, then $N'_1+N_2+\cdots+N_p=U+N_2+\cdots+N_p$ where U is a proper submodule of N_1 and thus is small in M. So $N'_2+\cdots+N'_q+U+N_2+\cdots+N_p=N'_1+\cdots+N'_q+N_2+\cdots N_p=M$. Thus $N'_2+\cdots+N'_q+N_2+\cdots+N_p=M$ since U is small. If $N'_2+N_2+\cdots+N_p\neq M$, we use the same process, this time adding $N'_3+\cdots+N'_q$ to get $N'_3+\cdots+N'_q+N_2+\cdots+N_p=M$. We continue in this way to find that if there is no $i\leq q-1$ with $N'_i+N_2+\cdots+N_p=M$, then $N'_q+N_2+\cdots+N_p=M$. In any case we now see there is i and $N'_i+N_2+\cdots+N_p=M$. Now we must consider the problem of deletion. It is obvious, from the previous construction, that if we delete N'_i, we no longer have M. Suppose we delete N_2. We then have $N'_i+N_3+\cdots+N_p$. If this equals M, then consider $N_2+\cdots+N_p=U+N_3+\cdots+N_p$ where U is a proper submodule of N'_i and thus is small. Then $N_1+U+N_3+\cdots+N_p=M$, so $N_1+N_3+\cdots+N_p=M$ since U is small. Thus we have successfully deleted N_2 from the first summation. This is a contradiction, so we cannot delete N_2. If we continue in this way, we find that we cannot delete any of the N_i's left in $N'_1+N_2+\cdots+N_p$.

Now after we have replaced N_1 by N_i' and changed neither the fact that the sum is M nor the fact that no term can be deleted, we replace N_2 by some N_j' and show the same two things. We continue this way, replacing all the possible N_i's. Then since $q<p$, we find that after replacing all of the N_i's that we have actually deleted some N_i''s. This is a contradiction, so $p=q$.

4. **Further observations.** From now on, we will term the integer determined in theorem 3.1 the spanning dimension of the module, M, and we will denote it by $\mathrm{Sd}(M)$. We would like to be able to relate $\mathrm{Sd}(M)$ to $\mathrm{Sd}(N)$ when N is a submodule of M. Unfortunately, we are unable to deal with an arbitrary submodule since an arbitrary submodule might not have finite spanning dimension. However, we can prove the following.

THEOREM 4.1. *Let M have finite spanning dimension and $K\subseteq M$ be a supplement. Then K has finite spanning dimension and if* $\mathrm{Sd}(K)=\mathrm{Sd}(M)$, $K=M$.

Proof. By a supplement, we mean that K is a supplement of some submodule L of M. Now if $X_1\supseteq X_2\supseteq\cdots$ is a sequence of submodules of K, then there is an i and X_j is small in M for $j\geq i$. If X_j is not small in K, there is $L_j(\neq K)$ and $X_j+L_j=K$. But then, $X_j+L_j+L=M$. Since X_j is small in M, $L_j+L=M$ and this contradicts the fact that K is a supplement of L.

Now suppose $\mathrm{Sd}(K)=\mathrm{Sd}(M)$. If $K\neq M$, pick L such that $K+L=M$ and K is a supplement for L. Then L contains L_1 which is a supplement for K. Clearly, then, K is also a supplement for L_1. Using the decomposition of theorem 3.1 on both K and L, we see that if $\mathrm{Sd}(L_1)>0$ we would have $\mathrm{Sd}(K+L_1)>\mathrm{Sd}(M)$ when, in fact, $\mathrm{Sd}(K+L_1)=\mathrm{Sd}(M)$. Thus $L_1=0$, so $K=M$.

THEOREM 4.2. *Let M have finite spanning dimension and let $K\subseteq M$ be a supplement. Then M/K has finite spanning dimension and* $\mathrm{Sd}(M/K)=\mathrm{Sd}(M)-\mathrm{Sd}(K)$.

Proof. Actually, we show that M/K is Artinian. Since every Artinian module has finite spanning dimension, the first part of the result will then follow. So suppose $X_1\supseteq X_2\supseteq\cdots$ is a strictly decreasing sequence of submodules of M/K. If we let f denote the natural map from M to M/K, then $f^{-1}(X_1)\supseteq f^{-1}(X_2)\supseteq\cdots$ is a strictly decreasing sequence of submodules containing K. Since K is not small, no $f^{-1}(X_i)$ can be small in M. Thus the sequence of inverse images must terminate and this implies that the original sequence had to terminate.

Now if K is a supplement, it is a supplement for some submodule L of M. Now L must contain L_1 which is a supplement for K so $K+L_1=M$. Once again, as in the previous proof, K is a supplement for L_1. Thus $\mathrm{Sd}(K)+\mathrm{Sd}(L_1)=\mathrm{Sd}(M)$. We will show $\mathrm{Sd}(L_1)=\mathrm{Sd}(M/K)$. By theorem 3.1, $L_1=N_1+\cdots+N_t$ where each N_i is hollow and no N_i may be deleted from the summation. Once again we denote the natural map from M to M/K by f. In that case, we see $M/K=f(M)=f(K+L_1)=f(K)+f(L_1)=f(L_1)=f(N_1)+\cdots+f(N_t)$. Since it is well known that the

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image of a small submodule of a module is again small, $f(N_i)$ is hollow for each i. Furthermore, no $f(N_i)$ can be deleted from the sum for deletion in M/K would imply the possibility of deletion in M. Thus $t=\mathrm{Sd}(M/K)=\mathrm{Sd}(M)-\mathrm{Sd}(K)$.

5. **The second decomposition.** There is a fault with the decomposition of theorem 3.1; it is not direct and, usually, it will not be direct. We would thus be interested in conditions under which some aspects of directness would be assured. There is, for example, the following theorem. In it and in all the following we use the word semi-simple to mean that the radical of a module, the intersection of its maximal proper submodules, is zero.

THEOREM 5.1. *If a module has finite spanning dimension and is semi-simple, then it is a finite direct sum of simple modules.*

Proof. Let $M=N_1+\cdots+N_t$ where each N_i is hollow. Each N_i is also simple because any submodule of N_i would have to be small and thus would be contained in the radical which is zero. Because no N_i can be deleted, it is easy to see that the sum is direct.

COROLLARY 5.2. *A module is semi-simple and Artinian if and only if it is a semi-simple with finite spanning dimension.*

We have now treated the case of the module being semi-simple but there are still theorems which will help when semi-simplicity is not assumed. First, we return to theorem 4.1. From this theorem, we can conclude that the maximum number of elements in a strictly increasing sequence of complements must be $\mathrm{Sd}(M)$, because the strict containment of K_1 in K_2 implies the dimension of K_1 is strictly less than the dimension of K_2. We have now proved the following.

PROPOSITION 5.3. *If M has finite spanning dimension, then M has the ascending chain condition on supplements.*

PROPOSITION 5.4. *If M has finite spanning dimension, then M has a maximal semi-simple supplement.*

Proof. Since the zero submodule is a semi-simple supplement, the set of all such is non-empty. Now, either we can use Zorn's lemma, or we can note that any strictly ascending chain of semi-simple supplements must end. Thus such a maximal supplement must exist.

DEFINITION. We shall say that a module is s^3-free if it contains no non-zero semi-simple supplements.

PROPOSITION 5.5. *If M is s^3-free, then* $\mathrm{Soc}(M)\subseteq\mathrm{Rad}(M)$.

Proof. Let A be a simple submodule of M. Let $A\nsubseteq\mathrm{Rad}(M)$, then there is a maximal submodule X and $A\nsubseteq X$. Thus $A+X=M$. Since A has no proper

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submodules, it must be the supplement of X. The simplicity of A guarantees semi-simplicity. Thus if $\mathrm{Soc}(M)\nsubseteq \mathrm{Rad}(M)$, M is not s^3-free.

The converse of the above theorem is true provided we assume that M has finite spanning dimension. We mention this for the sake of completeness since we don't really need it. We make two remarks which are easily proved and which we leave to the reader. First, it is easy to show that if $K+L=M$, then K is a supplement of L if and only if $K \cap L$ is small in K. Next, it can be shown that the radical of M is the sum of all the small submodules of M. Thus, if $a \in \mathrm{Rad}(M)$, Ra, the submodule generated by a, is in a finite sum of small submodules of M and thus it is small itself.

THEOREM 5.6. *Let M have finite spanning dimension. Then M is the direct sum of a maximal semi-simple supplement and an s^3-free submodule. Furthermore, if $M=K_1\oplus\cdots\oplus K_n\oplus P_1=L_1\oplus\cdots\oplus L_t\oplus P_2$ where $K_1\oplus\cdots\oplus K_n$ and $L_1\oplus\cdots\oplus L_t$ are both maximal semi-simple supplements and P_1, P_2 are s^3-free, then $n=t$.*

Proof. Suppose K is a maximal semi-simple supplement. Then it is a supplement for $L\subseteq M$. Now $L\supseteq P_1$ which is a supplement for K. Then $M=K+P_1$ and it is easy to see K is a supplement for P_1. Thus, by a preceding remark, $K \cap P_1$ is small in K. But K is semi-simple, so $P_1 \cap K=\{0\}$ making $M=P_1+K$ direct. Since K is semi-simple with finite spanning dimension, by theorem 5.1, $K=K_1\oplus\cdots\oplus K_n$ where each K_i is simple.

The submodule P_1 is then s^3-free, since, if it contains a semi-simple supplement, we would be able to find a larger submodule of M which is a semi-simple supplement and properly contains K.

Now suppose $M=K_1\oplus\cdots\oplus K_n\oplus P_1=L_1\oplus\cdots\oplus L\ \oplus P_2$ where the L's and K's are simple. First we note that $P_1=\bigcap_{i=1}^n K_1\oplus\cdots\oplus \hat{K}_i\oplus\cdots\oplus K_n\oplus P_1$ and each term in the intersection is maximal. Thus the radical of M is contained in P_1. Similarly, it is contained in P_2.

Now suppose $a \in P_1$ but $a \notin \mathrm{Rad}(M)$. Then Ra is not small in P_1, but if Ra were semi-simple, then we would be able to enlarge K and that is not possible. Thus, there is $r \in R$ and $o\neq ra \in \mathrm{Rad}(M)$.

Now we return to consideration of $K_1\oplus\cdots\oplus K_n$. We shall try to alter this sum to get it contained in $L_1\oplus\cdots\oplus L_t$ but we shall not change its supplementary property. Then we shall have proved that $n\leq t$ because of the dimensions of the two modules.

It is possible that some of the K's, say p of them, are already contained in $L_1\oplus\cdots\oplus L_t$. Then renumber them so that they become the first p of the K's. Thus K_{p+1} is the first summand not contained in $L_1\oplus\cdots\oplus L_t$. Now if $k \in K_{p+1}$, then $k=l_1+\cdots+l_t+m$ where $m\neq 0$ and $m \in P_2$. Since K_{p+1} is simple, we see that $Rk=K_{p+1}$. In fact, if $r \in R$ and $rk\neq 0$, then $Rrk=K_{p+1}$. We can also note that there is no r with $rm=0$ and $rk\neq 0$. If that were true, then $Rrk\subseteq L_1+\cdots+L_t$ and this is not so. Thus $\mathrm{ann}(k)\supseteq \mathrm{ann}(m)$. Since $rk=0$ but $rm\neq 0$ implies a non-trivial dependence relation among m and the l's, we also have $\mathrm{ann}(k)\subseteq \mathrm{ann}(m)$.

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Now we might as well assume $m \in \mathrm{Rad}(M)$. If not, there is $r \in R$ and $0 \neq rm \in \mathrm{Rad}(M)$. Thus $rk \neq 0$ and since $Rrk = K_{p+1}$, we find $k \in Rrk = Rrl_1 + \cdots + Rrl_t + Rrm$ and the last term in the sum is in the radical of M.

Now consider $k - m = l_1 + \cdots + l_t$. Since $\mathrm{ann}(k) = \mathrm{ann}(m)$ is maximal, we find that $\mathrm{ann}(k-m)$ is maximal. Thus $R(k-m)$ is simple. Now consider $K_1 + \cdots + K_p + R(k-m) + K_{p+2} + \cdots + K_n$. It is easily seen that this sum is direct. If we look at $(K_1 \oplus \cdots \oplus K_p \oplus R(k-m) \oplus K_{p+2} \oplus \cdots \oplus K_n) + Rm + P_1$ we find that we may delete Rm since, by a remark previous to the proof, Rm is small. It is now easy to see that $K_1 \oplus \cdots \oplus K_p \oplus R(k-m) \oplus K_{p+2} \oplus \cdots \oplus K_n$ is a supplement of P_1 and that the sum with P_1 is direct. Furthermore, the $p+1$'st term in the summation is now contained in $L_1 \oplus \cdots \oplus L_t$. Continuing in this way, we can change all of the K's until each is contained in the sum of the L's. Since even the altered K's form a supplement, we must have $n \leq t$. Similarly we show $t \leq n$. Thus $t = n$.

COROLLARY 5.7. *If M is an Artinian module over R, there is an integer n and $M = K_1 \oplus \cdots \oplus K_n \oplus N$ where* (i) *each K_i, $i = 1, \ldots, n$, is simple,* (ii) *N is s^3-free,* (iii) *$K_1 + \cdots + K_n$ is a maximal semi-simple direct summand, and* (iv) *any other semi-simple direct summand of M has at most n terms.*

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