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ON POINT-SYMMETRIC TOURNAMENTS

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1. **Introduction.** A *tournament* is a directed graph in which there is exactly one arc between any two distinct vertices. Let $\mathscr{A}(T)$ denote the automorphism group of T. A tournament T is said to be *point-symmetric* if $\mathscr{A}(T)$ acts transitively on the vertices of T. Let $g(n)$ be the maximum value of $|\mathscr{A}(T)|$ taken over all tournaments of order n. In [3] Goldberg and Moon conjectured that $g(n) \leq \sqrt{3}^{n-1}$ with equality holding if and only if n is a power of 3. The case of point-symmetric tournaments is what prevented them from proving their conjecture. In [2] the conjecture was proved through the use of a lengthy combinatorial argument involving the structure of point-symmetric tournaments. The results in this paper are an outgrowth of an attempt to characterize point-symmetric tournaments so as to simplify the proof employed in [2].

The construction discussed in §2 was used in [1] as a means of producing regular tournaments. The analogous construction for graphs was employed by J. Turner in [6] independent of any knowledge of [1]. There is an obvious generalization to directed graphs.

We list some of the terminology used in this paper. If there is an arc from the vertex u to the vertex v in T, we write $(u, v) \in T$. If S is a subset of the vertex set of T, then $\langle S \rangle$ denotes the subtournament whose vertex set is S. We use the symbol "$\simeq$" to denote isomorphism between tournaments. The sets $\mathscr{O}(u) = \{v \in T: (u, v) \in T\}$ and $\mathscr{I}(u) = \{v \in T: (v, u) \in T\}$ are called the *outset* and *inset* of u, respectively. The *score* $s(u)$ of the vertex u is given by $s(u) = |\mathscr{O}(u)|$. The *score sequence* of T is the sequence $(s_1, s_2, \ldots, s_{|T|})$ of scores of the respective vertices of T written so that $s_1 \leq s_2 \leq \cdots \leq s_{|T|}$. Throughout this paper all subscripts are understood modulo $2n+1$.

2. **Main results.** Consider a fixed integer of the form $2n+1$, $n \geq 1$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a set of n distinct integers chosen from $1, 2, \ldots, 2n$ with the property that $\alpha_i + \alpha_j \neq 2n+1$ for any two α_i, α_j in S. Construct a directed graph T with vertices $v_0, v_1, \ldots, v_{2n}$ as follows: There is an arc from v_i to v_j if and only if $j - i \equiv \alpha_k \pmod{2n+1}$ for some $\alpha_k \in S$. It is not difficult to see that T is a regular tournament of degree n. Any tournament that is constructible in the above manner is called a *rotation tournament* and S is called the *symbol* of the rotation tournament.

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It is easy to see that the permutation $(v_0\, v_1 \ldots v_{2n})$ is an automorphism of T and this proves the following result.

PROPOSITION 1. *A rotation tournament is a point-symmetric tournament.*

Moreover, if T is a tournament of order m and the automorphism group of T possesses an m-cycle $(v_0\, v_1 \ldots v_{m-1})$, then T is a rotation tournament with symbol $S=\{j: (v_0, v_j) \in T\}$. The following proposition has been proved.

PROPOSITION 2. *A tournament T of order m is a rotation tournament if and only if $\mathscr{a}(T)$ possesses an m-cycle.*

We are interested in the question of whether or not the construction given above produces all the point-symmetric tournaments. By Propositions 1 and 2 an equivalent question is the following: If T is a point-symmetric tournament of order $2n+1$, does $\mathscr{a}(T)$ possess a $(2n+1)$-cycle? In the case that $2n+1$ is a prime the latter question is easy to answer. For if $2n+1$ is a prime and $\mathscr{a}(T)$ acts transitively on the vertices of T, then $2n+1$ divides $|\mathscr{a}(T)|$ and, thus, $\mathscr{a}(T)$ contains a $(2n+1)$-cycle [6, Theorem 3.2 and Exercise 3.12]. We have proved the following result.

THEOREM 1. *A tournament T of prime order is point-symmetric if and only if it is a rotation tournament.*

The first non-prime case to consider is $2n+1=9$. There are 15 regular tournaments of order 9 of which three are point-symmetric. It can be verified directly that all three of them are also rotation tournaments. However, we shall give a proof that every point-symmetric tournament of order 9 is a rotation tournament as it employs a technique that is useful for point-symmetric tournaments of larger composite order.

Let T be a point-symmetric tournament of order 9. Let u_0 be a fixed vertex of T and let $\mathscr{a}_{u_0}$ denote the stabilizer of u_0, i.e., $\mathscr{a}_{u_0}=\{\sigma \in \mathscr{a}(T): \sigma(u_0)=u_0\}$. Notice that $\mathscr{a}_{u_0}$ is, in a very natural way, the automorphism group of $\langle T-u_0\rangle$. Since $\mathscr{O}(u_0)$ contains four elements and the orbits of the automorphism group of any tournament have odd cardinality because every permutation in an odd order permutation group is a product of disjoint cycles of odd length and any automorphism group of a tournament has odd order by [4], $\mathscr{a}_{u_0}$ must fix at least one element of $\mathscr{O}(u_0)$. Let u_1 be a vertex of $\mathscr{O}(u_0)$ fixed by $\mathscr{a}_{u_0}$. Therefore, if $H=\{\sigma \in \mathscr{a}(T): \sigma(u_0)=u', u'$ a fixed vertex of $T\}$, then each $\sigma \in H$ maps u_1 to the same vertex of T. In particular, every $\sigma \in \mathscr{a}(T)$ that maps u_0 to u_1 maps u_1 to the same vertex of T, call it u_2. Since $\langle T-u_0\rangle \simeq \langle T-u_1\rangle$, each $\sigma \in \mathscr{a}(T)$ that maps u_1 to u_2 maps u_2 to the same vertex of T, call it u_3. Either $u_3=u_0$ or we can continue this process until we obtain a sequence $u_0, u_1, \ldots, u_j$ of distinct vertices of T such that every $\sigma \in \mathscr{a}(T)$ for which $\sigma(u_0)=u_1$ also satisfies $\sigma(u_1)=u_2$, $\sigma(u_2)=u_3, \ldots, \sigma(u_{j-1})=u_j$, and $\sigma(u_j)=u_0$. If $u_0, u_1, \ldots, u_j$ does not exhaust all the vertices of T, pick a $u_0' \in T$ not appearing in the sequence. For every $\tau \in \mathscr{a}(T)$ such that $\tau(u_0)=u_0'$ we also have $\tau(u_1)=u_1', \ldots,$

$\tau(u_j)=u'_j$ with $\{u_0, u_1, \ldots, u_j\} \cap \{u'_0, u'_1, \ldots, u'_j\}=\varnothing$. Hence, either $u_0, u_1, \ldots, u_j$ exhausts all the vertices of T or the vertices of T can be decomposed into mutually disjoint sequences of vertices of T having the same property as $u_0, u_1, \ldots, u_j$ with respect to automorphisms and such that $\langle\{u_0, u_1, \ldots, u_j\}\rangle \simeq \langle\{v_0, v_1, \ldots, v_j\}\rangle$ via the isomorphism $u_0 \to v_0, \ldots, u_j \to v_j$, where $v_0, v_1, \ldots, v_j$ denotes any of the other sequences.

If $u_0, u_1, \ldots, u_8$ exhausts all nine vertices of T, then $\sigma=(u_0\, u_1 \ldots u_8)$ is an automorphism of T. There is only one possibility remaining, namely, $\mathscr{a}(T)$ is imprimitive with three blocks (see [7]), say $\{u_0, u_1, u_2\}$, $\{u_3, u_4, u_5\}$, and $\{u_6, u_7, u_8\}$. Every $\sigma \in \mathscr{a}_{u_0}$, the stabilizer of u_0, fixes u_0, u_1, and u_2. Suppose some $\sigma \in \mathscr{a}_{u_0}$ moves one of the other vertices. Without loss of generality assume $\sigma(u_3)\neq u_3$. Then either $\sigma(u_3)=u_4$ or $\sigma(u_3)=u_5$ for otherwise σ would contain an even cycle in its disjoint cycle decomposition which cannot happen.

By considering what happens to the remaining vertices of T under any automorphism containing the 3-cycle (u_0, u_1, u_2), we see that all arcs between two distinct 3-blocks must have the same orientation. Therefore, $(u_0\, u_3\, u_6\, u_1\, u_4\, u_7\, u_2\, u_5\, u_8) \in \mathscr{a}(T)$ and T is a rotation tournament.

We may assume every $\sigma \in \mathscr{a}_{u_0}$ fixes every vertex of T, that is, $|\mathscr{a}_{u_0}|=1$ which implies $|\mathscr{a}(T)|=9$. To within isomorphism there are two transitive permutation groups of order nine in S_9. One is cyclic and generated by a 9-cycle and, hence, corresponds to a rotation tournament. The other is generated by two permutations σ_1, σ_2, of the form $\sigma_1=(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$ and $\sigma_2=(1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)$. By using the process described in the addendum to [3] it can be shown that the latter permutation group is not the automorphism group of a tournament of order nine. Therefore, every point-symmetric tournament of order nine is a rotation tournament.

We now consider the point-symmetric tournaments of order fifteen. Let $\mathscr{a}_u$ denote the stabilizer of the vertex u in a point-symmetric tournament T of order fifteen. If the orbits of $\mathscr{a}_u$ are $\{u\}$, $\mathscr{O}(u)$, and $\mathscr{I}(u)$, then the transitive constituents of $\mathscr{a}_u$ [see 7] in $\mathscr{O}(u)$ and $\mathscr{I}(u)$ must each contain a 7-cycle. Therefore, $\langle\mathscr{O}(u)\rangle$ and $\langle\mathscr{I}(u)\rangle$ are both rotation tournaments of order seven of which there are two to within isomorphism. By considering the four possible cases for $\langle\mathscr{O}(u)\rangle$ and $\langle\mathscr{I}(u)\rangle$ it can be shown through a tedious argument that it is impossible for the orbits of $\mathscr{a}_u$ to be $\{u\}$, $\mathscr{O}(u)$, and $\mathscr{I}(u)$. Therefore, $\mathscr{a}_u$ fixes a point of either $\mathscr{O}(u)$ or $\mathscr{I}(u)$. We assume without loss of generality that $\mathscr{a}_u$ fixes a point of $\mathscr{O}(u)$ since T and T^* have the same automorphism group where T^* denotes the converse tournament of T.

We proceed as before via some fixed point of $\mathscr{O}(u)$ under $\mathscr{a}_u$. If $\mathscr{a}(T)$ is primitive, then T must be a rotation tournament. Otherwise there are three 5-blocks or five 3-blocks. Suppose we have the blocks $\{u_1, u_2, u_3, u_4, u_5\}$, $\{u_6, u_7, u_8, u_9, u_{10}\}$, and $\{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$. Let $B_1=\langle\{u_1, u_2, \ldots, u_5\}\rangle$, $B_2=\langle\{u_6, u_7, \ldots, u_{10}\}\rangle$, and $B_3=\langle\{u_{11}, u_{12}, \ldots, u_{15}\}\rangle$. If any automorphism $\sigma \in \mathscr{a}_{u_1}$, the stabilizer of u_1, moves some vertex in B_1 or B_2, then following the argument used in the order nine case we see that all arcs between two distinct B_i's must have the same orientation and, therefore, the permutation $(u_1\, u_6\, u_{11}\, u_2\, u_7\, u_{12} \ldots u_5\, u_{10}\, u_{15}) \in \mathscr{A}(T)$. Now suppose we have the decomposition $\{u_1, u_2, u_3\}$, $\{u_4, u_5, u_6\}$, $\{u_7, u_8, u_9\}$, $\{u_{10}, u_{11}, u_{12}\}$, and $\{u_{13}, u_{14}, u_{15}\}$ where each set of three vertices forms a 3-block in T. Notice that $\mathscr{A}(T)$ induces an odd order transitive permutation group on the five 3-blocks as the object set. Since no odd order transitive subgroup of S_5 contains a 3-cycle, there is no $\sigma \in \mathscr{A}(T)$ that maps exactly three of the 3-blocks onto different 3-blocks. If any $\sigma \in \mathscr{A}_{u_1}$ moves some vertex in another 3-block, say $\sigma(u_4) \neq u_4$, then by the preceding remark and the fact there are no even cycles appearing in the cycle decomposition of σ, we have that $\sigma(u_4) = u_5$ or u_6 and all the arcs between $\langle\{u_1, u_2, u_3\}\rangle$ and $\langle\{u_4, u_5, u_6\}\rangle$ have the same orientation. By examining σ's action on the remaining three 3-blocks we see that all the arcs between two distinct 3-blocks have the same orientation. Thus there is a 15-cycle in $\mathscr{A}(T)$. Therefore, we are left with the case that $|\mathscr{A}_{u_1}| = 1$, i.e., $|\mathscr{A}(T)| = 15$. However, to within isomorphism there is only one transitive permutation group in S_{15} of order fifteen and it is generated by a 15-cycle. Therefore, every point-symmetric tournament of order fifteen is a rotation tournament.

Consider the following three 7×7 matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Let T be the tournament of order twenty-one whose incidence matrix is given by

$$\left(\begin{array}{c|c|c} A_1 & A_2 & A_3 \\ \hline A_3 & A_1 & A_2 \\ \hline A_2 & A_3 & A_1 \end{array}\right).$$

The score sequence of $\langle \mathcal{O}(u_1) \rangle$ is (3, 3 4, 4, 4, 5, 5, 6, 7) and the four vertices with score four form a transitive quadruple implying the automorphism group of $\langle \mathcal{O}(u_1)\rangle$ is the identity group. Similarly, the score sequence of $\langle \mathcal{I}(u_1)\rangle$ is (2, 3, 4, 4, 5, 5, 5, 5, 6, 6) and the four vertices with score five form a strongly connected quadruple implying the automorphism group of $\langle \mathcal{I}(u_1)\rangle$ is the identity group. In particular, $|\mathcal{A}_{u_1}|=1$.

It is easy to check that the two permutations

$$\sigma = (u_1\, u_8\, u_{15})(u_2\, u_9\, u_{16})\ldots(u_7\, u_{14}\, u_{21})$$

and

$$\tau = (u_1u_7u_6u_5u_4u_3u_2)(u_8u_{13}u_{11}u_9u_{14}u_{12}u_{10})(u_{15}u_{18}u_{21}u_{17}u_{20}u_{16}u_{19})$$

are in $\mathcal{A}(T)$. Since $A_{u_1}=\{1\}$, the given permutations σ and τ, with $\tau^7=\sigma^3=1$, generate a group of order 21. Observing that $(\sigma\tau)(u_1)=u_{14}$ while $(\tau\sigma)(u_1)=u_{13}$, we see that the group is non-abelian, hence non-cyclic, and, therefore, contains no element of order 21. Hence, T is an example of a point-symmetric tournament that is not a rotation tournament.

An *anti-automorphism* of a tournament T is a mapping σ of the vertex set of T onto itself satisfying $(u, v)\in T$ if and only if $(\sigma(u), \sigma(v))\notin T$ for every pair of distinct vertices u and v belonging to T. A tournament T is said to be *self-converse* if it has an anti-automorphism, that is, if $T\simeq T^*$.

PROPOSITION 3. *A rotation tournament is self-converse.*

Proof. Let T be a rotation tournament with vertices $u_0, u_1, \ldots, u_{2n}$. The permutation σ defined by $\sigma(u_i)=u_{2n-i+1}$ is easily seen to be an anti-automorphism of T. Thus $T\simeq T^*$.

An anti-automorphism of a tournament T composed with an automorphism of T results in an anti-automorphism of T. Therefore, if T is point-symmetric and self-converse, there exists an anti-automorphism of T that fixes any vertex one chooses. Let T denote the order twenty-one tournament exhibited above. If T is self-converse, there exists an anti-automorphism of T that maps $\mathcal{O}(u)$ onto $\mathcal{I}(u)$ with the four vertices of score four in $\langle\mathcal{O}(u)\rangle$ going onto the four vertices of score five in $\langle\mathcal{I}(u)\rangle$. But since one quadruple is transitive and the other is strongly connected we see that no such anti-automorphism exists. Therefore, T is not self-converse.

This suggests the following question: If T is a self-converse point-symmetric tournament, is T a rotation tournament?

3. **Enumeration of rotation tournaments.** We now consider the problem of enumerating the rotation tournaments of a given order. Let C_n denote the set of all symbols of the rotation tournaments of order $2n+1$ so that $|C_n|=2^n$. For each integer m satisfying $1\le m<2n+1$ with m and $2n+1$ relatively prime define $P_{n,m}$ by $P_{n,m}(S)=mS=\{x_i\equiv m\alpha_i(\text{mod } 2n+1): 1\le x_i\le 2_n\}$ where $S\in C_n$. It is easy to see $P_{n,m}$ is a permutation on C_n. The set of all such $P_{n,m}$ form a permutation group,

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call it G_n, acting on C_n. The number of orbits in C_n under the group G_n is given by the result [5, Theorem 3.21]

$$\frac{1}{\varphi(2n+1)} \sum_{G_n} F(P_{n,m})$$

where $F(P_{n,m})$ is the number of symbols fixed by $P_{n,m}$ and φ denotes the Euler φ-function. Denote the number of orbits by $g(n)$.

If $S'=mS=P_{n,m}(S)$, then the mapping π defined on $(u_0, u_1, \ldots, u_{2n})$ by $\pi(u_i)=u'_{mi}$ is an isomorphism between the tournaments corresponding to S and S'. Thus, $g(n)$ gives us an upper bound for the number of non-isomorphic rotation tournaments of order $2n+1$. The results obtained in [6] apply equally well to circulant tournament matrices so that in case $2n+1$ is a prime we have that two rotation tournaments are isomorphic if and only if their corresponding symbols are in the same orbit. Theorem 1 then proves the following result.

THEOREM 2. *If* $2n+1$ *is a prime, then the number of non-isomorphic point-symmetric tournaments of order* $2n+1$ *is*

$$g(2n+1) = \frac{1}{2n} \sum_{G_n} F(P_{n,m}).$$

Letting $r(n)$ denote the number of non-isomorphic rotation tournaments of order $2n+1$ and $t(n)$ denote the number of non-isomorphic point-symmetric tournaments of order $2n+1$ we have the following table.

n	$g(n)$	$r(n)$	$t(n)$
1	1	1	1
2	1	1	1
3	2	2	2
4	4	3	3
5	4	4	4
6	6	6	6
7	16	16	16
8	16	16	16
9	30	30	30
10	88		

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