BULL. AUSTRAL. MATH. SOC. 30D35, 30D30
VOL. 41 (1990) [417–420]

ON A RESULT OF SINGH

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In this paper relations between $T(r,f)$ and $T\left(r,f^{(k)}\right)$ are established for a class of meromorphic functions $f(z)$, where $T(r,f)$ and $T\left(r,f^{(k)}\right)$ are the Nevanlinna characteristic functions $f(z)$ and $f^{(k)}(z)$ respectively. An example is provided to show that a result of Singh is not true. The conclusions obtained here correct and generalise the result of Singh.

We denote by C the set of all finite complex numbers and by $\overline{C}$ the extended complex plane consisting of all (finite) complex numbers and ∞. Let $f(z)$ be a transcendental meromorphic function in the complex plane. We use with their usual definitions the Nevanlinna functions $T(r,f)$, $N(r,f)$, et cetera (see [1]). If $f(z)-a$ has a finite number of simple zeros, we say that a is an exceptional value Picard (e.v.P.) for simple zeros of $f(z)$. If $f(z)$ has a finite number of simple poles, we say that ∞ is e.v.P. for simple zeros of $f(z)$. In [3] Singh obtained the following result:

Let $f(z)$ be a transcendental meromorphic function of finite order with four (finite or infinite) distinct e.v.P. for simple zeros. Then

$$\lim_{r\to\infty}\frac{T(r,f')}{T(r,f)}=\frac{3}{2}.$$

Let $f(z)=sn(z)$, where $sn(z)$ is the Jacobian elliptic function (see [2]). We know that $f(z)$ is a transcendental meromorphic function of finite order and that $(f')^2=\left(1-f^2\right)\left(1-t^2f^2\right)$, where $t\,(\neq 0,\ 1,\ -1)$ is a constant. It is easy to see that 1, -1, $1/t$ and $-1/t$ are four distinct e.v.P. for simple zeros of $f(z)$ and that

$$\lim_{r\to\infty}\frac{T(r,f')}{T(r,f)}=2.$$

This shows that the above result of Singh is wrong.

In this paper we obtain the following theorem which is a correction of the result of Singh.

Received 22 June 1989

Theorem 1. *Let $f(z)$ be a transcendental meromorphic function of finite order with four distinct e.v.P. for simple zeros.*

(i) *If ∞ is an e.v.P. for simple zeros of $f(z)$, then*

$$\lim_{r\to\infty}\frac{T(r,f')}{T(r,f)}=\frac{3}{2};$$

(ii) *If ∞ is not an e.v.P. for simple zeros of $f(z)$, then*

$$\lim_{r\to\infty}\frac{T(r,f')}{T(r,f)}=2.$$

Instead of Theorem 1, we prove the more general theorem:

Theorem 2. *Let $f(z)$ be a transcendental meromorphic function of finite order with four distinct e.v.P. for simple zeros and k be a positive integer.*

(i) *If ∞ is an e.v.P. for simple zeros of $f(z)$, then*

$$\lim_{r\to\infty}\frac{T\left(r,f^{(k)}\right)}{T(r,f)}=\frac{1}{2}k+1;$$

(ii) *If ∞ is not an e.v.P. for simple zeros of $f(z)$, then*

$$\lim_{r\to\infty}\frac{T\left(r,f^{(k)}\right)}{T(r,f)}=k+1.$$

In order to state our third theorem, we introduce the following notation.

Let $f(z)$ be a meromorphic function and $a\in\overline{C}$. We denote by $n_1(r,a,f)$ the number of simple zeros of $f(z)-a$ in $|z|\leqslant r$. $N_1(r,a,f)$ is defined in terms of $n_1(r,a,f)$ in the usual way. Further we define

$$\delta_1(a,f)=1-\limsup_{r\to\infty}\frac{N_1(r,a,f)}{T(r,f)}.$$

Yang [4] proved that there exists at most a denumerable number of complex numbers a for which $\delta_1(a,f)>0$ and

$$\sum_{a\in\overline{C}}\delta_1(a,f)\leqslant 4.$$

Theorem 3. *Let $f(z)$ be a transcendental meromorphic function of finite order and k be a positive integer. If*

$$\sum_{a\in\overline{C}}\delta_1(a,f)=4, \tag{1}$$

then

$$\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)} = k+1-\frac{1}{2}k\delta_1(\infty,f).$$

Obviously, if a *is an e.v.P. for simple zeros of* $f(z)$, *then* $\delta_1(a,f)=1$.

Thus Theorem 3 includes Theorem 2 as a very special case.

PROOF OF THEOREM 3: Let $\{a_i\}_{i=1}^{\infty}$ be an infinite sequence of distinct elements of C which includes every $a\in C$ satisfying $\delta_1(a,f)>0$. By the second fundamental theorem and noting that $f(z)$ is a transcendental meromorphic function of finite order, we have

$$(q-1)T(r,f) < \sum_{i=1}^{q}\overline{N}(r,a_i,f)+\overline{N}(r,f)+O(\log r), \tag{2}$$

where q is any positive integer. Again

$$\begin{aligned}\overline{N}(r,a_i,f) &\leqslant \frac{1}{2}N_1(r,a_i,f)+\frac{1}{2}N(r,a_i,f)\\ &\leqslant \frac{1}{2}N_1(r,a_i,f)+\frac{1}{2}T(r,f)+O(1).\end{aligned} \tag{3}$$

From (2) and (3) we obtain

$$(q-1)T(r,f) < \frac{1}{2}\sum_{i=1}^{q}N_1(r,a_i,f)+\frac{1}{2}qT(r,f)+\overline{N}(r,f)+O(\log r). \tag{4}$$

Thus

$$\liminf_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)} \geqslant \frac{1}{2}\sum_{i=1}^{q}\delta_1(a_i,f)-1. \tag{5}$$

Since (5) holds for all $q\geqslant 1$, letting $q\to\infty$, we get

$$\begin{aligned}\liminf_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)} &\geqslant \frac{1}{2}\sum_{a\in C}\delta_1(a,f)-1\\ &=1-\frac{1}{2}\delta_1(\infty,f),\end{aligned} \tag{6}$$

using (1).

On the other hand,

$$\overline{N}(r,f)\leqslant \frac{1}{2}N_1(r,f)+\frac{1}{2}N(r,f) \tag{7}$$

https://doi.org/10.1017/S0004972700018293 Published online by Cambridge University Press

and hence

$$\overline{N}(r,f) \leqslant \frac{1}{2}N_1(r,f) + \frac{1}{2}T(r,f). \tag{8}$$

Thus

$$\limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} \leqslant 1 - \frac{1}{2}\delta_1(\infty,f). \tag{9}$$

From (6) and (9) we obtain

$$\lim_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} = 1 - \frac{1}{2}\delta_1(\infty,f). \tag{10}$$

By (7) and (10) we have

$$\lim_{r\to\infty} \frac{N(r,f)}{T(r,f)} = 1, \tag{11}$$

using $N(r,f) \leqslant T(r,f)$.

Since

$$N\left(r,f^{(k)}\right) = N(r,f) + k\overline{N}(r,f)$$

and

$$m\left(r,f^{(k)}\right) < m(r,f) + O(\log r),$$

thus we have

$$N(r,f) + k\overline{N}(r,f) < T\left(r,f^{(k)}\right) < T(r,f) + k\overline{N}(r,f) + O(\log r). \tag{12}$$

From (10), (11) and (12), we get

$$\lim_{r\to\infty} \frac{T\left(r,f^{(k)}\right)}{T(r,f)} = k + 1 - \frac{1}{2}k\delta_1(\infty,f).$$

This completes the proof of Theorem 3. ∎

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