IDEMPOTENT-GENERATED REGULAR SEMIGROUPS

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Suppose S is a regular semigroup and E is its set of idempotents. If E is subsemigroup of S, then S has been called orthodox and studied recently by Hall [3], Meakin [6], and Yamada [8]. In this paper we assume that E is not (necessarily) a subsemigroup of S and consider the subsemigroup generated by E, denoted $\langle E \rangle$. If E^n denotes the set of all elements of S which can be written as the product of n (not necessarily distinct) idempotents of S, then $\langle E \rangle = \bigcup_{n=1}^{\infty} E^n$. We show that $\langle E \rangle$ is always a regular subsemigroup of S and investigate relationships between it and S. The case where $\langle E \rangle = S$ is of particular interest to us; such semigroups will be referred to as idempotent-generated regular semigroups.

Throughout S will denote a regular semigroup and E its set of idempotents. The set of inverses of an element x of S is denoted by $V(x)$. The regularity of S guarantees that each of its elements has at least one inverse. For other facts about regular semigroups see Clifford and Preston [2].

1. The regularity of $\langle E \rangle$

Our first lemma was motivated by a lemma of Howie and Lallement [4].

LEMMA 1.1. *Let S be a regular semigroup and let $n > 1$. An element x of S can be written as the product of n idempotents of S if, and only if, x has an inverse which can be written as the product of $n-1$ idempotents of S. In symbols,*

$$x \in E^n \Leftrightarrow V(x) \cap E^{n-1} \neq \varnothing .$$

PROOF. $\Rightarrow$: The case $n = 2$ is due to Howie and Lallement [4]. If $x = e_1 e_2$, then choose $y \in V(x)$ and let $f = e_2 y e_1$. One computes that $f \in V(x) \cap E$.

Inductively, suppose n is a positive integer > 1 and suppose it has been shown

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that each element of E^n has an inverse in E^{n-1}. Let $x \in E^{n+1}$, say $x = e_1 e_2 \cdots e_{n+1}$. Pick $g \in V(e_2 \cdots e_{n+1}) \cap E^{n-1}$ and $y \in V(x)$ and let $f = g(e_2 \cdots e_{n+1} y e_1) \in gE$. By computation, one sees that $f \in V(z) \cap E^n$; hence, the first implication is established.

$\Leftarrow$: Suppose $x \in S$ such that $V(x) \cap E^{n-1} \neq \varnothing$ for some positive integer $n > 1$. If $n = 2$, pick $e \in V(x) \cap E$ and note that $x = xex = (xe)(ex) \in E^2$. If $n > 2$, choose $w \in V(x) \cap E^{n-1}$ and use the first implication to choose $y \in V(w) \cap E^{n-2}$. Then

$$x = xwx = (xw)y(wx) \in E \cdot E^{n-2} \cdot E = E^n.$$

This concludes the proof of 1.1.

COROLLARY 1.2. *If S is a regular semigroup, then the subsemigroup $\langle E \rangle$ of S generated by the idempotents of S is also regular. In fact, if $x \in E^n$, then $V(x) \subseteq E^{n+1}$; so that every inverse of an element of $\langle E \rangle$ is also in $\langle E \rangle$.*

PROOF. The first assertion follows from the first implication of 1.1. To prove the second assertion, let $y \in V(x)$. Then $x \in V(y)$ and $x \in E^n$; hence by the second part of 1.1, $y \in E^{n+1}$.

A natural equivalence relation to define on any regular semigroup S is the following:

$x \mathscr{V} y \Leftrightarrow$ there exists a sequence $x_1, \cdots, x_n$ of elements of S such that $x_1 \in V(x)$, $x_i \in V(x_{i-1})$ for $i = 2, \cdots, n$ and $y \in V(x_n)$.

Note that $\mathscr{V}$ is simply the transitive closure of the 'inverse' relation on S $(x \sim y \Leftrightarrow x \in V(y))$. Note also that $\mathscr{V} \subseteq \mathscr{D}$ ($\mathscr{D}$ is the usual Green's relation on S) since all inverses of an element of S are $\mathscr{D}$-related to that element.

The next theorem gives a description of $\langle E \rangle$ in terms of the relation $\mathscr{V}$.

THEOREM 1.3. *Let S be regular with idempotents E. Let $x \in S$. Then*
(i) $x \in \langle E \rangle \Leftrightarrow$ there is an $e \in$ such that $x \mathscr{V} e$.
(ii) $x \in E^n \Rightarrow x \in (D_x \cap E)^{n+2}$.

Proof. (i) $\Rightarrow$: Let $x \in \langle E \rangle$, and let n be a positive integer such that $x \in E^n$. If $n = 1$, the conclusion holds. If $n > 1$, repeated application of the first part of 1.1 yields a sequence $x_1, x_2, \cdots, x_{n-1}$ of elements of S such that $x_1 \in V(x) \cap E^{n-1}$ and $x_i \in V(x_{i-1}) \cap E^{n-i}$ for $i = 2, \ldots, n-1$. Note that $x_{n-1} \in E$ and $x \mathscr{V} x_{n-1}$.

$\Leftarrow$: Suppose $x \in S$ and $x \mathscr{V} e$ for some $e \in E$. Choose a sequence $x_1, \cdots, x_n$ in S such that $x_1 \in V(x)$, $x_i \in V(x_{i-1})$ for $i = 2, \cdots, n$ and $e \in V(x_n)$. Note that $x = xx_1x = xx_1x_2 \cdots x_n e x_n \cdots x_2 x_1 x$. If n is odd, $x = (xx_1)(x_2x_3) \cdots (x_{n-1}x_n)e(x_n x_{n-1}) \cdots (x_3x_2)(x_1x) \in E^{n+2}$. If n is even, $x = (xx_1)(x_2x_3) \cdots (x_n e)(e x_n) \cdots (x_3 x_2)(x_1 x_2) \in E^{n+2}$.

This concludes the proof of (i). The proof of (ii) follows upon noting that all of the products $xx_1, xx_2x_3, \cdots$ are in the $\mathscr{D}$-class of x.

COROLLARY 1.4. *If S is an idempotent-generated regular semigroup and I is an ideal of S, then I is also an independent-generated regular semigroup.*

PROOF. This follows from 1.3(ii) and the fact that I is the union of all the $\mathscr{D}$-class of S which it meets.

2. Green's $\mathscr{D}$-relation on $\langle E\rangle$ and a characterization of idempotent-generated regular semigroups

It is well known that, in general, Green's relations are not well-behaved relative to subsemigroups; that is to say, if T is a subsemigroup S, then it it not necessarily the case that $K_T = K_S \cap (T \times T)$, where K_S denotes one of Green's relations on S and K_T denotes the corresponding relation on T.

In case T is regular, it follows from a lemma of Anderson, Hunter, and Koch [1] that the relations $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{H}$ do behave nicely in the above sense. However, it still may not be the case that $\mathscr{D}_T = \mathscr{D}_S \cap (T \times T)$; for example, let S be the bicyclic semigroup and T be its semilattice of idempotents.

In this section we shall describe the $\mathscr{D}$-relation in $\langle E\rangle$ when S is a regular semigroup and use this to characterize those regular semigroups which are idempotent generated.

LEMMA 2.1. *Let T be a regular subsemigroup of semigroup S. Then*

(i) $K_T = K_S \cap (T \times T)$ for $K = \mathscr{L}, \mathscr{R}$, or $\mathscr{H}$.

(ii) $x \mathscr{D}_T y \Leftrightarrow$ *there exists $z \in T$ such that $x \mathscr{R}_S z \mathscr{L}_S y$.*

The proof of (i) follows immediately from [1], whereas (ii) is a direct consequence of (i) together with the definition of $\mathscr{D}$. It will thus be necessary to denote the $\mathscr{D}$-relation on $\langle E\rangle$ by $\mathscr{D}_{\langle E\rangle}$, while the relations $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$ on $\langle E\rangle$, being restrictions of the corresponding relations on S, will not be subscripted.

In order to facilitate the statement of the next lemma we define the relation α on E as follows:

$e\alpha f \Leftrightarrow$ there is a sequence $g_1, \cdots, g_n$ in E such that $eK_i g_1, g_i K_{i+1} g_{i+1}$ for $i = 2, \cdots, n-1$ and $g_n K_{n+1} f$, where for each i, K_i is one of $\mathscr{L}$ or $\mathscr{R}$.

Note that α is simply the transitive closure of the symmetric reflexive relation

$$[\mathscr{L} \cap (E \times E)] \cup [\mathscr{R} \cap (E \times E)]$$

and is thus an equivalence. Note also that $\alpha \subseteq \mathscr{V}$ since if two idempotents are $\mathscr{L}$ or $\mathscr{R}$ related they are inverses of one another. Roughly speaking, two idempotents are α-related if it is possible to get from one to the other by going along the $\mathscr{R}$ and $\mathscr{V}$ classes of S, turning corners only when one is an H-class containing an idempotent.

LEMMA 2.2. *Let S be regular with idempotents E. Then $\alpha = \mathscr{D}_{\langle E\rangle} \cap (E \times E)$.*

PROOF. $\subset$: Since $\mathscr{R}_{\langle E\rangle} \cup \mathscr{L}_{\langle E\rangle} \subseteq \mathscr{D}_{\langle E\rangle}$ and since α is the transitive closure of $[\mathscr{L} \cap (E \times E)] \cup [\mathscr{R} \cap (E \times E)]$, the containment follows.

$\supset$: We say that a pair $(x, y) \in \mathscr{D}_{\langle E\rangle} \cap (E \times E)$ has *degree* n provided n is the smallest positive integer k for which there exists $z \in E_k$ such that $x\mathscr{L}z\mathscr{R}y$. Let $(e,f) \in \mathscr{D}_{\langle E\rangle} \cap (E \times E)$. Then clearly if the degree of (e,f) is 1, then $e\alpha f$. Inductively, suppose that n is a positive integer for which it has been shown that whenever a pair of idempotents (e', f') is of degree not exceeding n, then $e'\alpha f'$. Now if the degree of (e,f) is $n+1$, pick $z \in E^{n+1}$ such that $e\mathscr{L}z\mathscr{R}f$. By 1.1, pick $z' \in V(z) \cap E^n$, and let $e' = zz'$, $f' = z'z$. Note that $e', f' \in E$, $f\mathscr{R}z\mathscr{R}e'$ and $f'\mathscr{L}z\mathscr{L}e$; hence $f\mathscr{R}e'\mathscr{L}z'\mathscr{R}f'\mathscr{L}e$. By the induction hypothesis, since f' and e' have degree at most n, $e'\alpha f'$; hence $e\alpha f$. Since the pair (e,f) must have some degree, this completes the proof of the containment.

THEOREM 2.3. *Let S be a regular semigroup with idempotents E. Then* $\mathscr{D}_{\langle E\rangle} = \mathscr{V} \cap (\langle E\rangle \times \langle E\rangle)$.

PROOF. Suppose $(x, y) \in \mathscr{V} \cap (\langle E\rangle \times \langle E\rangle)$. Choose $x_1, x_2, \cdots, x_n$ in S so that $x_1 \in V(x)$, $x_i \in V(x_{i-1})$ for $i = 2, \cdots, n$, and $y \in V(x_n)$. Now since $x \in \langle E\rangle$, it follows from 1.2 that $x_1 \in \langle E\rangle$, and hence $x\mathscr{D}_{\langle E\rangle}x_1$. In the same way we find $x_1\mathscr{D}_{\langle E\rangle}x_2$, and hence $x\mathscr{D}_{\langle E\rangle}x_2$. Continuing in this manner we finally conclude that $x\mathscr{D}_{\langle E\rangle}y$.

Conversely suppose $x\mathscr{D}_{\langle E\rangle}y$. By 1.3(i), we can choose $e, f \in E$ such that $x\mathscr{V}e$ and $y\mathscr{V}f$. It follows from the first part of this proof that $x\mathscr{D}_{\langle E\rangle}e$ and $y\mathscr{D}_{\langle E\rangle}f$; hence $e\mathscr{D}_{\langle E\rangle}f$. Thus by 2.2, $e\alpha f$ and so $e\mathscr{V}f$. Hence $x\mathscr{V}y$ and the equality is established.

As a corollary we obtain the following characterization of idempotent-generated regular semigroups.

COROLLARY 2.4. *A regular semigroup S is idempotent-generated if, and only if,* $\mathscr{D} = \mathscr{V}$.

PROOF. $\Rightarrow$: This implication is immediate by setting $S = \langle E\rangle$ in 2.3.

$\Leftarrow$: Suppose $S \neq \langle E\rangle$. By 1.3(i), there is an element whose $\mathscr{V}$-class contains no idempotent. Since every $\mathscr{D}$-class contains at least one idempotent, it follows that $\mathscr{D} \neq \mathscr{V}$.

The next theorem describes when the $\mathscr{D}$-relation on $\langle E\rangle$ is well-behaved in the sense that $\mathscr{D}_{\langle E\rangle} = \mathscr{D}_S \cap (\langle E\rangle \times \langle E\rangle)$.

THEOREM 2.5. *For a regular semigroup S, the following are equivalent*:

(i) $\alpha = \mathscr{D}_S \cap (E \times E)$

(ii) $\mathscr{D}_{\langle E\rangle} = \mathscr{D}_S \cap (\langle E\rangle \times \langle E\rangle)$

(iii) *Each $\mathscr{H}$-class of S contains a product of idempotents S.*

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PROOF. (i) ⇒ (ii): By 2.1, $\mathscr{D}_{\langle E\rangle} \subseteq \mathscr{D}_S \cap (\langle E\rangle \times \langle E\rangle)$. If $(x, y) \in \mathscr{D}_S \cap (\langle E\rangle \times \langle E\rangle)$, choose idempotents e and f such that $e\mathscr{L}x$ and $y\mathscr{R}f$. Then $e\mathscr{D}_S f$ and so $(e,f) \in \mathscr{D}_S \cap E \times E$ and hence using (i) $e\alpha f$. But by 2.2 $\alpha = \mathscr{D}_{\langle E\rangle} \cup (E \times E)$ hence $e\mathscr{D}_{\langle E\rangle}f$. So $x\mathscr{L}e\mathscr{D}_{\langle E\rangle}f\mathscr{R}y$ from which we get that $x\mathscr{D}_{\langle E\rangle}y$.

(ii) ⇒ (iii): Let H be an $\mathscr{H}$-class of S. Then $H = R \cap L$ where R is an $\mathscr{R}$-class of S and L is an $\mathscr{L}$-class of S. Choose $e \in R \cap E$ and $f \in L \cap E$. So $e\mathscr{D}_S f$ and hence using (ii) $e\mathscr{D}_{\langle E\rangle}f$. Thus there is a $z \in \langle E\rangle$ such that $e\mathscr{R}z\mathscr{L}f$ and thus $z \in H \cap \langle E\rangle$.

(iii) ⇒ (i): We always have $\alpha \subseteq \mathscr{D}_S \cap (E \times E)$. If $(e,f) \in \mathscr{D}_S \cap (E \times E)$ then by (iii) there is a $z \in (R_e \cap L_f) \cap \langle E\rangle$. Hence $(e,f) \in \mathscr{D}_{\langle E\rangle} \cap (E \times E)$ and so by 2.2, $e\alpha f$. This concludes the proof.

3. The simple case

If S is a simple (no proper ideals) regular semigroup with idempotents E, then of course it need not be that $\langle E\rangle$ is simple; for example take S to be the bicycle semigroup.

In this section we shall first investigate the relationship between S and $\langle E\rangle$ when one of them is simple.

LEMMA 3.1. *Let S be a regular semigroup. Then*

(i) *S is simple if, and only if, for every $e, f \in E$, there is an $a \in S$ and an inverse a' of a such that $aa' = e$ and $a'af = a'a$.*

(ii) *S is bisimple (one $\mathscr{D}$-class) if, and only if, for each $e, f \in E$ there is an $a \in S$ an inverse a' of a such that $aa' = e$ and $a'a = f$.*

PROOF. These statements are proved for inverse semigroups in Clifford and Preston [2, II] and no essential use is made of the uniqueness of inverses.

THEOREM 3.2. *Let S be a regular semigroup. Then*

(i) *If $\langle E\rangle$ is simple, then S is simple*

(ii) *If S is simple and each $\mathscr{H}$-class of S meets $\langle E\rangle$, then $\langle E\rangle$ is simple.*

(iii) *$\langle E\rangle$ is bisimple if, and only if, S is bisimple and each $\mathscr{H}$-class of S meets $\langle E\rangle$.*

(iv) *$\langle E\rangle$ is completely simple if, and only if, S is completely simple.*

PROOF. (i) follows from 3.1(i).

(ii) let $e, f \in E$. By 3.1(i), choose $a \in S$ and $a' \in V(a)$ such that $aa' = e$ and $a'af = a'a$.

Now choose $x \in H_a \cap \langle E\rangle$, where H_a denotes the $\mathscr{H}$-class of a, and let x' denote the inverse of x which lies in $H_{a'}$. Then we have $xx' = aa' = e$ and $x'xf = a'af = a'a = x'x$. Since by 1.2, $x' \in \langle E\rangle$, we conclude from 3.1(i) that $\langle E\rangle$ is simple.

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(iii) If $\langle E\rangle$ is bisimple, then S is bisimple from 3.1 (ii). Now if H denotes an $\mathscr{H}$-class of S, let $e, f \in E$ such that $R_e \cap L_f = H$. Since $e\mathscr{D}_{\langle E\rangle}f$ such that $e\mathscr{R}z\mathscr{R}f$; hence $z \in H \cap \langle E\rangle$.

(iii) If $\langle E\rangle$ is completely simple, then $\langle E\rangle$ has a primitive idempotent; hence S has a primitive idempotent. By 3.2(i), S is simple. Hence S is completely simple. Conversely if S is completely simple, then S is simple and each $\mathscr{H}$-class of S meets $\langle E\rangle$; hence by 3.1(ii), $\langle E\rangle$ is simple. But also S has a primitive idempotent which must also then be a primitive idempotent of $\langle E\rangle$, so $\langle E\rangle$ is completely simple.

REMARK 3.3. The authors have so far been unable to find an example of a simple idempotent-generated regular semigroup which is not completely simple. The following theorem seems to indicate that possibly no such example exists.

THEOREM 3.4. *The only simple regular semigroup with identity which is idempotent-generated is the one element semigroup.*

PROOF. Let S denote such a semigroup and let e denote the identity of S. First we shall show that $E^n \cap L_e = \{e\} = E^n \cap R_e$, $n = 1,2,\cdots$: so suppose $f \in E \cap L_e$. Then $xf = e$ for some $x \in S$, so $e = xf = (xf)f = f$. Hence $E \cap L_e = \{e\}$. Similarly $E \cap R_e = \{e\}$. Inductively, suppose that n is a positive integer for which it has been shown that $E^n \cap L_e = \{e\} = E^n \cap R_e$. Let $x \in$ $x \in E^{n+1} \cap L_e$. By 1.1, there exist $y \in V(x) \cap E^n$. Moreover, since $E \cap L_x = E \cap L_e = \{e\}$, it must be that $y \in R_e$. But by the induction hypothesis, $R_e \cap E^n = \{e,\}$ hence $y = e$. Hence $e = y = yxy = exe = x$, and we conclude that $E_{n+1} \cap L_e = \{e\}$. Similarly $E^{n+1} \cap R_e = \{e\}$. Since $S = \langle E\rangle$, we conclude that $L_e = R_e = \{e\}$.

Now suppose $f \in E$. Since S is simple, there exists, by 3.1(i), an element $a \in S$ and an inverse a' of a such that $aa' = e$ and $a'af = a'a$. But $aa' = e$ implies that $a \in R_e = \{e\}$. Hence $a = e = a'$ and so $e = a'a = a'af = eef = f$. Thus $E = \{e\}$ and the proof is complete.

Suppose S is a completely simple semigroup and E is the set of idempotents of S. One may ask about the structure of $\langle E\rangle$ beyond the fact that $\langle E\rangle$ is completely simple. The next theorem shows among other things, that $\langle E\rangle$ is determined to a large extent by the number of $\mathscr{L}$ and $\mathscr{R}$ classes of S.

THEOREM 3.5. *Let α* and *β be non-zero cardinal numbers and let X and Y be sets of cardinal α and β respectively. Let $\mathfrak{F}_{XxY}$ denote the free semigroup on XxY and let ρ be the congruence on $\mathfrak{F}_{XxY}$ generated by the relations $(x,y)(x,z) = (x,z)$ and $(x,y)(z,y) = (x,y)$. Denote the quotient semigroup obtained by $T(\alpha,\beta)$. Then $T(\alpha,\beta)$ is an idempotent-generated completely simple semigroup having β $\mathscr{L}$-classes and α $\mathscr{R}$-classes. Further if S is an idempotent-generated completely simple semigroup having β $\mathscr{L}$-classes and α $\mathscr{R}$-classes then S is a homomorphic image of $T(\alpha,\beta)$.*

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PROOF. For $(x, y) \in XxY$, let $[x, y]$ be the ρ-class of (x, y). Suppose $[x, y] = [x', y']$, and fix $(x_0, y_0) \in X \times Y$. Consider the mapping $\phi: X \times Y \to x_0 \times Y$ defined by $\phi(x, y) = (x_0, y)$. This induces a homomorphism $\bar{\phi}$ from $T(\alpha, \gamma)$ onto $x_0 \times Y$ with right trivial multiplication. Since $(x_0, y) = \bar{\phi}[x, y] = \bar{\phi}[x', y'] = (x_0, y')$ we conclude that $y = y'$. Dually $x = x'$. Also note that $[x, y]$ is idempotent. Thus $T(\alpha, \beta)$ is idempotent-generated.

$T(\alpha, \beta)$ is a union of groups: If $z = \prod_{i=1}^{n}[x_1, y_i] \in T(\alpha, \beta)$, then $z \in [x_1, y_n]\, T(\alpha, \beta)\, [x_1, y_n]$. Also one computes that

$$z' = [x_1, y_n]\left(\prod_{i=0}^{n-2} [x_{n-i}, y_{n-i-1}]\right)[x_1, y_n]$$

is an inverse for z in $[x_1, y_n]T(\alpha, \beta)[x_1, y_n]$. Further $zz' = z'z = [x_1, y_n]$, so $[x_1, y_n]\, T(\alpha, \beta)\, [x_1, y_n]$ is a group. Note also that the idempotent of each group (and thus every idempotent of $T(\alpha, \beta)$) is of the form $[x, y]$.

$T(\alpha, \beta)$ is simple: Let $a, b \in T(\alpha, \beta)$. Choose inverses a' and b' for a and b respectively. Then $aa' = [x, y]$ and $bb' = [w, z]$ for some (x, y) and (w, x) in $X \times Y$. Now

$$a = aa'a = [x, y]a = [x, z][w, z][x, y]a = [x, z]bb'[x, y]a \in T(\alpha, \beta)bT(\alpha, \beta).$$

Hence $T(\alpha, \beta)$ is simple.

Thus we conclude that $T(\alpha, \beta)$ is completely simple. Now fix an idempotent $e = [x_0, y_0]$ in $T(\alpha, \beta)$. One computes that the idempotents in the same $\mathscr{R}$-class with e are precisely those of the form $[x_0, y]$, $y \in Y$. Hence there are β $\mathscr{L}$-classes A dual argument shows that there are α $\mathscr{R}$-classes.

To prove the last assertion of the theorem, fix an idempotent e in S and consider the sets $eS \cap E$ and $Ee \cap E$. By assumption these have cardinality β and α respectively. Let 1-1 onto mappings $X \xrightarrow{g} Se \cap E$ and $T \xrightarrow{f} eS \cap E$ be given and define a map $X \times Y \xrightarrow{h} E$ by $h(x, y)$ is the idempotent in $g(x)S \cap Sf(y)$ Then clearly h is 1-1 and onto. Further one verifies that the defining relations for ρ are satisfied. Hence there is a unique homomorphism h: $T(\alpha, \beta) \to \langle E \rangle$ which extends h. This completes the proof of 3.5.

We remark that we have been unable in general to determine the nature of the group of $T(\alpha, \beta)$. We conjecture that it is a free group on γ generators, where γ is a function of (α, β). For example, if $\alpha = \beta = 2$, then the group of $T(\alpha, \beta)$ can easily be shown to be the integers.

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