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A BK inequality for random matchings

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Abstract

Let $G = (S, T, E)$ be a bipartite graph. For a matching M of G, let $V(M)$ be the set of vertices covered by M, and let $B(M)$ be the symmetric difference of $V(M)$ and S. We prove that if M is a uniform random matching of G, then $B(M)$ satisfies the BK inequality for increasing events.

1. Introduction

Let V be a finite set. We will consider random subsets of V. Let $\mathcal{A}$ and $\mathcal{B}$ be upward closed subsets of 2^V; in other words, let $\mathcal{A}$ and $\mathcal{B}$ be increasing events. Let $\mathcal{A}\square\mathcal{B}$ be the event that $\mathcal{A}$ and $\mathcal{B}$ both occur disjointly. More formally, we define

$$\mathcal{A}\square\mathcal{B} = \{A \cup B| \quad A \in \mathcal{A}, B \in \mathcal{B}, A \cap B = \emptyset\}.$$

Let $G = (S, T, E)$ be a bipartite graph, and let $V = S \cup T$. Let $\mathcal{M}$ be the set of matchings in G. For a matching $M \in \mathcal{M}$, let $V(M)$ be the set of vertices covered by M, and let

$$B(M) = V(M)\Delta S,$$

where Δ denotes the symmetric difference. Note that we have $|B(M)| = |S|$ for any matching M.

Our main result is the following.

Theorem 1.1. *Let M be a uniform random element of $\mathcal{M}$. Then $B(M)$ satisfies the BK inequality for increasing events, that is, if $\mathcal{A}$ and $\mathcal{B}$ are upward closed subsets of 2^V, then*

$$\mathbb{P}(B(M) \in \mathcal{A}\square\mathcal{B}) \leq \mathbb{P}(B(M) \in \mathcal{A})\mathbb{P}(B(M) \in \mathcal{B}).$$

For a random subset with independent marginals, the BK inequality was proved by van den Berg and Kesten [5]. There is an extension of the notion $\mathcal{A}\square\mathcal{B}$ for arbitrary events, see Subsection 2.1. With this definition, the BK inequality holds for all events in the case of a random subset with independent marginals. This was conjectured by van den Berg and Kesten [5], and proved by Reimer [2]. Building on the results of Reimer, van den Berg and Jonasson proved that the BK inequality also holds for a uniform random k element subset if we only consider increasing events [4]. Our results extend the results in [4], see the discussion after Theorem 1.4. See also the paper of van den Berg and Gandolfi [3] for further results.

We say that an event $\mathcal{A}$ depends only on $V_0 \subseteq V$, if for any $A, B \subseteq V$ the conditions $A \cap V_0 = B \cap V_0$ and $A \in \mathcal{A}$ imply that $B \in \mathcal{A}$. Note that if $\mathcal{A}$ and $\mathcal{B}$ are increasing events depending on disjoint subsets of V, then $\mathcal{A} \square \mathcal{B} = \mathcal{A} \cap \mathcal{B}$. Thus, Theorem 1.1 has the following corollary.

Corollary 1.2. *Let $B(M)$ be as above, then $B(M)$ has negative associations, which means the following. Let $\mathcal{A}$ and $\mathcal{B}$ be events depending on disjoint subsets of V. If $\mathcal{A}$ and $\mathcal{B}$ are both increasing or both decreasing, then*

$$\mathbb{P}(B(M) \in \mathcal{A} \cap \mathcal{B}) \leq \mathbb{P}(B(M) \in \mathcal{A})\mathbb{P}(B(M) \in \mathcal{B}).$$

If $\mathcal{A}$ is increasing and $\mathcal{B}$ is decreasing, then

$$\mathbb{P}(B(M) \in \mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(B(M) \in \mathcal{A})\mathbb{P}(B(M) \in \mathcal{B}).$$

Now we give a few extensions of Theorem 1.1. Assume that every edge e of G has a positive weight $w(e)$. For a matching M, we define the weight of M as $w(M) = \prod_{e \in M} w(e)$. Let M be a random matching, where the probability of a matching is proportional to its weight. We have the following extension of Theorem 1.1.

Theorem 1.3. *Let M be as above. Then $B(M)$ satisfies the BK inequality for increasing events, that is, if $\mathcal{A}$ and $\mathcal{B}$ are upward closed subsets of 2^V, then*

$$\mathbb{P}(B(M) \in \mathcal{A} \square \mathcal{B}) \leq \mathbb{P}(B(M) \in \mathcal{A})\mathbb{P}(B(M) \in \mathcal{B}).$$

Furthermore, let V_+ and V_- be disjoint subsets of V. Let M' have the same distribution as M conditioned on the event that $V_+ \subseteq B(M)$ and $V_- \cap B(M) = \emptyset$. Let $V' = V \backslash (V_+ \cup V_-)$, and let $B'(M') = B(M') \cap V'$. Clearly, $B'(M')$ is a random subset of V'.

Theorem 1.4. *The random subset $B'(M')$ satisfies the BK inequality for increasing events.*

As a special case of Theorem 1.4, we can obtain the statement that a uniform random k element subset of an n element set satisfies the BK inequality for increasing event. Thus, our results generalize the result of van den Berg and Jonasson [4] mentioned above. Indeed, let G be a complete bipartite graph (with constant edge weights) such that $|S| = k$ and $|T| = n$. If we set $V_- = S$ and $V_+ = \emptyset$, then M' is chosen uniformly at random from the set of matchings covering S. By symmetry, it is clear that $B'(M')$ is a uniform random k element subset of T.

Theorem 1.4 also has the following corollary.

Corollary 1.5. *Let M be as above. Then for any subset X and Y of V, we have*

$$\mathbb{P}(X \subseteq B(M))\mathbb{P}(Y \subseteq B(M)) \geq \mathbb{P}(X \cap Y \subseteq B(M))\mathbb{P}(X \cup Y \subseteq B(M)).$$

In other words, the law of $B(M)$ satisfies the negative lattice condition. See [1], where various notions of negative dependence are discussed.

We can also deduce the following theorem from Theorem 1.3.

Theorem 1.6. *Let M be uniform random maximum size matching. Then the random subset $B(M)$ satisfies the BK inequality for increasing events.*

2. The proofs

2.1 The definition of $\mathcal{A} \square \mathcal{B}$ for arbitrary events

Let us recall how to extend the definition of $\mathcal{A} \square \mathcal{B}$ to arbitrary events. A subset C of V is in $\mathcal{A} \square \mathcal{B}$ if and only if there are disjoint subsets V_A and V_B of V such that

$$\{D \subseteq V | D \cap V_A = C \cap V_A\} \subseteq \mathcal{A}$$

and

$$\{D \subseteq V | D \cap V_B = C \cap V_B\} \subseteq \mathcal{B}.$$

If $\mathcal{A}$ and $\mathcal{B}$ are increasing, then this definition indeed coincides with our earlier definition.

2.2 The proof of Theorem 1.4

In this subsection, we prove Theorem 1.4. Note that Theorem 1.1 and Theorem 1.3 can be obtained as special cases of Theorem 1.4.

Our proof will use several ideas of Berg and Jonasson [4].

Let I be the set of tuples (W, K, L, R), where W is a subset of V, K and L are perfect matchings in the induced subgraph $G[W]$, R is a subgraph of $G[V \setminus W]$ consisting of vertex disjoint paths.[1]

Fix a linear ordering of the edges of G. Consider an $i = (W, K, L, R) \in I$. Then R is the vertex disjoint union of the paths $P_1, P_2, \ldots, P_k$, where we list the paths in increasing order of their lowest edge. We can write P_j as the union of the matchings $M_{j,0}$ and $M_{j,1}$. This decomposition is unique once we assume that $M_{j,0}$ contains the lowest edge of P_j. For $\omega = (\omega_1, \omega_2, \ldots, \omega_k) \in \{0, 1\}^k$, we define the matchings

$$C_{i,\omega} = K \cup \bigcup_{j=1}^k M_{j,\omega_j} \quad \text{and} \quad D_{i,\omega} = L \cup \bigcup_{j=1}^k M_{j,1-\omega_j}.$$

Moreover, we define

$$Y_i^C = \left\{ C_{i,\omega} \mid \omega \in \{0, 1\}^k \right\},$$

$$Y_i^D = \left\{ D_{i,\omega} \mid \omega \in \{0, 1\}^k \right\},$$

and

$$X_i = \left\{ (C_{i,\omega}, D_{i,\omega}) \mid \omega \in \{0, 1\}^k \right\}.$$

Let H_i be the set of endpoints of the paths $P_1, P_2, \ldots, P_k$. Let $V(R)$ be the vertex set of R. Let $B_i = ((W \cup V(R)) \Delta S) \setminus H_i$. Let $v_{j,0}$ and $v_{j,1}$ be the two endpoints of P_j. If we choose the indices in the right way, then we get that

$$B(C_{i,\omega}) = B_i \cup \left\{ v_{j,\omega_j} \mid j = 1, 2, \ldots, k \right\},$$

and

$$B(D_{i,\omega}) = B_i \cup \left\{ v_{j,1-\omega_j} \mid j = 1, 2, \ldots, k \right\}.$$

This immediately implies that

$$\left\{ B(C_{i,\omega}) \mid \omega \in \{0, 1\}^k \right\} = \left\{ B(D_{i,\omega}) \mid \omega \in \{0, 1\}^k \right\}$$
$$= \left\{ B_i \cup H \mid H \subseteq H_i \text{ and } \left| H \cap \{v_{j,0}, v_{j,1}\} \right| = 1 \text{ for all } j = 1, 2, \ldots, k \right\}. \tag{1}$$
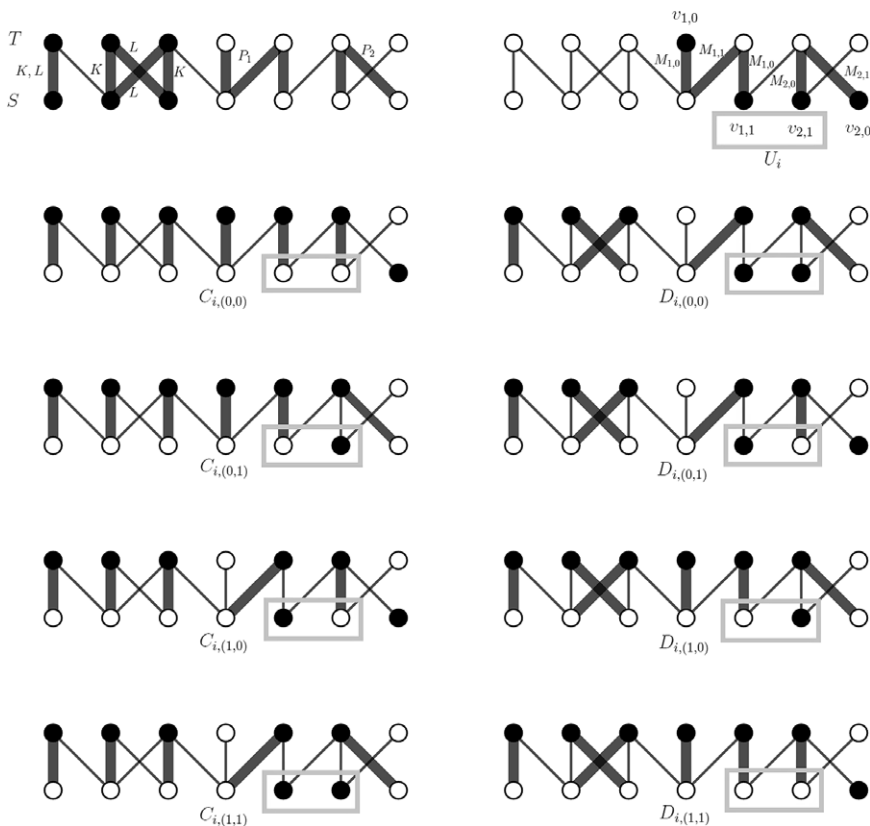
Let $U_i = \{v_{j,1} \mid j = 1, 2, \ldots, k\}$. We define the map $\tau_i : \mathcal{M} \to 2^{U_i}$ by $\tau_i(M) = B(M) \cap U_i$. It is clear from what is written above that the appropriate restriction of τ_i gives a bijection from Y_i^C to 2^{U_i}, and also from Y_i^D to 2^{U_i}. Moreover,

$$X_i = \left\{ (C, D) \in Y_i^C \times Y_i^D \mid \tau_i(C) = U_i \setminus \tau_i(D) \right\}. \tag{2}$$

We define

$$\mathcal{M}' = \{M \in \mathcal{M} \mid V_+ \subseteq B(M), V_- \cap B(M) = \emptyset\},$$

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[1] In our terminology, a path must have at least 1 edge.

Figure 1. The first figure describes a tuple $i = (W, K, L, R) \in I$: the vertices of W are coloured black; the bold edges correspond to the edges of $K \cup L \cup R$; the labels show the edges of the matchings K, L and the decomposition of R into two paths P_1 and P_2, and the two colour classes S and T of the bipartite graphs G. Note that the left most edge belongs to both K and L. In the second figure vertices of H_i are coloured black; the edges of $R = P_1 \cup P_2$ are bold; the labels show the indexing of the vertices of H_i, and also the decomposition of the paths P_1 and P_2 into the matchings $M_{1,0}, M_{1,1}$ and $M_{2,0}, M_{2,1}$. The vertical edges are in $M_{1,0}$ and $M_{2,0}$, the tilted edges are in $M_{1,1}$ and $M_{2,1}$. (Of course, depending on the linear ordering of the edges, the labels of $M_{1,0}$ and $M_{1,1}$ can be switched, we omitted the linear ordering from these figures.) We used a grey frame to indicate the elements of U_i. In the last four rows, the bold edges correspond to the matchings $C_{i,\omega}$ and $D_{i,\omega}$ as indicated. The vertices in $B(C_{i,\omega})$ (and $B(D_{i,\omega})$) are coloured black. The grey frame again contains the vertices of U_i.

and

$$I' = \left\{ i \in I \mid V_+ \subseteq B_i, \, V_- \cap (B_i \cup H_i) = \emptyset \right\}.$$

Lemma 2.1. *The sets $(X_i)_{i \in I'}$ give a partition of $\mathcal{M}' \times \mathcal{M}'$.*

Proof. Let $(C, D) \in \mathcal{M}' \times \mathcal{M}'$. Consider the multi-graph $C \cup D$, it is a vertex disjoint union of cycles and paths. Let R be the union of paths, and let Q be the union of cycles. Let W be the vertices covered by the cycles. Let $i = (W, C \cap Q, D \cap Q, R)$. One can easily prove that i is the unique element of I' such that $(C, D) \in X_i$.

Moreover, if $i \in I'$, then $X_i \subset \mathcal{M}' \times \mathcal{M}'$. Thus, the statement follows.   □

Given a subset $\mathcal{F}$ of $2^{V'}$, we define $\mathcal{M}_{\mathcal{F}}$ as $\{M \in \mathcal{M}' \mid B'(M) \in \mathcal{F}\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be upward closed subsets of $2^{V'}$.

Lemma 2.2. *If for all $i \in I'$, we have*

$$\left| (\mathcal{M}_{\mathcal{A} \square \mathcal{B}} \times \mathcal{M}') \cap X_i \right| \leq \left| (\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{B}}) \cap X_i \right|, \tag{3}$$

then

$$\mathbb{P}(B'(M') \in \mathcal{A} \square \mathcal{B}) \leq \mathbb{P}(B'(M') \in \mathcal{A})\mathbb{P}(B'(M') \in \mathcal{B}).$$

Proof. Consider any $i \in I'$. If $(C_1, D_1), (C_2, D_2) \in X_i$, then $C_1 + D_1 = C_2 + D_2$ as multisets. In particular, $w(C_1)w(D_1) = w(C_2)w(D_2)$. Thus, there is a w_i such that $w(C)w(D) = w_i$ for any $(C, D) \in X_i$. Multiplying both sides of Inequality (3) by w_i, we obtain that

$$\left|\left(\mathcal{M}_{\mathcal{A} \square \mathcal{B}} \times \mathcal{M}'\right) \cap X_i\right| \cdot w_i \leq \left|\left(\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{B}}\right) \cap X_i\right| \cdot w_i,$$

which is equivalent to

$$\sum_{(C,D) \in \left(\mathcal{M}_{\mathcal{A} \square \mathcal{B}} \times \mathcal{M}'\right) \cap X_i} w(C)w(D) \leq \sum_{(C,D) \in (\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{B}}) \cap X_i} w(C)w(D).$$

Summing these inequalities for all $i \in I'$ and using Lemma 2.1, we obtain that

$$\sum_{(C,D) \in \mathcal{M}_{\mathcal{A} \square \mathcal{B}} \times \mathcal{M}'} w(C)w(D) \leq \sum_{(C,D) \in (\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{B}})} w(C)w(D).$$

This can be rewritten as

$$\left(\sum_{M \in \mathcal{M}_{\mathcal{A} \square \mathcal{B}}} w(M)\right)\left(\sum_{M \in \mathcal{M}'} w(M)\right) \leq \left(\sum_{M \in \mathcal{M}_{\mathcal{A}}} w(M)\right)\left(\sum_{M \in \mathcal{M}_{\mathcal{B}}} w(M)\right).$$

Dividing both sides by $\left(\sum_{M \in \mathcal{M}'} w(M)\right)^2$, we obtain that

$$\mathbb{P}(B'(M') \in \mathcal{A} \square \mathcal{B}) \leq \mathbb{P}(B'(M') \in \mathcal{A})\mathbb{P}(B'(M') \in \mathcal{B}). \qquad \square$$

From Lemma 2.2, it follows that it is enough to prove that for any $i \in I'$, we have

$$\left|\left(\mathcal{M}_{\mathcal{A} \square \mathcal{B}} \times \mathcal{M}'\right) \cap X_i\right| \leq \left|\left(\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{B}}\right) \cap X_i\right|. \tag{4}$$

For a subset $\mathcal{F}$ of $2^{V'}$ and $i \in I'$, we define

$$\mathcal{F}^i = \left\{\tau_i(C) \big| C \in Y_i^C \cap \mathcal{M}_{\mathcal{F}}\right\} = \left\{B(C) \cap U_i \big| C \in Y_i^C, B'(C) \in \mathcal{F}\right\}.$$

From Equation (1), it follows that $\left\{B'(C) | C \in Y_i^C\right\} = \left\{B'(D) | D \in Y_i^D\right\}$. Therefore, $\mathcal{F}^i = \left\{\tau_i(D) | D \in Y_i^D \cap \mathcal{M}_{\mathcal{F}}\right\}$. (Note that, even for an increasing $\mathcal{F}$ it might happen that $\mathcal{F}^i$ is not increasing.) For a subset $\mathcal{J}$ of 2^{U_i}, we define $\overline{\mathcal{J}} = \{U_i \backslash J | J \in \mathcal{J}\}$.

Then

$$
\begin{aligned}
\left|(\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{B}}) \cap X_i\right| \\
= \left|\left\{(C, D) \in Y_i^C \times Y_i^D \big| \tau_i(C) \in \mathcal{A}^i, \tau_i(D) \in \mathcal{B}^i, \tau_i(C) = U_i \backslash \tau_i(D)\right\}\right| \\
= \left|\left\{(A, B) \in 2^{U_i} \times 2^{U_i} \big| A \in \mathcal{A}^i, B \in \mathcal{B}^i, A = U_i \backslash B\right\}\right| \\
= \left|\mathcal{A}^i \cap \overline{\mathcal{B}^i}\right|.
\end{aligned}
\tag{5}
$$

Similarly,

$$\left|\left(\mathcal{M}_{\mathcal{A} \square \mathcal{B}} \times \mathcal{M}'\right) \cap X_i\right| = \left|(\mathcal{A} \square \mathcal{B})^i\right|. \tag{6}$$

Lemma 2.3. *We have*

$$(\mathcal{A} \square \mathcal{B})^i \subseteq \mathcal{A}^i \square \mathcal{B}^i.$$

Proof. Let $F \in (\mathcal{A}\square\mathcal{B})^i$, then $F = \tau_i(C)$ for some $C \in Y_i^C$ such that $B'(C) \in \mathcal{A}\square\mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are upward closed, there are disjoint sets $V_A \in \mathcal{A}$ and $V_B \in \mathcal{B}$ such that $B'(C) = V_A \cup V_B$. We define

$$U_A = \{v_{j,1} | \quad \{v_{j,0}, v_{j,1}\} \cap V_A \neq \emptyset, j \in \{1, 2, \ldots, k\}\}$$

and

$$U_B = \{v_{j,1} | \quad \{v_{j,0}, v_{j,1}\} \cap V_B \neq \emptyset, j \in \{1, 2, \ldots, k\}\}.$$

Since V_A and V_B are disjoint and $|B'(C) \cap \{v_{j,0}, v_{j,1}\}| = 1$ for all j, we obtain that U_A and U_B are disjoint.

Moreover, if for some $C' \in Y_i^C$, we have $\tau_i(C) \cap U_A = \tau_i(C') \cap U_A$, then $V_A \subseteq B'(C')$. Consequently $B'(C') \in \mathcal{A}$ and $\tau_i(C') \in \mathcal{A}^i$. The analogous statement is true for V_B and U_B. Therefore, the pair U_A, U_B witnesses that $F = \tau_i(C) \in \mathcal{A}^i\square\mathcal{B}^i$. $\square$

Recall the following theorem of Reimer [2]. See also [4].

Theorem 2.1. *(Reimer) Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of 2^U, where U is a finite set. Then*

$$|\mathcal{X}\square\mathcal{Y}| \leq |\mathcal{X} \cap \overline{\mathcal{Y}}|.$$

Combining Theorem 2.1 with Equations (5) and (6) and Lemma 2.3, we obtain that

$$\left|(\mathcal{M}_{\mathcal{A}\square\mathcal{B}} \times \mathcal{M}') \cap X_i\right| = |(\mathcal{A}\square\mathcal{B})^i| \leq |\mathcal{A}^i\square\mathcal{B}^i| \leq |\mathcal{A}^i \cap \overline{\mathcal{B}^i}| = \left|(\mathcal{M}_\mathcal{A} \times \mathcal{M}_\mathcal{B}) \cap X_i\right|.$$

This proves Inequality (4).

2.3 The proof Corollary 1.5

Let $X_0 = X\backslash Y$ and $Y_0 = Y\backslash X$. Clearly the events $X_0 \subseteq B(M)$ and $Y_0 \subseteq B(M)$ depend on disjoint sets. Theorem 1.4 gives us

$$\mathbb{P}(X_0 \subseteq B(M)|X \cap Y \subseteq B(M))\mathbb{P}(Y_0 \subseteq B(M)|X \cap Y \subseteq B(M))$$
$$\geq \mathbb{P}(X_0 \subseteq B(M), Y_0 \subseteq B(M)|X \cap Y \subseteq B(M)),$$

and this is equivalent with the statement of the corollary.

2.4 The proof Theorem 1.6

Let $t > 0$, and set all the edge weights to be equal to t. Let M_t be the corresponding random matching. By Theorem 1.3, if $\mathcal{A}$ and $\mathcal{B}$ are increasing events, then

$$\mathbb{P}(B(M_t) \in \mathcal{A}\square\mathcal{B}) \leq \mathbb{P}(B(M_t) \in \mathcal{A})\mathbb{P}(B(M_t) \in \mathcal{B}).$$

Observe that

$$\lim_{t\to\infty} \mathbb{P}(B(M_t) \in \mathcal{A}) = \mathbb{P}(B(M) \in \mathcal{A}), \quad \lim_{t\to\infty} \mathbb{P}(B(M_t) \in \mathcal{B}) = \mathbb{P}(B(M) \in \mathcal{B})$$
$$\text{and } \lim_{t\to\infty} \mathbb{P}(B(M_t) \in \mathcal{A}\square\mathcal{B}) = \mathbb{P}(B(M) \in \mathcal{A}\square\mathcal{B}).$$

Thus, the statement follows.

Acknowledgements

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