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SHARP BOUNDS ON THE DIAMETER OF A GRAPH

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ABSTRACT. Let $D_{n,m}$ be the diameter of a connected undirected graph on $n \geqslant 2$ vertices and $n - 1 \leqslant m \leqslant s(n)$ edges, where $s(n) = n(n-1)/2$. Then $D_{n,s(n)} = 1$, and for $m < s(n)$ it is shown that

$$2 \leqslant D_{n,m} \leqslant n - [(\sqrt{8(m-n)+17} - 1)/2].$$

The bounds on $D_{n,m}$ are sharp.

Introduction. Let $D_{n,m}$ be the diameter of a connected undirected graph on n vertices and m edges, where $n - 1 \leqslant m \leqslant s(n) = n(n-1)/2$. There is no known $0(m)$ algorithm for the determination of the diameter of a given graph [3], and even the specification of useful bounds on $D_{n,m}$ has so far seemed to be a difficult task. Klee and Larman [4] and Bollobás [1] have described the asymptotic behaviour of $D_{n,m}$ as $n \to \infty$, where $m = m(n)$ is regarded as a given function of n. Klee and Larman quote a result due to Korśunov, that for sufficiently large n and almost every graph $G_{n,\lambda n}$ on n vertices and λn edges ($\lambda \geqslant 2$ a small constant),

$$\frac{1}{2} \log_\lambda n < D_{n,\lambda n} < 10 \log_\lambda n.$$

All these results require lengthy and intricate proofs. More recently, Chung and Garey [2] have derived bounds on the diameter of the graph resulting from the addition/deletion of edges to/from a graph of known diameter.

In this paper a straightforward elementary argument is used to derive a sharp upper bound on $D_{n,m}$ in closed form. This result has been suggested by computer experiments:

(1) The testing of algorithms for the determination of diameter and "pseudo-diameter" of random graphs [5] made it clear that the diameter of "most" graphs was much more narrowly bounded than Korśunov's results indicated;

(2) exhaustive runs on all graphs on n vertices, $2 \leqslant n \leqslant 8$, led directly to conjectures [6] which in turn led directly to the results described here.

Upper bound on $D_{n,m}$. Since by definition of $D_{n,m}$ the graph is assumed to be connected, it follows that $m \geqslant n - 1$. Since $D_{n,s(n)} = 1$, we may assume that $m < s(n)$.

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We remark then that $D_{n,m} \geqslant 2$, and moreover that for every integer $m \in [n - 1, s(n) - 1]$ there exists a graph $G_{n,m}$ on n vertices and m edges whose diameter is exactly 2. Then the lower bound is sharp. We now prove

LEMMA A. *For* $n \geqslant 3$ *and* $j = 1, \ldots, n - 2$,

$$D_{n,s(n)-j} = 2.$$

PROOF. Suppose that $n - 2$ edges are deleted from a complete graph $G_{n,s(n)}$. Then $D_{n,s(n)-n+2} \geqslant 2$. But in $G_{n,s(n)}$ there is one path of length 1 and $n - 2$ disjoint paths of length 2 connecting every pair of vertices. Hence $D_{n,s(n)-n+2} \leqslant 2$ and the lemma follows.

THEOREM B. *For* $n \geqslant 2$ *and* $i = 0, \ldots, n - 2$,

(a) $D_{n,s(n-i)+i} \leqslant i + 1$;

(b) $D_{n,s(n-i)+i-j} \leqslant i + 2, j = 1, \ldots, n - i - 2$.

Every bound is sharp.

PROOF. Observe that the result is true for $n = 2$ and by Lemma A for $n > 2$ and $i = 0$. Observe further that the bound $i + 1$ for (a) is attained by the graph $G_{n,s(n-i)+i}$ consisting of a complete subgraph on $n - i$ vertices $\{v_{i+1}, \ldots, v_n\}$ together with the chain

$$v_1 \text{ —— } v_2 \text{ —— } \cdots \text{ —— } v_i \text{ —— } v_{i+1}$$

The bound $i + 2$ for (b) is attained by removing $1 \leqslant j \leqslant n - i - 2$ of the $n - i - 1$ edges incident at v_{i+1} (an application of Lemma A). The proof is by induction: we suppose that the result is true for n and show that therefore it holds for $n + 1$.

(a) Consider any connected graph $G_{n+1,\sigma(n,i)}$, where $\sigma(n,i) = s[(n + 1) - (i + 1)] + (i + 1)$ and $0 < i \leq n - 2$. Observe that $D_{n+1,\sigma(n,i)} < i + 4$, for otherwise removal of a single vertex and its j incident edges from the graph would yield $D_{n,s(n-i)+i-(j-1)} \geqslant i + 3$, in contradiction to the inductive hypothesis. Suppose then that $D_{n+1,\sigma(n,i)} = i + 3$. Then there exist vertices u, v such that $d(u, v) = i + 3$, and the vertices of the graph may be arranged into $i + 4$ levels including at least one shortest path from $u = x_0$ to $v = x_{i+3}$:

$$x_0 \text{ —— } x_1 \text{ —— } x_2 \text{ —— } \cdots \text{ —— } x_{i+2} \text{ —— } x_{i+3}$$

Suppose then that one vertex $w \neq x_k$, $k = 0, \ldots, i + 3$, is removed from the graph together with all edges incident at w. From the level structure it is clear that the number j of edges deleted satisfies $1 \leqslant j = n - i - 1$. Then the reduced graph $G_{n,s(n-i)+i-(j-1)}$ has diameter $i + 3$, in contradiction to the inductive hypothesis. Then it cannot be true that $D_{n+1,\sigma(n,i)} = i + 3$. This proves (a) for $i > 1$.

(b) Assuming that $D_{n+1,\sigma(n,i)-j} = i + 4$, for some $1 \leqslant j \leqslant n - i - 2$, we use the inductive hypothesis as in (a) to establish (b) by contradiction.

Table 1 presents an interpretation of Theorem B. The values of m are displayed in classes $c = 1, \ldots, n - 2$, corresponding to the upper bound $D_{n,m}^{\max}$ on the diameter $D_{n,m}$.

TABLE 1
No. of edges classified according to maximum diameter $D_{n,m}^{\max}$

Class C	Range of Edges m		$k = m - n + 2$		$D_{n,m}^{\max}$
1	$n - 1$	$n - 1$	1	1	$n - 1$
2	n	$n + 1$	2	3	$n - 2$
3	$n + 2$	$n + 4$	4	6	$n - 3$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
$n - 2$	$s(n) - (n - 2)$	$s(n) - 1$	$s(n - 2) + 1$	$s(n - 1)$	2
$n - 3$	$s(n)$	$s(n)$	$s(n - 1) + 1$	$s(n - 1) + 1$	1

We see from the table that given $G_{n,m}$, $m \leqslant s(n)$, we can determine $D_{n,m}^{\max} = n - c$ by determining c such that $s(c) < k \leqslant s(c + 1)$. This requires the solution of the quadratic equation $c^2 + c - 2k = 0$, yielding

$$c = \lceil (\sqrt{8k + 1} - 1)/2 \rceil$$

from which

$$D_{n,m}^{\max} = n - \lceil (\sqrt{8(m - n) + 17} - 1)/2 \rceil.$$

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