A NOTE ON *L*-GROUPS

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The subject of this note is the study of conditions under which an l-group is simply ordered. We start with two definitions: A *po*-group is called *positively related* if every pair of elements $a, b>0$ has a lower bound $c>0$. Evidently a *po*-group is positively related if and only if it is *negatively related* in the sense that every pair of negative elements a, b has a negative upper bound. A *po*-group is called a *weak l-group* if every pair of elements a, b has a maximal lower bound c in the sense that $a, b \geqq c$ and $d>c$ imply that d is not a lower bound of a, b. Evidently every weak l-group is directed and every l-group is a weak l-group.

Theorem. *A weak l-group G is simply ordered if it is positively related.*

Proof. Let $a, b>0$ and c a maximal lower bound of a, b. Then $a-c \geqq 0$, $b-c \geqq 0$. We assert that either $a-c=0$ or $b-c=0$. For $a-c$, $b-c>0$ implies the existence of an element $d>0$ such that $a-c$, $b-c \geqq d$, i.e. $a, b \geqq d+c>c$, which contradicts the definition of c. Therefore we have either $a-c=0$, whence $b \geqq a$ or $b-c=0$, whence $a \geqq b$. Therefore every two positive elements of G are comparable, and, since G is directed, it is simply ordered. This proves the theorem.

Corollary 1. *An l-group is simply ordered if it is positively related.*

Corollary 2. *An l-group is simply ordered if* $a, b>0$ *implies* $a \cap b \neq 0$.

Corollary 3. *An l-group is simply ordered if* $a \cap b = 0$ *implies* $a=0$ *or* $b=0$; *that is, if every positive element is a weak unit.*

Corollary 4. *An l-group is simply ordered if* $a, b>0$ *implies* $a+b>a \cup b$.

Proof. This follows from the well-known equality $a \cup b = a-(a \cap b)+b$.

Corollary 5. *An l-group is simply ordered if it has a positive element* $e>0$ *such that for each* $a>0$ *there exists some integer n such that* $na \geqq e$.

Proof. For each $a, b>0$, let $na, mb \geqq e$. Evidently $na \cap mb \geqq e>0$, which implies that $a \cap b>0$, since it is well known that $a \cap b=0$ implies $pa \cap qb = 0$ for all $p, q = 1, 2, \ldots$.

Corollary 6. *Every strongly Archimedean l-group is simply ordered.*

(By strongly Archimedean we mean that for every two elements $a, b>0$ there exists some integer n such that $na \geqq b$.)

To end this note we give an example to show that there exists a positively related *po*-group which is not simply ordered. Let $G = \{(a, b, c) \mid a, b, c$ integers$\}$, then G forms a group under component-wise addition. Define a partial order in G by writing $(a, b, c) \geqq (0, 0, 0)$ if and only if either $a > 0$, $b \geqq 0$ or $a \geqq 0$, $b > 0$ or $a = 0$, $b = 0$, $c \geqq 0$. It is then easily verified that G is a positively related *po*-group which is not simply ordered.

REFERENCE

(1) G. Birkhoff, *Lattice Theory*, Revised edn. (New York, 1960).

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