Introduction and Main Examples

Since the introduction of the derived category of an abelian category, triangulated structures have become an integral part of homological algebra. Abelian model structures provide a convenient method for implementing and studying triangulated structures arising as a type of localization of an abelian (or exact) category. Indeed the homotopy category associated to any abelian model structure is a triangulated category. This section is meant to give the reader a broad overview of this idea along with a survey of the most fundamental examples appearing on R-Mod, the category of (say left) R-modules over a ring R, and $\mathrm{Ch}(R)$, the associated category of chain complexes of R-modules. These categories are the simplest ones with meaningful applications and they serve as a common ground for anyone that might be interested in learning some of the theory of abelian model categories. These examples will also be referenced throughout the book, to illustrate the general theory as it is being developed.

Cotorsion pairs are the cornerstone of the theory of abelian model categories. Let $(\mathcal{X}, \mathcal{Y})$ be a pair of classes of objects in an abelian category $\mathcal{A}$, such as R-Mod or $\mathrm{Ch}(R)$, and consider short exact sequences

$$0 \to Y \to Z \to X \to 0 \tag{1}$$

in $\mathcal{A}$. We say that $(\mathcal{X}, \mathcal{Y})$ is a *complete cotorsion pair* if it satisfies the following.

- $X \in \mathcal{X}$ if and only if every such short exact sequence (1) with $Y \in \mathcal{Y}$ splits, thus inducing a direct sum decomposition $Z \cong Y \bigoplus X$.
- Dually, $Y \in \mathcal{Y}$ if and only if every such short exact sequence (1) with $X \in \mathcal{X}$ splits.
- Given any $A \in \mathcal{A}$, there exists a short exact sequence $0 \to Y \to X \to A \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say $(\mathcal{X}, \mathcal{Y})$ *has enough projectives*.
- Dually, there exists a short exact sequence $0 \to A \to Y' \to X' \to 0$ with $Y' \in \mathcal{Y}$ and $X' \in \mathcal{X}$. We say $(\mathcal{X}, \mathcal{Y})$ *has enough injectives*.

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The first two conditions can be expressed more succinctly by writing $\mathcal{X}^{\perp} = \mathcal{Y}$, and ${}^{\perp}\mathcal{Y} = \mathcal{X}$. The usual definition of this orthogonality is given in terms of the Yoneda Ext functor, $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$, discussed in Section 1.6.

Note that the concept of a complete cotorsion pair generalizes the fundamental idea from homological algebra that $\mathcal{A} = R$-Mod has enough projectives and enough injectives. Indeed this can be summarized in the language of cotorsion pairs by saying that the pair $(\mathcal{A}, \mathcal{I})$, where $\mathcal{I}$ is the class of all injective R-modules, is a complete cotorsion pair. Dually, $(\mathcal{P}, \mathcal{A})$ is a complete cotorsion pair, where $\mathcal{P}$ is the class of projective R-modules. We call these, respectively, the *canonical injective* and *canonical projective* cotorsion pairs in R-Mod. In Chapter 2 we define and study cotorsion pairs in the quite general setting of Quillen exact categories, in terms of Yoneda's Ext functor.

A result known as Hovey's Correspondence reduces an abelian model structure (whatever that means precisely) on $\mathcal{A}$ to a triple of classes of objects, $\mathfrak{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$, where (i) $\mathcal{W}$ satisfies the property that if two out of three terms in a short exact sequence are in $\mathcal{W}$ then so must be the third, and (ii) $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are each complete cotorsion pairs. Objects in $\mathcal{Q}$ (resp. $\mathcal{R}$) are called *cofibrant* (resp. *fibrant*), and objects in $\mathcal{W}$ are called *trivial*. Hovey's Correspondence, Theorem 4.25, shows that such a triple $\mathfrak{M}$ is equivalent to the seemingly more complicated notion of an abelian model structure. Thus one could even define an abelian model structure to be such a *Hovey triple*, and in fact this is the philosophy and approach taken in this book. Since a Hovey triple packages a great amount of data in a very simple way, this perspective has proven to be beneficial. It also makes the subject more accessible.

Before proceeding, let us now give what are perhaps the two most fundamental examples of abelian model categories. Here we take $\mathcal{A} = \mathrm{Ch}(R)$, the category of chain complexes of R-modules. By a chain complex X, we mean a $\mathbb{Z}$-indexed sequence of R-module homomorphisms

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

satisfying $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. A morphism $f\colon X \to Y$ of chain complexes is a *chain map*, that is, a collection of R-module homomorphisms $f_n\colon X_n \to Y_n$ making all squares commute with the d_n.

- The *standard projective model structure* on $\mathrm{Ch}(R)$ is most easily described by a triple

$$\mathrm{Ch}(R)_{proj} = \left(dg\widetilde{\mathcal{P}}, \widetilde{\mathcal{E}}, All\right) \tag{2}$$

  of classes of chain complexes. Here, $\mathcal{W} = \widetilde{\mathcal{E}}$ is the class of all acyclic (i.e. exact) chain complexes of R-modules; those for which all homology groups

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are 0. The chain complexes in the class $\mathcal{Q} = dg\widetilde{\mathcal{P}}$ of cofibrant objects are usually called *DG-projective*, (or sometimes *semiprojective*, or *homotopically projective*). More details on this model structure are given shortly in Example 3. But we note now that the homotopy category of $\mathrm{Ch}(R)_{proj}$ is $\mathcal{D}(R)$, the derived category of R. A fundamental idea, discussed a bit more shortly and made precise in Section 6.7, is that $\mathcal{D}(R)$ is the triangulated localization of $\mathrm{Ch}(R)$ with respect to the class $\widetilde{\mathcal{E}}$ of acyclic complexes. Moreover, each chain complex X is isomorphic in the homotopy category, $\mathcal{D}(R)$, to a DG-projective complex, QX.

- Dually, there exists the *standard injective model structure* on $\mathrm{Ch}(R)$, given by the triple

$$\mathrm{Ch}(R)_{inj} = \left(All, \widetilde{\mathcal{E}}, dg\widetilde{\mathcal{I}}\right) \tag{3}$$

of classes of chain complexes. Again, $\mathcal{W} = \widetilde{\mathcal{E}}$ denotes the class of all acyclic chain complexes. But this time every chain complex is cofibrant while the fibrant objects form the class $dg\widetilde{\mathcal{I}}$ of *DG-injective* chain complexes. This gives an injective model for the derived category $\mathcal{D}(R)$.

We will explain how to establish the existence of these two model structures later, in Examples 3 and 4. We will also relate them to the standard fact that $\mathrm{Ext}^n_R(M, N)$ may be computed by way of either a projective resolution of M, or an injective coresolution of N.

The first big idea behind model categories is the construction of $\mathrm{Ho}(\mathfrak{M})$, the homotopy category of $\mathfrak{M}$. In general, we want it to be a category with the same objects as $\mathcal{A}$, and in the abelian case it should, at the very least, be an additive category for which the trivial objects have been "killed". That is, each $W \in \mathcal{W}$ should identify as a 0 object in $\mathrm{Ho}(\mathfrak{M})$. But the additive category obtained by setting the objects of $\mathcal{W}$ to 0 is merely $\mathrm{St}(\mathcal{A})$, a category we will call the *stable category* of $\mathfrak{M}$. By definition, the category $\mathrm{St}(\mathcal{A})$ has the same objects as $\mathcal{A}$, but its morphism sets are defined by

$$\underline{\mathrm{Hom}}(A, B) := \mathrm{Hom}_{\mathcal{A}}(A, B)/ \sim,$$

where $\sim$ is the equivalence relation defined by $f \sim g$ if and only if $g - f$ factors through some trivial object $W \in \mathcal{W}$. The stable category $\mathrm{St}(\mathcal{A})$ is not the homotopy category $\mathrm{Ho}(\mathfrak{M})$, but it is an important first step. While the stable category $\mathrm{St}(\mathcal{A})$ may be thought of as the additive localization with respect to $\mathcal{W}$, the homotopy category, $\mathrm{Ho}(\mathfrak{M})$, may be thought of as the triangulated localization with respect to $\mathcal{W}$. This is made precise in Section 6.7, but let us now note this: We also want any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ to identify as a *distinguished triangle* in $\mathrm{Ho}(\mathfrak{M})$, with the idea being that

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distinguished triangles hold onto much of the homological data encoded within the class of short exact sequences. This condition implies in particular that any monomorphism $i\colon A \to B$, with trivial cokernel, $\operatorname{Cok} i \in \mathcal{W}$, shall become an isomorphism in $\operatorname{Ho}(\mathfrak{M})$. We call such a morphism i a trivial monic. Similarly, the dual statement will be true: Each trivial epic, that is, epimorphism $p\colon B \to C$ with $\operatorname{Ker} p \in \mathcal{W}$, will become an isomorphism in $\operatorname{Ho}(\mathfrak{M})$. This leads to the idea that we wish to invert the class of all morphisms f which factor as $f = pi$, where i is a trivial monic and p is a trivial epic. Such morphisms make up the class of **weak equivalences**, which we denote by We.

Following a purely categorical approach that does not involve the classes $\mathcal{Q}$ or $\mathcal{R}$, one may formally invert the morphisms of We. With this approach one defines the homotopy category $\operatorname{Ho}(\mathfrak{M}) := \mathcal{A}\left[We^{-1}\right]$, by keeping the objects the same as $\mathcal{A}$, but defining the morphisms to be finite "zig-zags" of $\mathcal{A}$-morphisms where we allow the reversal of any arrow in We; see Exercise 5.4.1 for more details. This forces all weak equivalences to become isomorphisms and one obtains a canonical functor $\gamma\colon \mathcal{A} \to \mathcal{A}\left[We^{-1}\right]$ which is universally initial with respect to the property of inverting the morphisms of We. The standard result about Quillen model categories is that this construction does indeed produce a category in the sense that we still have small hom-sets, but more importantly that there is a more useful way to construct the homotopy category, $\operatorname{Ho}(\mathfrak{M})$. For our case of an abelian model category, $\mathfrak{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$, we get the following elegant theorem. To describe it, we need the notion of (co)fibrant approximations. Their existence follows immediately from the definition of a complete cotorsion pair, as given above. Indeed since $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ is a complete cotorsion pair, there exists for each object A, a short exact sequence

$$0 \to R_A \xrightarrow{i_A} QA \xrightarrow{p_A} A \to 0 \tag{4}$$

with $QA \in \mathcal{Q}$ and $R_A \in \mathcal{W} \cap \mathcal{R}$. Although such a short exact sequence is not unique, any such morphism p_A is unique in $\operatorname{St}(\mathcal{A})$, up to a canonical isomorphism. We call any such QA a *cofibrant approximation*, or a *cofibrant replacement*, of A. Note that p_A is a trivial epic, so it will provide an isomorphism $QA \cong A$ in $\operatorname{Ho}(\mathfrak{M})$. On the other hand, the dual notion is that of a *fibrant approximation* (or *fibrant replacement*), RA, obtained by taking a short exact sequence

$$0 \to A \xrightarrow{j_A} RA \xrightarrow{q_A} Q_A \to 0 \tag{5}$$

with $RA \in \mathcal{R}$ and $Q_A \in \mathcal{Q} \cap \mathcal{W}$. Objects in $\mathcal{Q} \cap \mathcal{R}$ are called *bifibrant* and by a *bifibrant approximation* of A we mean RQA, that is, a fibrant approximation of a cofibrant approximation of A.

Theorem 1 (The Fundamental Theorem of Abelian Model Categories) *Let* $\mathfrak{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ *be an abelian model category (i.e. Hovey triple) on an abelian category* $\mathcal{A}$*. Then there is an additive category,* $\mathrm{Ho}(\mathfrak{M})$*, called the* **homotopy category of** $\mathfrak{M}$*, whose objects are the same as those of* $\mathcal{A}$ *but whose morphisms are given by*

$$\mathrm{Ho}(\mathfrak{M})(A, B) := \underline{\mathrm{Hom}}(RQA, RQB),$$

where these are morphism sets in the stable category, $\mathrm{St}(\mathcal{A})$*, between any choice of bifibrant approximations. Moreover, we have the following.*

(1) *The inclusion* $\mathcal{Q} \cap \mathcal{R} \hookrightarrow \mathcal{A}$ *induces an equivalence of categories*

$$\mathrm{St}(\mathcal{Q} \cap \mathcal{R}) \xrightarrow{\cong} \mathrm{Ho}(\mathfrak{M}),$$

whose inverse is given by any bifibrant approximation assignment RQ.

(2) *There is a functor* $\gamma \colon \mathcal{A} \to \mathrm{Ho}(\mathfrak{M})$ *which is the identity on objects but bifibrant approximation on morphisms. It is a localization of* $\mathcal{A}$ *with respect to the class We (of all weak equivalences), and hence there is a canonical isomorphism of categories*

$$\mathrm{Ho}(\mathfrak{M}) \cong \mathcal{A}\left[We^{-1}\right].$$

(3) *For any choice of cofibrant approximation QA and fibrant approximation RB, there is a natural isomorphism* $\mathrm{Ho}(\mathfrak{M})(A, B) \cong \underline{\mathrm{Hom}}(QA, RB)$*.*

These results are all proved in Chapter 5, where our initital study of the homotopy category takes place. Chapter 6 then studies the triangulated structure that exists on the homotopy category. Our proofs are in the more general setting of any Quillen exact category for which every split monomorphism admits a cokernel. These are the so-called *weakly idompotent complete* exact categories which we find to be the most natural abstract setting to develop the theory of abelian model categories.

Examples: Model Structures for the Derived Category of R

As promised, the remainder of this introductory section will illustrate some of the most well-known examples of abelian model structures for the cases $\mathcal{A} = R$-Mod, and $\mathcal{A} = \mathrm{Ch}(R)$. We start by describing the construction of the standard projective and injective model structures on $\mathrm{Ch}(R)$.

Through Hovey's Correspondence, the problem of showing a cotorsion pair to be complete corresponds to proving the Factorization Axiom for model categories. Quillen's small-object argument is typically the tool used to construct such factorizations. Translating back to the abelian case, this corresponds to the

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notion of a cotorsion pair that is cogenerated by a set (as opposed to a proper class). Using the notation alluded to above, after the definition of a cotorsion pair, a set $\mathcal{S}$ is said to **cogenerate** a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ if $\mathcal{S}^\perp = \mathcal{Y}$. It just means $Y \in \mathcal{Y}$ if and only if every short exact sequence $0 \to Y \to Z \to S \to 0$ in $\mathcal{A}$, with $S \in \mathcal{S}$, splits. Said another way, $Y \in \mathcal{Y}$ if and only if $\mathrm{Ext}^1_{\mathcal{A}}(Y, S) = 0$ for all $S \in \mathcal{S}$. Chapter 9 details a version of Quillen's small object argument that is useful for constructing complete cotorsion pairs in very general exact categories. Our approach is inspired by the *efficient exact categories* introduced in Saorín and Šťovíček [2011]. The following is a special case of the powerful Theorem 9.34; see also Corollary 9.40 and Corollary 12.4.

Theorem 2 (Eklof and Trlifaj [2001]) *Let $\mathcal{A} = R$-Mod, or $\mathrm{Ch}(R)$. Then any set $\mathcal{S}$ (not a proper class) of objects in $\mathcal{A}$ cogenerates a complete cotorsion pair $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$.*

With this, we can easily construct the standard projective model structure $\mathrm{Ch}(R)_{proj} = \left(dg\widetilde{\mathcal{P}}, \widetilde{\mathcal{E}}, All\right)$, described earlier in (2). A comment on notation: Given any R-module M, we denote by $S^n(M)$ the chain complex consisting only of M in degree n and 0 elsewhere. We call $S^n(M)$ the *n-sphere on M*.

Example 3 (The Standard Projective Model Structure for $\mathcal{D}(R)$) Again, $\widetilde{\mathcal{E}}$ denotes the class of all acyclic (i.e. exact) chain complexes of R-modules. Let $\mathcal{S} = \{S^n(R)\}$ be the set of all n-spheres on R, where here R is considered as a (left) R-module over itself. There is an isomorphism $\mathrm{Ext}^1_{\mathrm{Ch}(R)}\left(S^{n+1}(R), X\right) \cong H_n X$, from which it follows that $\mathcal{S}^\perp = \widetilde{\mathcal{E}}$. So by Theorem 2 we have a complete cotorsion pair, $\left({}^\perp\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}\right)$. This is the key to the existence of the Hovey triple

$$\mathrm{Ch}(R)_{proj} = \left(dg\widetilde{\mathcal{P}}, \widetilde{\mathcal{E}}, All\right),$$

where $dg\widetilde{\mathcal{P}} := {}^\perp\widetilde{\mathcal{E}}$ is the class of *DG-projective* chain complexes. The DG-projective complexes are characterized as those complexes P such that each P_n is a projective R-module and any chain map $P \to E$ is null homotopic whenever E is an exact chain complex. The latter condition is automatic if P is a bounded below complex of projectives.

The chain homotopy category of R, denoted $K(R)$, is the category whose objects are chain complexes but whose morphisms are chain homotopy classes of chain maps. Results of Sections 10.5 and 10.6 relate abelian model structures on $\mathrm{Ch}(R)$ to the classical notion of Verdier quotients of $K(R)$. In particular, the formalities associated to the triangulated structure on $\mathrm{Ho}(\mathrm{Ch}(R))$ (such as the suspension functor, Σ, the mapping cone, $\mathrm{Cone}(f)$, etc.) will typically coincide with the classical notions in $K(R)$. In the current case, $\mathrm{Ho}\left(\mathrm{Ch}(R)_{proj}\right)$ identifies as the Verdier quotient $\mathcal{D}(R) := K(R)/\widetilde{\mathcal{E}}$, and shows the category to be equivalent to the isomorphic closure of $dg\widetilde{\mathcal{P}}$ in $K(R)$. In the language of Spaltenstein [1988], these are the precisely the K-projective complexes. □

Example 3 is a special case of Corollary 10.42 where the projective model structure is constructed in a far more general setting.

It is often stated that model categories encompass homological algebra. Let us give an example, beyond the above construction of $\mathcal{D}(R)$, supporting this sentiment. A well-known fact in algebra is that for two given R-modules M and N, the cohomology groups $\mathrm{Ext}^n_R(M,N)$ may be computed by taking a projective resolution

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0 \tag{6}$$

of M, and then taking the nth-cohomology group of the cochain complex obtained by applying $\mathrm{Hom}_R(-,N)$ to the projective resolution

$$\mathcal{P}_\circ \;\equiv\; \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0.$$

That is,

$$\mathrm{Ext}^n_R(M,N) \cong H^n[\mathrm{Hom}_R(\mathcal{P}_\circ, N)] \cong K(R)(\mathcal{P}_\circ, S^n(N)).$$

From the abelian model category perspective, the projective resolution $\mathcal{P}_\circ \xrightarrow{\epsilon} M \to 0$ of (6) is a cofibrant approximation of M in the projective model structure $\mathrm{Ch}(R)_{proj} = \left(dg\widetilde{\mathcal{P}}, \widetilde{\mathcal{E}}, All\right)$. To see this, we identify M with the 0-sphere complex $S^0(M)$. Note that the projective resolution of (6) may then be construed as a short exact sequence of chain complexes

$$0 \to E \to \mathcal{P}_\circ \xrightarrow{\epsilon} S^0(M) \to 0.$$

The projective resolution $\mathcal{P}_\circ$ is indeed a DG-projective complex while the kernel E is an exact complex. Referring to the very definition of a cofibrant approximation, given previously in Equation (4), this means that the projective resolution $\mathcal{P}_\circ$ is a cofibrant approximation of $S^0(M)$ in the projective model structure $\mathrm{Ch}(R)_{proj}$. So by part (3) of the Fundamental Theorem 1,

$$\mathrm{Ho}\left(\mathrm{Ch}(R)_{proj}\right)\left(S^0(M), S^n(N)\right) \cong \underline{\mathrm{Hom}}(\mathcal{P}_\circ, S^n(N)).$$

But in this case we have $\underline{\mathrm{Hom}}(\mathcal{P}_\circ, S^n(N)) = K(R)(\mathcal{P}_\circ, S^n(N))$; in the stable category $\mathrm{St}(\mathrm{Ch}(R)_{proj})$, morphisms with cofibrant domain are precisely chain homotopy classes of chain maps. Putting all this together we have

$$\mathrm{Ho}\left(\mathrm{Ch}(R)_{proj}\right)\left(S^0(M), S^n(N)\right) \cong \mathrm{Ext}^n_R(M,N).$$

Note that this identifies $\mathrm{Ext}^n_R(M,N)$ with a morphism set in the homotopy category.

On the other hand, recall that $\mathrm{Ext}^n_R(M,N)$ may be computed by taking an injective coresolution in the second variable N. This corresponds to the existence of the dual standard injective model structure described earlier in (3). However, the dual of Theorem 2 doesn't hold, simply because R-modules don't satisfy the dual properties needed to carry out the constructions. The following example indicates the set-theoretic flavor of arguments that are typically used to construct (cofibrantly generated) abelian model structures.

Example 4 (The Standard Injective Model Structure for $\mathcal{D}(R)$) Again, let $\widetilde{\mathcal{E}}$ denote the class of all exact chain complexes. Let κ be a regular cardinal number satisfying $\kappa \geq \max\{|R|, \omega\}$. Up to isomorphism, there exists a set (that is not a proper class) of exact chain complexes E with each $|E_n| \leq \kappa$. Let $\widetilde{\mathcal{E}}_\kappa$ denote a set of isomorphism representatives for all the exact chain complexes with this property. It is not too hard to argue that for each exact chain complex $E \in \widetilde{\mathcal{E}}$, there exists an exact subcomplex $E' \subseteq E$ with each $|E'_n| \leq \kappa$. It follows from this that the set $\widetilde{\mathcal{E}}_\kappa$ cogenerates a complete cotorsion pair $\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}^\perp\right)$; see Exercise 10.9.1. This provides the abelian model structure,

$$\mathrm{Ch}(R)_{inj} = \left(All, \widetilde{\mathcal{E}}, dg\widetilde{\mathcal{I}}\right),$$

where this time, the class $dg\widetilde{\mathcal{I}} := \widetilde{\mathcal{E}}^\perp$ is the class of *DG-injective* chain complexes. Such complexes are characterized as chain complexes I of injective R-modules such that any chain map $E \to I$ is null homotopic whenever E is an exact chain complex. In other words, these are Spaltenstein's K-injective complexes but with injective components. □

Example: Modules over Iwanaga–Gorenstein Rings

The category of R-modules over an Iwanaga–Gorenstein ring possesses a beautiful homotopy theory which can nicely be described in terms of abelian model structures. This corresponds to part of the subject known as Gorenstein homological algebra, as presented in the book by Enochs and Jenda [2000].

Let's first consider some immediate consequences of having an abelian model structure $\mathfrak{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ on R-Mod, regardless of the ring R we are considering. If we are to have such an $\mathfrak{M}$, then since $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ is a cotorsion pair, it is immediate that $\mathcal{Q} \cap \mathcal{W}$ must contain all projective R-modules. On the other hand, $\mathcal{W} \cap \mathcal{R}$ must contain all injective modules, and so the class $\mathcal{W}$ must contain all projective and injective modules. But then by the 2 out of 3 property on short exact sequences, $\mathcal{W}$ must contain any module of finite injective dimension, or of finite projective dimension. Consequently, the smallest class of trivial objects possible is the class of all modules having either finite injective dimension, or finite projective dimension.

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In the case of an Iwanaga–Gorenstein ring R, there exist abelian model structures on R-Mod for which $\mathcal{W}$ is nothing more than this minimal class of modules. In fact, *Iwanaga–Gorenstein* rings are characterized as the two-sided Noetherian rings for which the class of all (left and right) modules of finite injective dimension coincides with the class of all (left, resp. right) modules of finite projective dimension. This class of Noetherian rings includes the simplest case of quasi-Frobenius rings which are characterized by the property that a module is injective if and only if it is projective. A key example is the group ring $R = k[G]$ where k is a field and G is a finite group. In representation theory, the stable module category $\mathrm{St}(k[G])$ naturally arises from $k[G]$-Mod by killing the projective–injective modules. Formally, the objects of $\mathrm{St}(k[G])$ are just $k[G]$-modules, but morphisms are identified by the relation $f \sim g$ if and only if $g - f$ factors through a projective–injective $k[G]$-module. The Tate cohomology groups of G reside as morphism sets in $\mathrm{St}(k[G])$. Beyond quasi-Frobenius rings, Iwanaga–Gorenstein rings include, for instance, integral group rings $\mathbb{Z}[G]$ and p-adic group rings $\mathbb{Z}_p[G]$ over finite groups G.

Example 5 (Hovey [2002]) Let R be an Iwanaga–Gorenstein ring and let $\mathcal{W}$ denote the class of all (say left) R-modules of finite projective dimension, equivalently, finite injective dimension. Then we have the following.

- There is an abelian model structure $\mathfrak{M}_{inj} = (All, \mathcal{W}, \mathcal{GI})$ on R-Mod in which every module is cofibrant. The modules in $\mathcal{GI}$ are called *Gorenstein injective*, and they are precisely the modules appearing as a cycle in some (possibly unbounded) exact chain complex of injective modules.
- There is an abelian model structure $\mathfrak{M}_{proj} = (\mathcal{GP}, \mathcal{W}, All)$ on R-Mod in which every module is fibrant. The modules in $\mathcal{GP}$ are called *Gorenstein projective*, and they are precisely the modules appearing as a cycle in some exact chain complex of projective modules.

Since these two models share the same class of trivial objects they each model the same homotopy category which is a generalization of $\mathrm{St}(R)$, the stable module category of a quasi-Frobenius ring R. Referring to the Fundamental Theorem 1, the homotopy category is equivalent to each of the two stable module categories, $\mathrm{St}(\mathcal{GI})$, and $\mathrm{St}(\mathcal{GP})$. □

Examples: Frobenius Categories and Chain Homotopy Categories

Let R be a quasi-Frobenius ring, such as $R = k[G]$. Since the injective modules and projective modules coincide we get that the model structures $\mathfrak{M}_{inj}$ and $\mathfrak{M}_{proj}$ of Example 5 each coincide and become the simple Hovey triple

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$\mathfrak{M} = (All, \mathcal{W}, All)$ on R-Mod. So the associated homotopy category, $\mathrm{Ho}(\mathfrak{M})$, is exactly the stable module category $\mathrm{St}(k[G])$. These properties are reflecting that R-Mod is a *Frobenius category* whenever R is a quasi-Frobenius ring. Such structures will appear throughout this book. Let us give another standard example of a Frobenius category, one arising in the context of chain complexes of modules over a general ring R. Since model categories capture the idea of homotopy it is not surprising that $K(R)$, the chain homotopy category of R, is itself the homotopy category of an abelian model structure on $\mathrm{Ch}(R)$.

Example 6 (Frobenius Model Structure for $K(R)$) Let $\mathrm{Ch}(R)_{dw}$ denote the category of chain complexes along with the class of all short exact sequences

$$0 \to W \xrightarrow{f} X \xrightarrow{g} Y \to 0$$

that are *degreewise* split exact. That is, each $0 \to W_n \xrightarrow{f_n} X_n \xrightarrow{g_n} Y_n \to 0$ is a split exact sequence of R-modules. Then $\mathrm{Ch}(R)_{dw}$ is an example of a Quillen exact category in the sense studied in Chapter 1. We get a model structure on $\mathrm{Ch}(R)$ which is abelian relative to $\mathrm{Ch}(R)_{dw}$, and it may be described quite succinctly by a Hovey triple

$$\mathfrak{M}_{K(R)} = (\mathrm{Ch}(R), \mathcal{W}, \mathrm{Ch}(R)). \tag{7}$$

This time $\mathcal{W}$ denotes the class of all contractible chain complexes; they are precisely the projective–injective objects of $\mathrm{Ch}(R)_{dw}$. Since two chain maps are chain homotopic if and only if their difference factors through a contractible complex, it follows again from the above Fundamental Theorem 1 that $\mathrm{Ho}(\mathfrak{M}_{K(R)}) = K(R)$. □

Chain complexes over general additive categories are studied in Chapter 10. The above example is a special case of part of Theorem 10.20.

The relevance of Frobenius categories to the theory of abelian model categories goes far beyond providing easy examples of such model structures. The vast majority of the abelian model structures that have arisen in applications have been *hereditary* model structures. These are studied in Chapter 8. A notable feature is that their homotopy categories are triangle equivalent to the stable category of a Frobenius category with its classical triangulation from Happel [1988]. See Theorem 8.6.

Examples: Flat Model Structures on Modules and Complexes

Note that Example 3 lifts the canonical projective cotorsion pair, $(\mathcal{P}, \mathcal{A})$ on R-Mod, to the standard projective model structure on $\mathrm{Ch}(R)$. On the other hand, Example 4 lifts the canonical injective cotorsion pair, $(\mathcal{A}, \mathcal{I})$, to the standard injective model structure on Ch(R). In general, there are infinitely many intermediate cotorsion pairs that will lift to an abelian model structure on Ch(R). The following is a special case of Theorem 10.49 which is stated for Grothendieck categories.

Example 7 (General Model Structures for $\mathcal{D}(R)$) Let $(\mathcal{X}, \mathcal{Y})$ be an hereditary cotorsion pair on R-Mod, that is cogenerated by a set. Then it lifts to an hereditary model structure on chain complexes,

$$\mathrm{Ch}(R)_{(\mathcal{X},\mathcal{Y})} = \left(dg\widetilde{\mathcal{X}}, \widetilde{\mathcal{E}}, dg\widetilde{\mathcal{Y}}\right), \tag{8}$$

whose homotopy category again is equivalent to the derived category, $\mathcal{D}(R)$. The complexes in $dg\widetilde{\mathcal{X}}$ (resp. $dg\widetilde{\mathcal{Y}}$) are built up from R-modules in $\mathcal{X}$ (resp. $\mathcal{Y}$). □

An important example comes from Enochs' flat cotorsion pair, $(\mathcal{F}, \mathcal{C})$, in R-Mod. Here, $\mathcal{F}$ is the class of all flat R-modules and $\mathcal{C} := \mathcal{F}^{\perp}$ is the class of all *cotorsion* R-modules. Using theory developed in Chapter 9, the reader will be asked to prove the well-known fact that $(\mathcal{F}, \mathcal{C})$ is a complete hereditary cotorsion pair, cogenerated by a set $\mathcal{S}$; see Exercise 9.9.4. As a special case of Example 7, we have a flat model structure $\mathrm{Ch}(R)_{flat} = \left(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{E}}, dg\widetilde{\mathcal{C}}\right)$ for the derived category. Complexes in $dg\widetilde{\mathcal{F}}$ are called *DG-flat* and these include all DG-projective complexes. However, we develop technical tools in Section 9.10 to simplify the description of the fibrant objects. It turns out that, unlike the situation for the classes $dg\widetilde{\mathcal{I}}$, $dg\widetilde{\mathcal{P}}$, and $dg\widetilde{\mathcal{F}}$, every (even unbounded) chain complex of cotorsion modules is in the class $dg\widetilde{\mathcal{C}}$ of fibrant objects. This result was proved by Bazzoni, Cortés-Izurdiaga, and Estrada [2020].

Example 8 (The Flat Model Structure for $\mathcal{D}(R)$) Let $dw\widetilde{\mathcal{C}}$ denote the class of all chain complexes that are *degreewise* cotorsion, meaning $C \in dw\widetilde{\mathcal{C}}$ if and only if each C_n is a cotorsion R-module. Then Enochs' flat cotorsion pair, $(\mathcal{F}, \mathcal{C})$, lifts to an hereditary model structure on chain complexes,

$$\mathrm{Ch}(R)_{flat} = \left(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{E}}, dw\widetilde{\mathcal{C}}\right), \tag{9}$$

for the derived category, $\mathcal{D}(R)$. In particular, $dw\widetilde{\mathcal{C}} = dg\widetilde{\mathcal{C}}$ is the class of fibrant objects. The interested reader will be able to prove this result from the tools we develop in Section 9.10; see Exercises 10.9.4 and 10.9.6. □

Although sheaves and schemes are beyond the scope of this book, it should be pointed out that the significance of the flat model structure is that it generalizes to quasi-coherent sheaves over quite general schemes. Its existence

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on the category of complexes of quasi-coherent sheaves over such a scheme provides an abelian *monoidal* model structure which is a nice way to put the derived tensor product functor on solid theoretical ground. We study abelian monoidal model structures in Chapter 7, showing in particular that their homotopy categories are always tensor triangulated in the sense of Balmer [2005]. See Example 10.51 for further comments on the flat model structure for quasi-coherent sheaves.

Returning to the example of modules over an Iwanaga–Gorenstein ring R, there is another interesting point. We call any abelian model structure such as $\mathfrak{M}_{inj} = (All, \mathcal{W}, \mathcal{GI})$ from Example 5 an *injective model structure*. Here, every object is cofibrant, equivalently, the trivially fibrant objects are precisely the injectives. It follows from a technical result, Corollary 9.57, that if there is to exist an abelian model structure of the form $\mathfrak{M} = (All, \mathcal{W}, \mathcal{R})$ on R-Mod, then the class $\mathcal{W}$ must be closed under direct limits. So it then follows from the Govorov–Lazard Theorem that $\mathcal{W}$ must even contain all flat R-modules. Then again, the 2 out of 3 property for $\mathcal{W}$ implies that it must contain all modules of finite flat dimension. In particular, for Iwanaga–Gorenstein rings, any flat module has finite projective/injective dimension, and the class $\mathcal{W}$ of trivial objects is precisely the class of modules of finite flat dimension.

Example 9 Let R be an Iwanaga–Gorenstein ring and let $\mathcal{W}$ denote the class of all R-modules of finite projective (equivalently finite injective, or finite flat) dimension. There is an abelian model structure $\mathfrak{M}_{flat} = (\mathcal{GF}, \mathcal{W}, \mathcal{C})$ on R-Mod in which the fibrant objects form the class $\mathcal{C}$ of cotorsion R-modules. The modules in $\mathcal{GF}$ are called *Gorenstein flat*, and they are precisely the modules appearing as a cycle in some exact chain complex of flat modules. Its homotopy category coincides with those of Example 5 and shows them to also be equivalent to $\mathrm{St}(\mathcal{GF} \cap \mathcal{C})$.

For which rings R can we define such nice stable module categories? Such questions have motivated much work in Gorenstein homological algebra. For instance, the results of Examples 5 and 9 generalize to the larger class of Ding–Chen rings [Gillespie, 2017b]. More amazing is that it was shown in Šaroch and Šťovíček [2020] that for a general ring R, there exists a complete cotorsion pair $(\mathcal{GF}, \mathcal{GF}^{\perp})$ where $\mathcal{GF}$ is the general class of all Gorenstein flat modules, as in Enochs and Jenda [2000]. On the other hand, they show that we always have a complete cotorsion pair $({}^{\perp}\mathcal{GI}, \mathcal{GI})$, where $\mathcal{GI}$ is the general class of all Gorenstein injective modules. Both of these cotorsion pairs do indeed correspond to abelian model structures on R-Mod although their trivial objects may not coincide for general rings. As of the writing of this book, it is an open question whether or not the dual statement about Gorenstein projectives holds.