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ON THE η FUNCTION OF BROWN AND PEARCY AND THE NUMERICAL FUNCTION OF AN OPERATOR

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1. Introduction. Throughout this paper $\mathfrak{H}$ will denote an infinite dimensional, separable complex Hilbert space, and $\mathfrak{S}$ will denote the unit sphere of $\mathfrak{H}$ (i.e. $\mathfrak{S} = \{x \in \mathfrak{H}: ||x|| = 1\}$). Also $\mathfrak{L}(\mathfrak{H})$ will represent the algebra of all bounded linear operators on $\mathfrak{H}$, and $\mathfrak{K}$ will represent the ideal of all compact operators on $\mathfrak{H}$. Furthermore $\mathfrak{P}$ will denote the set of all (orthogonal) projections on $\mathfrak{H}$ and $\mathfrak{P}_f$ will denote the sublattice of $\mathfrak{P}$ consisting of all finite rank projections. In most of the cases (especially when limits are involved) $\mathfrak{P}_f$ will be regarded as a directed set with the usual order relation inherited from $\mathfrak{P}$.

Brown and Pearcy in [**1**] define the non-negative function η on $\mathfrak{L}(\mathfrak{H})$ by

$$\eta(T) = \inf_{P \in \mathfrak{P}_f} \sup_{x \in \mathfrak{S} \cap (1-P)\mathfrak{H}} ||Tx - (Tx, x)x||. \tag{1.1}$$

They showed [**1**, Theorem 1] that $\eta(T) = 0$ if and only if T can be written as $T = \lambda + K$ where $K \in \mathfrak{K}$ and $\lambda \in \mathbf{C}$ (as usual, $\mathbf{C}$ denotes the complex field). Following the notation of [**3**], we denote by (T) the set

$$(T) = \{K + \lambda: K \in \mathfrak{K}, \lambda \in \mathbf{C}\},$$

and we denote the complement of (T) in $\mathfrak{L}(\mathfrak{H})$ by (F) [**1**]. Our first task in this paper (§ 2) is to study some of the properties enjoyed by the function η. In particular we prove (§ 2, Theorem 3) that $\eta(T) = \eta(T^*)$ for every $T \in \mathfrak{L}(\mathfrak{H})$, which was conjectured by Brown and Pearcy. In § 3 we define the essential numerical range $W_e(T)$ of an operator T, and we show (Lemma 3.3) that our definition is equivalent to the one given by Stampfli and Williams in [**5**]. Also we prove that the diameter $d_e(T)$ of $W_e(T)$ is zero if and only if $T \in (T)$ (Theorem 4), which constitutes another characterization of the class (T). Finally, in § 4, we introduce the numerical function, ϕ_T, of the operator T. This function is defined by the formula

$$\phi_T(x) = (Tx, x)/||x||^2, \quad 0 \neq x \in \mathfrak{H}.$$

The function ϕ_T seems to have an important relation with the operator T; for example, the range of ϕ_T is the numerical range $W(T)$ of T.

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Furthermore, let $w^{(1)}(T)$ (the differential numerical radius of T) be defined by

$$w^{(1)}(T) = \sup_{z \in \mathfrak{S}} ||D\phi_T(z)||,$$

where $D\phi_T(z)$ denotes the differential of the function ϕ_T at z. Also, set

$$w_e^{(1)}(T) = \inf_{P \in \mathfrak{P}_f} w^{(1)}([1 - P]T[1 - P]).$$

Using some standard techniques provided by the differential calculus on Banach spaces [**2**, Chapter VIII] we prove in Theorem 6 that

$$(1/2)d_e(T) \leqq w_e^{(1)}(T) \leqq 2\eta(T).$$

This inequality (in conjunction with Theorem 4) produces an alternative proof of the above mentioned theorem of Brown and Pearcy [**1**, Theorem 1] and gives a sharper estimate for the diameter of the essential numerical range of T, than that given by [**1**, Lemma 2.2].

In the last part of Section 4 we make some remarks concerning the higher order differentials of the numerical function ϕ_T.

2. Properties of the η function. We begin with some preliminary notation and remarks. Since the function $Tz - (Tz, z)z$ plays an important role in the definition (1.1) of the function η, in what follows we adopt the notation

$$E_T(z) = Tz - (Tz, z)z.$$

The following are some of the properties enjoyed by the function $E_T(z)$, for any $z \in \mathfrak{S}$.

(i) $E_{T+\lambda}(z) = E_T(z), \quad \lambda \in \mathbf{C}$,

(ii) $E_T(z) = 0$ if and only if z is an eigenvector of T,

(iii) $||E_T(z)|| \leqq ||Tz||$.

Given any bounded function $F : \mathfrak{S} \to \mathfrak{H}$ and any $Q \in \mathfrak{P}$, we will write $||F||_Q = \sup_{x \in \mathfrak{S} \cap Q\mathfrak{H}} ||F(x)||$, and simply $||F||$ if $Q = 1$.[1] Then formula (1.1) takes the form

$$\eta(T) = \inf_{(1-Q) \in \mathfrak{P}_f} ||E_T||_Q = \lim_{(1-Q) \in \mathfrak{P}_f} ||E_T||_Q.$$

Let $\pi : \mathfrak{L}(\mathfrak{H}) \to \mathfrak{L}(\mathfrak{H})/\mathfrak{K}$ be the canonical projection onto the (Calkin) quotient algebra, and recall that

$$||\pi(T)|| = \inf_{K \in \mathfrak{K}} ||T + K||.$$

The following lemma gives another characterization of $||\pi(T)||$, which will be used without explicit mention.

[1]The notation $||.||$ is usually reserved for the norm of a bounded *linear* transformation. However, since we are working with *non-linear* functions, like the function E_T, we extend such a notation to any bounded function on $\mathfrak{S}$ as indicated.

LEMMA 2.1. *If $T \in \mathfrak{L}(\mathfrak{H})$, then*

$$(2.1) \qquad \|\pi(T)\| = \inf_{P \in \mathfrak{P}_f} \|(1-P)T(1-P)\| = \lim_{P \in \mathfrak{P}_f} \|T\|_{(1-P)}.$$

Proof. Let

$$\nu(T) = \lim_{P \in \mathfrak{P}_f} \|T\|_{(1-P)} = \inf_{P \in \mathfrak{P}_f} \|T(1-P)\|.$$

It is clear that

$$\|\pi(T)\| \leqq \inf_{P \in \mathfrak{P}_f} \|(1-P)T(1-P)\| \leqq \nu(T);$$

thus it remains to prove that $\nu(T) \leqq \|\pi(T)\|$. For any $K \in \mathfrak{K}$, there exists an increasing sequence $P_n \in \mathfrak{P}_f$ such that $\lim_{n\to\infty}\|K(1-P_n)\| = 0$. Therefore

$$\nu(K) = \lim_{P \in \mathfrak{P}_f} \|K(1-P)\| \leqq \lim_{n\to\infty} \|K(1-P_n)\| = 0.$$

Since ν is a seminorm on $\mathfrak{L}(\mathfrak{H})$, we observe that $\nu(T+K) = \nu(T)$, for every $K \in \mathfrak{K}$. Thus $\nu(T) \leqq \|T+K\|$, $K \in \mathfrak{K}$ and hence $\nu(T) \leqq \|\pi(T)\|$.

Now, we list some elementary properties of the function η,

(i) η is a seminorm on $\mathfrak{L}(\mathfrak{H})$,

(ii) $\eta(T+\lambda) = \eta(T)$, $\lambda \in \mathbf{C}$,

(iii) $\eta(T) \leqq \|\pi(T)\|$,

and hence

$$(2.2) \qquad \eta(\lambda + K) = 0 \text{ for all } \lambda \in \mathbf{C},\ K \in \mathfrak{K},$$

$$(2.3) \qquad \eta(T+K) = \eta(T) \text{ for all } K \in \mathfrak{K}.$$

We remark that nothing like a power inequality is true for the function η. For example, if $\eta(T^2) \leqq C\eta^2(T)$ were valid for some constant $C > 0$, and every $T \in \mathfrak{L}(\mathfrak{H})$, then for every $\lambda \in \mathbf{C}$, we would have that $\eta(T^2 + 2\lambda T) = \eta[(T+\lambda)^2] \leqq C\eta^2(T+\lambda) = C\eta^2(T)$, which is false if we take any $T \in \mathfrak{L}(\mathfrak{H})$ with $\eta(T) > 0$ and λ sufficiently large (the same reasoning applies to higher powers). The following result is a geometric lemma, which we will need in the sequel.

LEMMA 2.2. *Let $\mathfrak{M}$ be a (closed) subspace of $\mathfrak{H}$. Then*

(a) *if U is a unitary operator, $U(\mathfrak{M})^\perp = U(\mathfrak{M}^\perp)$,*

(b) *if H is a self-adjoint invertible operator, then $H(\mathfrak{M})^\perp = H^{-1}(\mathfrak{M}^\perp)$,*

(c) *if $S \in \mathfrak{L}(\mathfrak{H})$ is invertible and $S = UH$ is its polar decomposition, then $S(\mathfrak{M})^\perp = U(H^{-1}(\mathfrak{M}^\perp))$.*

THEOREM 1. *If $T \in \mathfrak{L}(\mathfrak{H})$ is invertible, then*

$$(2.4) \qquad \eta(T)/\|\pi(T)\|^2 \leqq \eta(T^{-1})\|\pi(T^{-1})\|^2\eta(T).$$

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Proof. If $x \in \mathfrak{S}$ and $y = Tx/||Tx||$, we have

$$
\begin{aligned}
||E_T(x)||^2 &= ||Tx||^2 - |(Tx, x)|^2 \\
&= ||Tx||^4(1/||Tx||^2 - |(Tx, x)|^2/||Tx||^4) \\
&= ||Tx||^4(||T^{-1}y||^2 - |(T^{-1}y, y)|^2).
\end{aligned} \tag{2.5}
$$

On the other hand, given $Q \in \mathfrak{P}$ with $(1 - Q) \in \mathfrak{P}_f$, by hypothesis we see that $x \in \mathfrak{S} \cap Q\mathfrak{H}$ if and only if $y = Tx/||Tx|| \in \mathfrak{S} \cap TQ\mathfrak{H}$. Therefore, using formula (2.5) we obtain

$$||E_T||_Q \leqq ||T||_Q{}^2||E_T||_{Q_T}, \tag{2.6}$$

where Q_T is the projection onto the subspace $TQ\mathfrak{H}$. Employing Lemma 2.2, we see that since T is invertible, the mapping $Q \to Q_T$ establishes a lattice preserving correspondence in $\mathfrak{P}$, and also that $(1 - Q)\mathfrak{H}$ is finite dimensional if and only if $(1 - Q_T)\mathfrak{H}$ is so. Therefore, taking limits on both sides of (2.6) we conclude that the first inequality of (2.4) is valid. Interchanging T and T^{-1} we see also that the second inequality is valid.

We next state without proof the following characterization of the function η given by Douglas and Pearcy in [**3**, Theorem 1].

LEMMA 2.3. *For every* $T \in \mathfrak{L}(\mathfrak{H})$,

$$\eta(T) = \limsup_{P \in \mathfrak{P}_f} ||PT(1 - P)||.$$

The following lemma tells us that the η function is invariant under unitary equivalences.

LEMMA 2.4. *For every unitary* $U \in \mathfrak{L}(\mathfrak{H})$ *and every* $T \in \mathfrak{L}(\mathfrak{H})$,

$$\eta(UTU^*) = \eta(T). \tag{2.7}$$

Proof. Let $P \in \mathfrak{P}_f$. Then

$$||PUTU^*(1 - P)|| = ||(U^*PU)T[1 - (U^*PU)]||.$$

Set $P_U = U^*PU$. Then the correspondence $P \to P_U$ is bijective and lattice preserving in $\mathfrak{P}_f$ (by Lemma 2.2), and therefore using Lemma 2.3, we have

$$
\begin{aligned}
\eta(UTU^*) &= \limsup_{P \in \mathfrak{P}_f} ||P_UT(1 - P_U)|| \\
&= \limsup_{P \in \mathfrak{P}_f} ||PT(1 - P)|| \\
&= \eta(T).
\end{aligned}
$$

Hence (2.7) is valid.

THEOREM 2. *If* $T \in \mathfrak{L}(\mathfrak{H})$ *and S is an invertible operator, then*

$$\eta(T)/(||S^{-1}||\ ||\pi(S)||) \leqq \eta(STS^{-1}) \leqq ||S||\ ||\pi(S^{-1})||\eta(T). \tag{2.8}$$

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Proof. Let $S = UH$ be the polar decomposition of S. Since S is invertible, U is unitary and H is invertible. From Lemma 2.4, we obtain

$$\eta(STS^{-1}) = \eta(UHTH^{-1}U^*) = \eta(HTH^{-1}).$$

Also it is easy to see that

$$||\pi(S)|| = ||\pi(H)||, \quad ||\pi(S^{-1})|| = ||\pi(H^{-1})||,$$
$$||S|| = ||H||, \quad ||S^{-1}|| = ||H^{-1}||.$$

Thus it remains to prove (2.8) in the case that S is replaced by an invertible self-adjoint operator H. Let $P \in \mathfrak{P}_f$, $Q = 1 - P$. Then

$$||PHTH^{-1}Q|| = \sup_{\substack{x \in \mathfrak{S} \cap Q\mathfrak{H} \\ y \in \mathfrak{S} \cap P\mathfrak{H}}} |(HTH^{-1}x, y)| = \sup_{\substack{x \in \mathfrak{S} \cap Q\mathfrak{H} \\ y \in \mathfrak{S} \cap P\mathfrak{H}}} (TH^{-1}x, Hy). \tag{2.9}$$

Now, let P_H, Q_H be the projections onto the subspaces $HP\mathfrak{H}$ and $H^{-1}Q\mathfrak{H}$ respectively. From Lemma 2.2, we have $P_H + Q_H = 1$ and $P_H \in \mathfrak{P}_f$. From (2.9) we deduce that

$$||PHTH^{-1}Q|| \leqq ||H|| \, ||H^{-1}||_Q \sup_{\substack{x \in \mathfrak{S} \cap Q_H\mathfrak{H} \\ y \in \mathfrak{S} \cap P_H\mathfrak{H}}} |(Tx, y)| = ||H|| \, ||H^{-1}||_Q ||P_H T Q_H||. \tag{2.10}$$

Now using Lemma 2.2, as in Lemma 2.4 and Theorem 1, we observe that the mapping $P \rightarrow P_H$ sets up a lattice preserving bijective correspondence in $\mathfrak{P}_f$, and then taking lim sup in (2.10) we get

$$\begin{aligned} \eta(HTH^{-1}) &= \limsup_{\substack{P \in \mathfrak{P}_f \\ Q = 1-P}} ||PHTH^{-1}Q|| \\ &\leqq ||H|| \lim_{P \in \mathfrak{P}_f} ||H^{-1}||_{(1-P)} \limsup_{\substack{P \in \mathfrak{P}_f \\ Q = 1-P}} ||P_H T Q_H|| \\ &= ||H|| \, ||\pi(H^{-1})|| \eta(T). \end{aligned}$$

This proves the second inequality of (2.8), the first one follows in a similar way.

THEOREM 3. *For every* $T \in \mathfrak{L}(\mathfrak{H})$,

$$\eta(T) = \eta(T^*). \tag{2.11}$$

Proof. If $\mathfrak{Q}$ is any subset of $\mathfrak{H}$ we denote by $[\mathfrak{Q}]$ the projection onto the subspace generated by $\mathfrak{Q}$. From Lemma 2.3, for any $\delta > 0$ there exists $P \in \mathfrak{P}_f$ such that, if $P' \in \mathfrak{P}_f$, $P \leqq P'$, then $||P'T^*(1 - P')|| \leqq \eta(T^*) + \delta$. Since $[T^*P\mathfrak{H}] \in \mathfrak{P}_f$, setting $P_1 = P \vee [T^*P\mathfrak{H}]$ we see that $P_1 \in \mathfrak{P}_f$. Given $\epsilon > 0$, by definition of the function η there exists $x \in \mathfrak{S} \cap (1 - P_1)\mathfrak{H}$ such that $\eta(T) - \epsilon < ||E_T(x)||$. Set $P_2 = P \vee [x]$. Therefore, $P \leqq P_2$ and $P_2 \in \mathfrak{P}_f$. Now we observe that $[E_T(x)]$ is orthogonal to P_2. In fact, $[E_T(x)]$ is orthogonal to $[x]$; on the other hand $[E_T(x)]$ is orthogonal to P, for, $y \in P\mathfrak{H}$ implies $(E_T(x), y) = (Tx, y) = (x, T^*y) = 0$ (because $x \in (1 - P_1)\mathfrak{H}$). By the above remark, $E_T(x) \in (1 - P_2)\mathfrak{H}$, and then we have

$$\begin{aligned}\eta(T) - \epsilon &< ||E_T(x)|| = ||(1 - P_2)E_T(x)||\\ &= ||(1 - P_2)E_T(P_2x)|| = ||(1 - P_2)TP_2x||\\ &\leqq ||(1 - P_2)TP_2|| = ||P_2T^*(1 - P_2)||\\ &< \eta(T^*) + \delta.\end{aligned}$$

Since ϵ and δ are arbitrary positive numbers we conclude that $\eta(T) \leqq \eta(T^*)$. Interchanging T and T^* in the last inequality we obtain (2.11).

Remark. The sets (T) and (F) are invariant under similarities, and under the maps $S \to S^*$ and $S \to S^{-1}$ (from [**1**, Theorem 1]). We observe that Theorems 1, 2 and 3 show such invariant properties in a more precise fashion. On the other hand, (T) (and hence (F)) is not invariant under quasi-similarities.[2] In fact Hoover showed [**4**, Chapter 1, § 4] that there exists a compact weighted shift which is quasi-similar to a noncompact one. Thus we cannot expect that an analogous property to that of (2.8) holds for quasi-similar operators.

3. Some other seminorms on $\mathfrak{L}(\mathfrak{H})/\mathfrak{K}$. Let $T \in \mathfrak{L}(\mathfrak{H})$. As usual, $W(T)$ will denote the numerical range of T, i.e.

$$W(T) = \{(Tx, x), \quad x \in \mathfrak{S}\}.$$

Also, $w(T)$ will represent the numerical radius of T, i.e.

$$w(T) = \sup_{x \in \mathfrak{S}} |(Tx, x)|,$$

and $d(T)$ will denote the numerical diameter of T, i.e.

$$d(T) = \sup_{x, y \in \mathfrak{S}} |(Tx, x) - (Ty, y)|.$$

In what follows we adopt the following notation: if $T \in \mathfrak{L}(\mathfrak{H})$, $Q \in \mathfrak{P}$ then by T_Q we mean the restriction of the operator QTQ to the subspace $Q\mathfrak{H}$. Thus

$$||T_Q|| = ||QT||_Q.$$

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[2]Two operators T and S on $\mathfrak{H}$ are said to be quasi-similar [**4**] if there exist two dense range injective operators X and Y satisfying $TX = XS$, $YT = SY$.

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Now, we define the following two seminorms

$$\begin{aligned} w_e(T) &= \inf_{P\in\mathfrak{P}_f} w(T_{(1-P)}) \\ &= \inf_{P\in\mathfrak{P}_f} \sup_{x\in\mathfrak{S}\cap(1-P)\mathfrak{H}} |(Tx,x)| \\ &= \lim_{P\in\mathfrak{P}_f} w([1-P]T[1-P]); \end{aligned}$$

$$\begin{aligned} d_e(T) &= \inf_{P\in\mathfrak{P}_f} d(T_{(1-P)}) \\ &= \inf_{P\in\mathfrak{P}_f} \sup_{x,y\in\mathfrak{S}\cap(1-P)\mathfrak{H}} |(Tx,x)-(Ty,y)| \\ &= \lim_{P\in\mathfrak{P}_f} d(T_{(1-P)}). \end{aligned}$$

It is easy to verify that the following properties are valid for any $T \in \mathfrak{L}(\mathfrak{H})$.

(a_1) $w_e(T) = w_e(T^*)$;

(a_2) $(1/2)\|\pi(T)\| \leqq w_e(T) \leqq \|\pi(T)\|$;

and hence

(a_3) $w_e(K) = 0$ if and only if $K \in \mathfrak{K}$;

(a_4) $w_e(T^n) \leqq [w_e(T)]^n$;

(a_5) If $w_e(1-P) < 1$, then $\pi(T)$ is invertible (in $\mathfrak{L}(\mathfrak{H})/\mathfrak{K}$). Actually, more is true, i.e. dim[null T] = dim[null T^*];

(b_1) $d_e(T) = d_e(T^*)$;

(b_2) $d_e(T+\lambda) = d_e(T)$, $\lambda \in \mathbf{C}$;

(b_3) $d_e(T) \leqq 2w_e(T)$;

and hence

(b_4) $d_e(\lambda + K) = 0$, $\lambda \in \mathbf{C}$, $K \in \mathfrak{K}$.

LEMMA 3.1. *If* $T \in \mathfrak{L}(\mathfrak{H})$, *then*

(i) $w_e(T) = \inf_{K\in\mathfrak{K}} w(T+K)$;

(ii) $d_e(T) = \inf_{K\in\mathfrak{K}} d(T+K)$.

Proof. From (a_3) and (b_3), it follows that

$$w_e(T+K) = w_e(T),\ d_e(T+K) = d_e(T),\ K \in \mathfrak{K}.$$

Therefore, $w_e(T) \leqq \inf_{K\in\mathfrak{K}} w(T+K)$, $d_e(T) \leqq \inf_{K\in\mathfrak{K}} d(T+K)$. Thus it remains to prove the reverse inequalities. But

$$w_e(T) = \inf_{P\in\mathfrak{P}_f} w([1-P]T[1-P]) \geqq \inf_{K\in\mathfrak{K}} w(T+K),$$

and (i) follows. On the other hand, let $Q \in \mathfrak{P}$ be such that $(1-Q) \in \mathfrak{P}_f$, and let $\lambda_0 \in W(T_Q) = \{(Tx,x): x \in \mathfrak{S}\cap Q\mathfrak{H}\}$. Then,

$$W(QTQ + \lambda_0(1-Q)) = W(T_Q). \tag{3.1}$$

Therefore, $d(T_Q) = d(QTQ + \lambda_0(1 - Q)) \geqq \inf_{K\in\mathfrak{K}} d(T + K)$, and hence

$$d_e(T) \geqq \inf_{K\in\mathfrak{K}} d(T+K),$$

which completes the proof of (ii).

Next, we introduce a set valued function defined on $\mathfrak{L}(\mathfrak{H})$. For $T \in \mathfrak{L}(\mathfrak{H})$,

$$W_e(T) = \bigcap_{P\in\mathfrak{P}_f} \overline{W(T_{(1-P)})}\,.$$

Since $\{\overline{W(T_{(1-P)})}\}_{P\in\mathfrak{P}_f}$ constitutes a filter base of nonempty compact, convex sets, $W_e(T)$ is a nonempty compact, convex set.

LEMMA 3.2. *If* $T \in \mathfrak{L}(\mathfrak{H})$, *then*

(i) $w_e(T) = \sup_{\lambda\in W_e(T)} |\lambda|$,

and

(ii) $d_e(T) = \sup_{\lambda,\mu\in W_e(T)} |\lambda - \mu|$.

Proof. It is clear that $w_e(T) \leqq \sup_{\lambda\in W_e(T)}|\lambda|$, $d_e(T) \leqq \sup_{\lambda\in W_e(T)}|\lambda - \mu|$. On the other hand, let C be the boundary of any disk whose interior contains $W_e(T)$. Also, let δ be the diameter of C, and $\rho = \sup_{\lambda\in C}|\lambda|$. Since $W_e(T) \cap C = \emptyset$, there exists $P \in \mathfrak{P}_f$ such that $\overline{W(T_{(1-P)})} \cap C = \emptyset$. Therefore $w_e(T) < \rho$ and $d_e(T) < \delta$. These imply that $w_e(T) \leqq \sup_{\lambda,\mu\in W_e(T)}|\lambda|$, and $d_e(T) \leqq \sup_{\lambda,\mu\in W_e(T)}|\lambda - \mu|$.

LEMMA 3.3. *If* $T \in \mathfrak{L}(\mathfrak{H})$, *then*

$$W_e(T) = \bigcap_{K\in\mathfrak{K}} \overline{W(T+K)}\,.$$

Proof. From (3.1), we see that

$$\bigcap_{K\in\mathfrak{K}} \overline{W(T+K)} \subset W_e(T).$$

To prove the other inclusion, let $K \in \mathfrak{K}$ and $\epsilon > 0$. It follows that there exists $P \in \mathfrak{P}_f$ such that

$$||K_{(1-P)}|| = ||(1 - P)K(1 - P)|| \leqq ||K(1 - P)|| < \epsilon.$$

Therefore, $w(K_{(1-P)}) < \epsilon$ and hence

$$\begin{aligned} W_e(T) = W_e(T + K - K) &\subset W([T + K]_{(1-P)}) + W(K_{(1-P)}) \\ &\subset W(T + K) + \{\lambda : |\lambda| < \epsilon\}.^3 \end{aligned}$$

Since ϵ is arbitrary, $W_e(T) \subset \overline{W(T + K)}$ and hence

$$W_e(T) \subset \bigcap_{K\in\mathfrak{K}} \overline{W(T+K)}\,,$$

which completes the proof.

[3]If A, B are subsets of $\mathbf{C}$, then $A + B = \{\alpha + \beta : \alpha \in A, \beta \in B\}$.

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In view of the above Lemma and according to [**5**, § 3], the set $W_e(T)$ will be called the essential numerical range of the operator T. We saw in Lemma 3.2 that $w_e(T)$ is the radius of $W_e(T)$ and that $d_e(T)$ is its diameter. Furthermore, if $\sigma(\pi(T))$ denotes the spectrum of $\pi(T)$ (in $\mathfrak{L}(\mathfrak{H})/\mathfrak{K}$), then

$$\sigma(\pi(T)) \subset \overline{W(T+K)},$$

for every $K \in \mathfrak{K}$ and therefore, $\sigma(\pi(T)) \subset W_e(T)$. Also, it can be proved (using the relation $W_e(T+S) \subset W_e(T) + W_e(S)$, which is valid for every $T, S \in \mathfrak{L}(\mathfrak{H})$) that $W_e(T)$ is a continuous set valued function of $\pi(T)$. More precisely, if $S, T \in \mathfrak{L}(\mathfrak{H})$ then $\Delta(W_e(T), W_e(S)) \leqq ||\pi(T-S)||$, where $\Delta(. , .)$ denotes the Hausdorff metric for compact subsets of the complex plane.

THEOREM 4. *For $T \in \mathfrak{L}(\mathfrak{H})$ we have*

$$d_e(T) = 0 \textit{ if and only if } T \in (T).$$

Proof. If $T \in (T)$, it follows from (b_3) that $d_e(T) = 0$. Conversely, assume $d_e(T) = 0$, then $W_e(T) = \{\lambda\}$, for some $\lambda \in \mathbf{C}$, and hence $W_e(T-\lambda) = \{0\}$. Therefore $w_e(T-\lambda) = 0$, which, in conjunction with (a_3), proves that $K = T - \lambda \in \mathfrak{K}$, completing the proof of the theorem.

Remark. From 2.3 we see that

$$(*) \qquad \eta(T) \leqq \inf_{K \in \mathfrak{K}} ||E_{T+K}||,$$

where, as before, $||E_{T+K}|| = \sup_{||x||=1} ||E_{T+K}(x)||$. According to Lemma 3.1 it is reasonable to raise the following question, the answer to which is still unknown to us. Is the reverse inequality of (*) valid?

4. Some estimates on the numerical function of an operator. Given an operator T on $\mathfrak{H}$ the complex valued function ϕ_T, defined on $\mathfrak{H} - \{0\}$ by the formula

$$\phi_T(x) = (Tx, x)/||x||^2,$$

will be called the numerical function associated with T. The following are some of the properties enjoyed by ϕ_T.

(a) $W(T) =$ range of ϕ_T,

(b) ϕ_T is a continuous function on $\mathfrak{H} - \{0\}$ (with the norm topology),

(c) ϕ_T is homogeneous of degree zero, i.e. $\phi_T(\alpha x) = \phi_T(x)$, for every $\alpha > 0$.

Definition. Let $\mathfrak{U}$ be an open subset of $\mathfrak{H}$ and let g be a continuous real-valued function defined on $\mathfrak{U}$. We say that g is differentiable on $\mathfrak{U}$ if for every $z \in \mathfrak{U}$, there exists a real linear functional, L_z, on $\mathfrak{H}$, such that

$$(**) \qquad \lim_{||y|| \to 0} ||g(z+y) - g(z) - L_z y||/||y|| = 0.$$

If such a real linear functional L_z exists, it is the only bounded real linear functional satisfying (**), for each $z \in \mathfrak{U}$, and it is called the differential of g

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at z, $Dg(z)$. The value $Dg(z)$ at $x \in \mathfrak{H}$ is denoted by $Dg(z; x)$. If f is a continuous complex valued function defined on $\mathfrak{U}$, i.e. $f = g + ih$, where g, h are continuous real-valued functions on $\mathfrak{U}$, we say that f is differentiable on $\mathfrak{U}$ if g and h are differentiable on $\mathfrak{U}$. In this case $Df(z)$ is defined by $Df(z) = Dg(z) + iDh(z)$, $z \in \mathfrak{U}$. We observe that $Df(z)$ can also be characterized by

$$(4.1) \qquad \lim_{||y|| \to 0} ||f(z + y) - f(z) - Df(z; y)||/||y|| = 0,$$

where $Df(z; y) = Dg(z; y) + iDh(z; y)$.

We will use the next two lemmas to prove that the numerical function ϕ_T of $T \in \mathfrak{L}(\mathfrak{H})$ is differentiable on $\mathfrak{H} - \{0\}$ and to compute $D\phi_T(z)$ for every $0 \neq z \in \mathfrak{H}$.

LEMMA 4.1. *Let $\mathfrak{U}$ be an open subset of $\mathfrak{H}$ and let the functions $f : \mathfrak{U} \to \mathbf{C}$, $g : \mathfrak{U} \to \mathbf{C}$ be differentiable, such that $g(x) \neq 0$ for all $x \in \mathfrak{U}$. Then the function f/g is differentiable on $\mathfrak{U}$, and*

$$(4.2) \qquad D(f/g)(z; x) = [g(z)Df(z; x) - f(z)Dg(z; x)]/g^2(z),$$

for all $z \in \mathfrak{U}$, $x \in \mathfrak{U}$.

LEMMA 4.2. *For any $T \in \mathfrak{L}(\mathfrak{H})$, let $\psi_T : \mathfrak{H} \to \mathbf{C}$ be the function defined by*

$$(4.3) \qquad \psi_T(x) = (Tx, x).$$

Then ψ_T is differentiable on $\mathfrak{H}$ and

$$(4.4) \qquad D\psi_T(z; x) = (Tz, x) + (x, T^*z), \; z, x \in \mathfrak{H}.$$

Proof. The statement follows from (4.1) and the following identity

$$(T(z + y), z + y) - (Tz, z) - [(Tz, y) + (y, T^*z)] = (Ty, y),$$

valid for $T \in \mathfrak{L}(\mathfrak{H})$, $y, z \in \mathfrak{H}$.

THEOREM 5. *For any $T \in \mathfrak{L}(\mathfrak{H})$ the numerical function ϕ_T is differentiable on $\mathfrak{H} - \{0\}$ and the value of its differential at $0 \neq z \in \mathfrak{H}$, $x \in \mathfrak{H}$ is given by*

$$(4.5) \qquad D\phi_T(z; x) = [(E_T(z/||z||), x) + (x, E_{T^*}(z/||z||))]/||z||.$$

Proof. Using formula (4.3) we see that $\phi_T(x) = \psi_T(x)/\psi_1(x)$. Therefore from Lemma 4.1 and Lemma 4.2, ϕ_T is differentiable in $\mathfrak{H} - \{0\}$, and

$$\begin{aligned} D\phi_T(z; x) &= D(\psi_T/\psi_1)(z; x) \\ &= [\psi_1(z)D\psi_T(z; x) - \psi_T(z)D\psi_1(z; x)]/\psi_1^2(z) \\ &= [(Tz, x) + (x, T^*z) - \phi_T(z)(z, x) - \phi_T(z)(x, z)]/||z||^2, \end{aligned}$$

from which (4.5) follows.

COROLLARY 4.3. *For $T \in \mathfrak{L}(\mathfrak{H})$, $D\phi_T z = 0$ if and only if z is an eigenvector of both, T and T^*.*

Proof. The statement is a consequence of (4.5) and the following fact. Let $z_1, z_2 \in \mathfrak{H}$ such that $(z_1, x) + (x, z_2) = 0$ for all $x \in \mathfrak{H}$. Then $z_1 = z_2 = 0$.

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Now it is natural to introduce the following terminology. Given $T \in \mathfrak{L}(\mathfrak{H})$ we define the first differential numerical radius of T by

$$w^{(1)}(T) = \sup_{z \in \mathfrak{S}} ||D\phi_T(z)||.$$

We observe that if $Q \in \mathfrak{P}$, then

$$w^{(1)}(T_Q) = w^{(1)}(QTQ) = \sup_{z \in \mathfrak{S} \cap Q\mathfrak{H}} ||D\phi_T(z)||_Q,$$

where $||D\phi_T(z)||_Q = \sup_{x \in \mathfrak{S} \cap Q\mathfrak{H}} |D\phi_T(z\,;x)|$, and as before $T_Q = QT|_{Q\mathfrak{H}}$. Next we define a new seminorm on $\mathfrak{L}(\mathfrak{H})$ by setting

$$w_e^{(1)}(T) = \inf_{(1-Q) \in \mathfrak{P}_f} w^{(1)}(T_Q).$$

It is easy to verify that $w_e^{(1)}$ has the following properties:

$$w_e^{(1)}(T) = w_e^{(1)}(T^*),$$

$$w_e^{(1)}(T) \leqq 2||\pi(T)||,$$

$$w_e^{(1)}(T + \lambda) = w_e^{(1)}(T),\ \lambda \in \mathbf{C}, \tag{4.6}$$

$$w_e^{(1)}(K + \lambda) = 0,\ \lambda \in \mathbf{C},\ K \in \mathfrak{K}. \tag{4.7}$$

THEOREM 6. *For any* $T \in \mathfrak{L}(\mathfrak{H})$ *we have*

$$(1/2)d_e(T) \leqq w_e^{(1)}(T) \leqq 2\eta(T). \tag{4.8}$$

Proof. From (4.5) we see that, for any $z \in \mathfrak{S}$,

$$||D\phi_T(z)|| \leqq ||E_T(z)|| + ||E_{T^*}(z)||.$$

Taking supremum on $z \in \mathfrak{S} \cap (1 - P)\mathfrak{H}$ and then infimun over $P \in \mathfrak{P}_f$ we get

$$w_e^{(1)}(T) \leqq \eta(T) + \eta(T^*).$$

Using Theorem 3, we conclude that the second inequality of (4.8) is valid. To prove the left inequality of (4.8), let $P \in \mathfrak{P}_f$ and let $\lambda, \mu \in W(T_{(1-P)})$. There exists $x, y \in \mathfrak{S} \cap (1 - P)\mathfrak{H}$ such that $\phi_T(x) = \lambda$, $\phi_T(y) = \mu$. Furthermore (replacing y by $-y$, if necessary) we may assume that

$$||x - y|| \leqq \sqrt{2}. \tag{4.9}$$

Therefore the segment $[x, y]$ joining x and y lies entirely in $(1 - P)\mathfrak{H} - \{0\}$ and we can apply the Mean Value Theorem of Differential Calculus [**2**, Chapter VIII, Theorem 8.5.4] to obtain

$$|\lambda - \mu| = |\phi_T(x) - \phi_T(y)| \leqq \sup_{z \in [x,y]} ||\phi_T(z)||_{(1-P)} ||x - y||. \tag{4.10}$$

On the other hand, from an elementary geometric fact,

$$\sup_{z \in [x,y]} (1/||z||) = 1/(\inf_{z \in [x,y]} ||z||) = 2/||x + y||. \tag{4.11}$$

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Also, from (4.9) and the parallelogram law, we get

(4.12) $$||x + y|| \geqq \sqrt{2}.$$

Now from (4.9), (4.10), (4.11), (4.12) and the fact that $||z|| \; ||D\phi_T(z)||$ is homogeneous of degree zero, we can obtain

$$|\lambda - \mu| \leqq \sup_{z \in \mathfrak{S} \cap (1-P)\mathfrak{H}} ||D\phi_T(z)||_{(1-P)} \sup_{z \in [x,y]} (1/||z||)||x - y||$$
$$\leqq 2 \sup_{z \in \mathfrak{S} \cap (1-P)\mathfrak{H}} ||D\phi_T(z)||_{(1-P)},$$

and thus

(4.13) $$d(T_{(1-P)}) \leqq 2w^{(1)}(T_{(1-P)}).$$

The proof of (4.8) can be completed by taking limits in (4.13), when $P \in \mathfrak{P}_f$.

The next corollary is a consequence of (2.2), Theorem 4, and (4.8).

COROLLARY 4.4 (Brown and Pearcy). *For any* $T \in \mathfrak{L}(\mathfrak{H})$, $\eta(T) = 0$ *if and only if* $T = \lambda + K$, $\lambda \in \mathbf{C}$, $K \in \mathfrak{K}$.

We observe that $||\pi(T)||^2 \leqq \eta^2(T) + w_e^2(T)$ (recall that $||Tz||^2 = ||E_T(z)||^2 + |(Tz, z)|^2$, for every $z \in \mathfrak{S}$) implies that

$$||\pi(T - \lambda)||^2 \leqq \eta^2(T - \lambda) + w_e^2(T - \lambda)$$
$$\leqq \eta^2(T) + d_e^2(T), \lambda \in W_e(T)$$

and therefore, using (4.8) we obtain

(4.14) $$||\pi(T - \lambda)||^2 \leqq 17\eta^2(T), \lambda \in W_e(T),$$

which constitutes a sharper estimate than that given in [**1**, Lemma 2.3] (in the limit).

Remark. As we did previously for $n = 1$, we define the nth differential numerical radius of an operator T by

$$w^{(n)}(T) = \sup_{z \in \mathfrak{S}} ||D^n\phi_T(z)||.$$

Also we set

$$d^{(n)}(T) = \sup_{x,y \in \mathfrak{S}} ||D^n\phi_T(x) - D^n\phi_T(y)||.$$

Here $D^n\phi_T(z)$ denotes the nth differential of the function ϕ_T at z (for definition and properties of higher order differentials of a function, see [**2**, Chapter VIII, § 12]). Now, we define the following essential quantities

$$w_e^{(n)}(T) = \inf_{P \in \mathfrak{P}_f} W^{(n)}(T_{(1-P)}),$$

and

$$d_e^{(n)}(T) = \inf_{P \in \mathfrak{P}_f} d^{(n)}(T_{(1-P)}).$$

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Obviously,

$$d_e^{(n)}(T) \leqq 2w_e^{(n)}(T), n = 0, 1, 2, \ldots .$$

Next, we observe that since ϕ_T is an even function, i.e. $\phi_T(z) = \phi_T(-z)$, $z \neq 0$, $D^{2k}\phi_T$ is also an even function, and $D^{2k+1}\phi_T$ is an odd function (i.e. $D^{2k+1}\phi_T(-z) = -D^{2k+1}\phi_T(z)$). Thus

$$d_e^{(2k+1)}(T) = 2w_e^{(2k+1)}(T), k = 0, 1, 2, \ldots .$$

On the other hand, since ϕ_T is homogeneous of degree zero, $D^n\phi_T$ is homogeneous of degree $-n$, and hence $||z||^n D^n\phi_T(z)$ is homogeneous of degree zero, $n = 0, 1, 2, \ldots$. It can be proved (with arguments similar to those used to show (4.13)) that for $Q \in \mathfrak{P}$ we have

$$d^{(2k)}(T_Q) \leqq 2^{(1+k)}w^{(2k+1)}(T_Q),$$

and hence

$$d_e^{(2k)}(T) \leqq 2^{(1+k)}w_e^{(2k+1)}(T), k = 0, 1, 2, \ldots .$$

Also it is not difficult to see that for each $n = 0, 1, 2, \ldots$ there exists a constant $C_n > 0$ such that

$$w_e^{(n)}(T) \leqq C_n||\pi(T)||.$$

Therefore for any $n = 1, 2, \ldots$

$$w_e^{(n)}(\lambda + K) = 0, \lambda \in \mathbf{C}, K \in \mathfrak{K}.$$

Thus it is natural to pose the following problem.

Problem. Let $n \geqq 1$ and $T \in \mathfrak{L}(\mathfrak{H})$ such that $w_e^{(n)}(T) = 0$. Do there exist $\lambda \in \mathbf{C}$ and $K \in \mathfrak{K}$ such that $T = \lambda + K$? Observe that from (2.2), Theorem 4, and (4.8), Corollary 4.4 may be stated

$$w_e^{(1)}(T) = 0 \text{ if and only if } T = \lambda + K, \lambda \in \mathbf{C}, K \in \mathfrak{K}.$$

Hence Corollary 4.4 tells us that the answer to this problem is yes, in case $n = 1$. On the other hand, it can be shown that if $D^2\phi_T(z) = 0$, for every $z \in \mathfrak{S}$, then T is a scalar operator. Thus if $w^{(2)}(T_{(1-P)}) = 0$ for some $P \in \mathfrak{P}_f$, then $T = \lambda + K$ for some $\lambda \in \mathbf{C}$, $K \in \mathfrak{K}$.

Note. Let $\mathfrak{G}$ be any nonseparable Hilbert space, and let $\aleph_\alpha$ be any (infinite) cardinal number such that $\aleph_\alpha \leqq \dim \mathfrak{G}$. We denote by $\mathfrak{P}_\alpha$ the set of all (orthogonal) projections $P \in \mathfrak{L}(\mathfrak{G})$ such that, $\dim P\mathfrak{G} < \aleph_\alpha$, and we let $\mathfrak{J}_\alpha$ be the uniform closed ideal generated by $\mathfrak{P}_\alpha$. Then all the definitions and results of §§ 2, 3, and 4 can be extended, without any modifications, to nonseparable spaces, if we replace (in all the cases) $\mathfrak{P}_f$ and $\mathfrak{K}$ by $\mathfrak{P}_\alpha$ and $\mathfrak{J}_\alpha$, respectively. We omit the details of such extensions to avoid irrelevant repetitions.

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