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On Hyperbolicity of Domains with Strictly Pseudoconvex Ends

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Abstract. This article establishes a sufficient condition for Kobayashi hyperbolicity of unbounded domains in terms of curvature. Specifically, when $\Omega \subset \mathbb{C}^n$ corresponds to a sub-level set of a smooth, real-valued function Ψ, such that the form $\omega = \mathbf{i}\partial\bar{\partial}\Psi$ is Kähler and has bounded curvature outside a bounded subset, then this domain admits a hermitian metric of strictly negative holomorphic sectional curvature.

1 Introduction

It is well known that any domain in $\mathbb{C}^n$ biholomorphically equivalent to a bounded domain is Kobayashi-hyperbolic. The main result of this note, proven in Section 2, provides a sufficient condition for hyperbolicity of unbounded domains in terms of curvature. In general, a complex space is Kobayashi-hyperbolic if it can be shown to possess a hermitian metric, the holomorphic sectional curvature of which is bounded by a negative constant (*cf.* [3]). With this in mind, we assume that

$$\Omega = \left\{ Z \in \mathbb{C}^n \mid \Psi(Z) < 1 \right\}$$

for a smooth function $\Psi \colon \Omega' \to [0, +\infty)$ ($\overline{\Omega} \subset \Omega'$) that is strongly plurisubharmonic outside a bounded subset of Ω.

Theorem 1.1 *Suppose there exists a bounded subset of Ω outside which the real form $\omega := \mathbf{i}\partial\bar{\partial}\Psi$ has bounded curvature. Then Ω is Kobayashi-hyperbolic.*

The proof of this result is not significantly altered if the role of $\mathbb{C}^n$ (*i.e.,* as the ambient space containing Ω) is taken by an arbitrary Stein manifold. Section 3 provides an example of a weakly pseudoconvex unbounded domain satisfying the above hypotheses, which we introduce as follows. Note first that the orthogonal group acts holomorphically on vectors $Z = X + \mathbf{i}Y \in \mathbb{C}^n$ according to the natural rule

$$\sigma(Z) = \sigma(X) + \mathbf{i}\sigma(Y) \text{ for all } \sigma \in \mathbb{O}(n).$$

Beginning with a projection map

$$\pi \colon \mathbb{C}^n \to [0, +\infty) \times [0, +\infty)$$

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such that $\pi(Z) = (|\Re(Z)|, |\Im(Z)|) = (|X|, |Y|)$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, we compose with any smooth function $\psi\colon [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ to define domains of the form

$$\Omega = \{Z \in \mathbb{C}^n \mid \Psi(Z) < c\},$$

where $\Psi = \psi \circ \pi$, for some $c > 0$. These domains clearly possess orthogonal symmetry, though in general they are not symmetric with respect to the full unitary group. They are introduced in [2] as the building blocks of a theory of cellular decomposition of Stein manifolds, based on a famous theorem of Andreotti and Frankel [1]. Our calculation of holomorphic sectional curvature will be carried out specifically for $\psi(r,s) = r^2s^2$ and when $n = 2$. In this case Ψ is seen to be strongly plurisubharmonic (hence ω is Kähler) outside a pair of transversely intersecting discs, which are extremally embedded with respect to the Kobayashi metric and are bounded by the unique $\mathbb{O}(2)$-orbit of weakly pseudoconvex points on the boundary of Ω.

2 Hyperbolicity and Pseudoconvex Ends

Let $\Psi\colon \mathbb{C}^n \to [0,+\infty)$ be a smooth function and let $B \subset \mathbb{C}^n$ be a bounded subset. Consider a domain $\Omega \subseteq \mathbb{C}^n$ defined, without significant loss of generality, by the inequality $\Psi < 1$ and contained in a slightly larger domain Ω' corresponding to $\Psi < 1 + \varepsilon$ (in particular, $\overline{\Omega} \subset \Omega'$). If it is assumed that $\Psi\mid_{\Omega' \setminus B}$ is strongly plurisubharmonic, then the real closed form $\omega := \mathbf{i}\partial\bar{\partial}\Psi$ is Kähler on $\Omega' \setminus B$, and we may denote by g the associated metric. As always, the curvature tensor of g is defined by the formula

$$R_{i,\bar{j},k,\bar{l}} = -\frac{\partial^2 g_{i,\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{\alpha,\beta} g^{\alpha,\bar{\beta}} \frac{\partial g_{i,\bar{\beta}}}{\partial z_k} \frac{\partial g_{\alpha,\bar{j}}}{\partial \bar{z}_l}$$

away from B. In the following theorem, all closures are taken with respect to Ω'.

Theorem 2.1 *Suppose $\omega\mid_{\overline{\Omega}\setminus B}$ has bounded curvature. Then Ω is Kobayashi-hyperbolic.*

Proof We begin by defining a metric h_K associated with the form $e^{K\Psi}\partial\bar{\partial}\Psi$ on $\Omega \setminus B$. Given a holomorphic map $f(\zeta)$ from the unit disc D into Ω, such that $f(0) = Z \in \Omega \setminus B$, we assume the presence of Kähler normal coordinates $W = (w_1, \dots, w_n)$ in a neighbourhood of Z ($W = 0$) in which

$$g_{i,\bar{j}}(0) = \delta_{i,j} \quad \text{and} \quad \frac{(\partial g_{i,\bar{j}})}{(\partial w_k)}(0) = 0, \quad 1 \le k \le n.$$

Now consider

$$f^* h_K = \left(e^{K\Psi\circ f} \sum_{i,j} \frac{\partial^2 \Psi}{\partial w_i \partial \bar{w}_j} f_i' \bar{f}_j' \right) d\zeta \wedge d\bar{\zeta} = \mu_f d\zeta \wedge d\bar{\zeta}$$

(for K an as yet unspecified constant) on $T^{1,0}_\zeta D \otimes T^{0,1}_\zeta D$. Now

$$\log(\mu_f) = K\Psi \circ f + \log\Big(\sum_{i,j} g_{i,\bar{j}} f_i' \bar{f}_j'\Big)$$

implies

$$\frac{\partial \log(\mu_f)}{\partial \bar{\zeta}} = K\frac{\partial \Psi \circ f}{\partial \bar{\zeta}} + \frac{1}{\sum_{i,j} g_{i,\bar{j}} f_i' \bar{f}_j'} \sum_{l=1}^{n} \Big(\frac{\partial g_{i,\bar{j}}}{\partial \bar{w}_l} \bar{f}_l' f_i' \bar{f}_j' + \delta_{j,l} g_{i,\bar{j}} f_i' \bar{f}_j''\Big),$$

and hence

$$\begin{aligned}
&\frac{\partial^2 \log(\mu_f)}{\partial \zeta \partial \bar{\zeta}}\\
&= K\frac{\partial^2 \Psi \circ f}{\partial \zeta \partial \bar{\zeta}}\\
&\quad - \frac{1}{(\sum_{i,j} g_{i,\bar{j}} f_i' \bar{f}_j')^2} \sum_{k,l} \Big(\frac{\partial g_{i,\bar{j}}}{\partial w_k} f_k' f_i' \bar{f}_j' + \delta_{i,k} g_{i,\bar{j}} f_i'' \bar{f}_j'\Big)\Big(\frac{\partial g_{i,\bar{j}}}{\partial \bar{w}_l} \bar{f}_l' f_i' \bar{f}_j' + \delta_{j,l} g_{i,\bar{j}} f_i' \bar{f}_j''\Big)\\
&\quad + \frac{1}{\sum_{i,j} g_{i,\bar{j}} f_i' \bar{f}_j'} \sum_{i,j,k,l} \Big(\frac{\partial^2 g_{i,\bar{j}}}{\partial w_k \partial \bar{w}_l} f_k' \bar{f}_l' f_i' \bar{f}_j' + g_{i,\bar{j}} \delta_{j,l} \delta_{i,k} f_i'' \bar{f}_j''\Big)\\
&= K|f'|^2 - \frac{1}{|f'|^4} \sum_{k,l} f_k'' \bar{f}_k' f_l' \bar{f}_l'' + \frac{1}{|f'|^2} \sum_{i,j,k,l} \Big(\frac{\partial^2 g_{i,\bar{j}}}{\partial w_k \partial \bar{w}_l} f_k' \bar{f}_l' f_i' \bar{f}_j' + |f''|^2\Big)\\
&= K|f'|^2 - \frac{|\langle f'', f'\rangle|^2}{|f'|^4} + \frac{|f''|^2}{|f'|^2} - \frac{1}{|f'|^2} \sum_{i,j,k,l} R_{i,\bar{j},k,\bar{l}} f_k' \bar{f}_l' f_i' \bar{f}_j'\\
&\geq K|f'|^2 - \frac{1}{|f'|^2} \sum_{i,j,k,l} R_{i,\bar{j},k,\bar{l}} f_k' \bar{f}_l' f_i' \bar{f}_j',
\end{aligned}$$

where $R_{i,\bar{j},k,\bar{l}}$ as above denotes the curvature tensor of ω. Since $R_{i,\bar{j},k,\bar{l}}$ is uniformly bounded on $\Omega \setminus B$, we may choose the constant $K \gg \sup_{\Omega\setminus B} \|R\|$. To complete the estimation of holomorphic sectional curvature at $Z \in \Omega \setminus B$, it remains to note $\mu_f = e^{K\Psi\circ f}|f'|^2$, and therefore

$$-\frac{1}{\mu_f}\frac{\partial^2 \log(\mu_f)}{\partial \zeta \partial \bar{\zeta}} \leq -\big(K - \sup_{\Omega\setminus B} \|R\|\big)\, e^{-K},$$

given $\Psi(Z) < 1$. Without loss of generality, let B correspond to

$$\overline{\Omega} \cap B_M(0) = \{Z \in \overline{\Omega} \mid |Z| < M\}.$$

Consider also the slightly larger ball $B_{M+\varepsilon}(0)$. We now introduce a C^∞ cut-off function χ, such that

$$\chi(|Z|) = \begin{cases} 1 & \text{if } Z \in \overline{B_M(0)},\\ 0 & \text{if } Z \in \mathbb{C}^n \setminus B_{M+\varepsilon}(0).\end{cases}$$

Letting ω_0 again denote the standard Kähler metric form on $\mathbb{C}^n$, we then define a hermitian metric h on Ω associated with the form

$$e^{K'\Psi+\chi\cdot|Z|^2}\big((1-\chi)\omega+\chi\cdot\omega_0\big).$$

(In particular, the adjusted constant $K' \geq K$ will be defined below.) The upper bound on holomorphic sectional curvature remains essentially the same for $Z \in \Omega\setminus B_{M+\varepsilon}(0)$, where the metric is identified with $h_{K'}$ above. Two further regions of Ω must now be examined. First, note that if $Z \in \overline{B_M(0)}\cap\Omega$, we may consider h to be associated with the form $e^{K'\Psi+|Z|^2}\omega_0$. Hence to f^*h we associate the function

$$\log(\mu_f) = K'\Psi\circ f + |f|^2 + \log\big(|f'|^2\big)$$

and obtain

$$\begin{aligned}-\frac{1}{\mu_f}\frac{\partial^2\log(\mu_f)}{\partial\zeta\partial\bar\zeta} &\leq -\frac{e^{-(K'\Psi\circ f+|f|^2)}}{|f'|^2}(K'g_{i,\bar j}+\delta_{i,j})f'_i\bar f'_j\\ &\leq -e^{-(K'+M^2)},\end{aligned}$$

if we recall that $g_{i,\bar j}f'_i\bar f'_j \geq 0$. It remains now to estimate the holomorphic sectional curvature in the region

$$A_\varepsilon = \big\{Z\in\Omega \mid M<|Z|<M+\varepsilon\big\}.$$

For $Z\in A_\varepsilon$ we will simply write

$$\mu_f = e^{K'\Psi\circ f}\,\Sigma_{i,j}(G\circ f)_{i,\bar j}f'_i\bar f'_j,$$

where $G_{i,\bar j} = ((1-\chi)g_{i,\bar j}+\chi\delta_{i,j})$ is understood to be smooth, positive definite, and bounded, with bounded derivatives in the region A_ε, which is relatively compact in Ω'. The calculation of holomorphic sectional curvature is then formally carried out as in the case of h_K, producing a curvature tensor R' that is bounded on $\Omega\setminus B_M(0)$. The leading term of this calculation corresponds to

$$-\frac{K'}{\mu_f}\frac{\partial^2\Psi}{\partial\bar z_j\partial z_i}f'_i\bar f'_j = -K'e^{-(K'\Psi\circ f)}\frac{g_{i,\bar j}f'_i\bar f'_j}{(G\circ f)_{i,\bar j}f'_i\bar f'_j} \leq -K'e^{-K'}\inf_{A_\varepsilon}\frac{g_{i,\bar j}f'_i\bar f'_j}{(G\circ f)_{i,\bar j}f'_i\bar f'_j}.$$

To see that the infimum above is strictly positive, we write

$$\frac{g_{i,\bar j}f'_i\bar f'_j}{(G\circ f)_{i,\bar j}f'_i\bar f'_j} = \frac{e^{-\chi\cdot|Z|^2}g_{i,\bar j}f'_i\bar f'_j}{((1-\chi)g_{i,\bar j}+\chi\delta_{i,j})f'_i\bar f'_j} = \frac{e^{-\chi\cdot|Z|^2}g(\frac{f'}{|f'|},\frac{f'}{|f'|})}{(1-\chi)g(\frac{f'}{|f'|},\frac{f'}{|f'|})+\chi},$$

noting that $g(\mathbf{v},\mathbf{v})$ is positive definite for all $Z\in\overline{A_\varepsilon}$, and all $\mathbf{v}\in\mathbb{S}^{2n-1}$. It remains now to choose

$$K'\cdot\inf_{A_\varepsilon}\frac{g_{i,\bar j}f'_i\bar f'_j}{(G\circ f)_{i,\bar j}f'_i\bar f'_j} \gg \sup_{\Omega\setminus B_M(0)}\|R'\|,$$

as in the construction of h_K, so that the holomorphic sectional curvature of h is uniformly bounded above by a strictly negative constant on Ω. It follows at once that Ω is Kobayashi-hyperbolic. ■

3 An Example: $\Psi(Z) = |\Re(Z)|^2|\Im(Z)|^2$

Let $Z = (z_1, z_2, \dots, z_n)$ be coordinates in $\mathbb{C}^n$ $(n \geq 2)$, $z_j = x_j + \mathbf{i}y_j$, and let $\Re(Z) = X = (x_1, x_2, \dots, x_n)$, $\Im(Z) = Y = (y_1, y_2, \dots, y_n)$. Then $\pi\colon \mathbb{C}^n \to \mathbb{R}^2$ will denote the natural projection

$$\pi(Z) = \big(|X|, |Y|\big),$$

where $|\cdot|$ denotes the Euclidean norm. Given $\psi\colon [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ smooth, we define special complex domains with $\mathbb{O}(n)$-symmetry, of the form

$$\Omega = \{Z \in \mathbb{C}^n \mid \Psi(Z) < c\}; \quad \Psi = \psi \circ \pi.$$

A sufficient condition for local plurisubharmonicity of a general function $\Psi = \psi \circ \pi$ is examined in [2]. If it is assumed that the critical locus of ψ is contained in $\psi^{-1}(0)$, then $d\psi \neq \mathbf{0}$ at all points (r, s) outside that locus, and the level set $\psi(r, s) = c$ passing through a given regular point locally admits an implicit function $s = \phi(r)$. Let $r = |\Re(Z)|$, $s = |\Im(Z)|$, and $Z \in \Psi^{-1}(c)$, $0 < c \leq 1$. The following statement is given in [2, Lemma 3.3.1].

Lemma ([2, Lemma 3.3.1]) *If, in a neighbourhood of* $(r_0, s_0) = (|\Re(Z)|, |\Im(Z)|)$, $Z \in \Psi^{-1}(c)$, *we have*

(a) $\phi'(r) > 0$, $\phi''(r) \leq 0$, $\phi'(r) > \frac{r}{\phi(r)}$, *and*

$$\phi''(r) + \frac{(\phi'(r))^3}{r} - \frac{1}{\phi(r)}(1 + (\phi'(r))^2) \geq 0,$$

then the hypersurface is pseudoconvex at Z, co-oriented from above;

(b) $\phi'(r) \leq 0$, $\phi''(r) \geq 0$, *and*

$$\phi''(r) + \frac{(\phi'(r))^3}{r} - \frac{1}{\phi(r)} \leq 0,$$

then the hypersurface is pseudoconvex at Z, co-oriented from below.

In the following, we examine the case $\psi(r, s) = r^2 s^2$, for which the pseudoconvexity condition above is easily checked to hold. More directly, we can perform some routine calculations based on the formula

$$\Psi(Z) = |\Re(Z)|^2|\Im(Z)|^2 = \frac{1}{16}\left(4|Z|^4 - \Big[\sum_{k=1}^{n} z_k^2 + \bar{z}_k^2\Big]^2\right).$$

In particular,

$$\begin{aligned}\frac{\partial^2\Psi}{\partial\bar{z}_j\partial z_i} &= \frac{1}{16}\frac{\partial}{\partial\bar{z}_j}\Big(8|Z|^2\cdot\bar{z}_i - 4\sum_k (z_k^2 + \bar{z}_k^2)\cdot z_i\Big) \\ &= \frac{1}{2}\big(|Z|^2\delta_{i,j} + z_j\bar{z}_i - z_i\bar{z}_j\big).\end{aligned}$$

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Given any tangent vector $\mathbf{v} \in T_Z\mathbb{C}^n$, note that

$$\frac{\partial^2\Psi}{\partial\bar{z}_j\partial z_i}(\mathbf{v},\bar{\mathbf{v}}) = \sum_{i,j}\frac{\partial^2\Psi}{\partial\bar{z}_j\partial z_i}\bar{v}_j\cdot v_i = \frac{1}{2}\sum_{i,j}\big(|Z|^2\delta_{i,j}+z_j\bar{z}_i-z_i\bar{z}_j\big)\,\bar{v}_j\cdot v_i$$
$$= \frac{1}{2}\big(|Z|^2|\mathbf{v}|^2+|\langle Z,\mathbf{v}\rangle|^2-|\langle\bar{Z},\mathbf{v}\rangle|^2\big)$$
$$\geq \frac{1}{2}|\langle Z,\mathbf{v}\rangle|^2 \geq 0,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard hermitian inner product on $\mathbb{C}^n$. Hence Ψ is plurisubharmonic. Note also that

$$(\nabla^{1,0}\Psi)_i = (\overline{\nabla^{0,1}\Psi})_i = \frac{1}{2}|Z|^2\cdot\bar{z}_i - \frac{1}{4}\sum_k(z_k^2+\bar{z}_k^2)\cdot z_i = 0$$

if and only if $\Psi(Z)=0$, $1\leq i\leq n$, hence the critical locus of this plurisubharmonic function coincides with $\mathbb{R}^n\cup\mathbf{i}\mathbb{R}^n$. Let $\Gamma\subset\Omega$ denote the set of points at which the form $\omega=\mathbf{i}\partial\bar{\partial}\Psi$ is degenerate (*i.e.,* ω restricted to $\Omega\setminus\Gamma$ is Kähler). Then ω induces the Levi form on restriction to the complex tangent space of the Ψ-level set through any point $Z\in\Omega$, hence weak pseudoconvexity of the level set at Z implies $Z\in\Gamma$. Let $g_{i,\bar{j}}$ denote the i, j-component of the associated Kähler metric corresponding to

$$\frac{\partial^2\Psi}{\partial z_i\partial\bar{z}_j} = \frac{1}{2}\big(|Z|^2\delta_{i,j}+z_j\bar{z}_i-z_i\bar{z}_j\big)\,.$$

In the remainder we will specialize to the case $n=2$, where Γ is defined simply by the vanishing of

$$\det(g) = \frac{1}{4}\big(|Z|^4-|z_1\bar{z}_2-\bar{z}_1z_2|^2\big) = \frac{1}{4}|z_1^2+z_2^2|^2,$$

hence $\det(g)=0$ if and only if $z_1=\pm\mathbf{i}z_2$. Now $g_{i,\bar{j}}=\frac{1}{2}(|Z|^2\delta_{i,j}+z_j\bar{z}_i-z_i\bar{z}_j)$, implies

$$\frac{\partial^2 g_{i,\bar{j}}}{\partial z_k\partial\bar{z}_l} = \frac{1}{2}(\delta_{k,l}\delta_{i,j}+\delta_{l,i}\delta_{k,j}-\delta_{k,i}\delta_{l,j}).$$

Moreover,

$$R_{i,\bar{j},k,\bar{l}} = -\frac{\partial^2 g_{i,\bar{j}}}{\partial z_k\partial\bar{z}_l}+\sum_{\alpha,\beta}g^{\alpha,\bar{\beta}}\frac{\partial g_{i,\bar{\beta}}}{\partial z_k}\frac{\partial g_{\alpha,\bar{j}}}{\partial\bar{z}_l}$$

is bounded on $\Omega\setminus\Gamma_\varepsilon$, where $\Gamma_\varepsilon=\{Z \mid |z_1^2+z_2^2|<\varepsilon\}$ for ε arbitrarily small and positive. A uniform bound can be determined reasonably explicitly in this case if we introduce a parameter $\xi=re^{\mathbf{i}\theta}$ such that $z_1=\xi z_2$, and hence

$$\det(g) = \frac{|z_2|^4}{4}\big((r^2+1)^2-4r^2\sin^2(\theta)\big) \geq \frac{|z_2|^4}{4}(r^2-1)^2.$$

Then

$$|g^{\alpha,\bar{\beta}}| = \frac{|z_2|^2}{2\det(g)}\left((r^2+1)^2\delta_{\alpha,\beta} + 4(1-\delta_{\alpha,\beta})r^2\sin^2(\theta)\right)^{\frac{1}{2}}$$
$$\leq \frac{2(r^2+1)}{|z_2|^2(r^2-1)^2}$$

(if we note that $(r^2+1)^2 \geq 4r^2$). Similarly,

$$\frac{\partial g_{i,\bar{\beta}}}{\partial z_k} = \frac{1}{2}(\bar{z}_k\delta_{i,\beta} + \bar{z}_i\delta_{k,\beta} - \bar{z}_\beta\delta_{i,k})$$

for which the substitution $z_k = z_2(\delta_{2,k} + (1-\delta_{2,k})\xi)$, etc., and the inequality

$$|\delta_{2,k} + (1-\delta_{2,k})\xi| \leq \max\{1, r\}$$

etc., yields

$$\Big|\frac{\partial g_{i,\bar{\beta}}}{\partial z_k}\Big|, \Big|\frac{\partial g_{\alpha,\bar{j}}}{\partial \bar{z}_l}\Big| \leq \frac{3}{2}\max\{1, r\}|z_2|.$$

Hence

$$\Big|\sum_{\alpha,\beta} g^{\alpha,\beta}\frac{\partial g_{i,\bar{\beta}}}{\partial z_k}\frac{\partial g_{\alpha,\bar{j}}}{\partial \bar{z}_l}\Big| \leq 18\frac{r^2+1}{(r^2-1)^2}\max\{1, r\}^2.$$

From the continuity in r of this last expression for all $r \geq 1+\varepsilon$, and the fact that its limit as $r \to \infty$ is finite, we conclude that $R_{i,\bar{j},k,\bar{l}}$ is uniformly bounded for $|1-|\xi|| \geq \varepsilon$.

In the light of this example we make a final remark. Let Ψ be any plurisubharmonic function on $\mathbb{C}^2$ satisfying the following:

- $\det(\frac{\partial^2\Psi}{\partial\bar{z}_j\partial z_i}) = |\langle Z, \bar{Z}\rangle|^s e^{f(Z,\bar{Z})}$, ($s \in \mathbb{R}\setminus\{0\}$, f smooth) on Ω;
- outside the locus corresponding to $\{\langle Z, \bar{Z}\rangle = 0\}$, the form $\omega = \mathbf{i}\partial\bar{\partial}\Psi$ has bounded curvature on the domain $\Omega = \{\Psi < 1\}$.

Then the pair of transversely intersecting discs corresponding to the locus above is extremally embedded in the sense that the Kobayashi distance between two points $(\pm\mathbf{i}\zeta, \zeta)$ and $(\pm\mathbf{i}\zeta', \zeta')$ in the same disc is equal to the Poincaré distance between ζ and ζ'. In other words, the $\mathbb{O}(2)$-orbit of weakly pseudoconvex points on $\partial\Omega$ bounds a pair of extremally embedded discs. Moreover, the Ricci form $\mathbf{i}\partial\bar{\partial}\log(\det(g))$ associated with the metric vanishes identically when f is constant.

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References

[1] A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*. Ann. of Math. **69**(1959), 713–717. http://dx.doi.org/10.2307/1970034

[2] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2. Internat. J. Math. **1**(1990), no. 1, 29–46. http://dx.doi.org/10.1142/S0129167X90000034

[3] S. Kobayashi, *Hyperbolic complex spaces.*Grundlehren der Mathematischen Wissenschaften, 318, Springer-Verlag, Berlin, 1998.

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