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DIVISION THEOREMS FOR INVERSE AND PSEUDO-INVERSE SEMIGROUPS

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Abstract

We show that every inverse semigroup is an idempotent separating homomorphic image of a convex inverse subsemigroup of a P-semigroup $P(G, L, L)$, where G acts transitively on L. This division theorem for inverse semigroups can be applied to obtain a division theorem for pseudo-inverse semigroups.

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We follow the terminology and the notation of Howie (1976). If T is a semigroup endowed with a partial order $\leqslant$, then we shall say that a subsemigroup S of T is a convex subsemigroup of T if S constitutes a convex subset of $(T, \leqslant)$. In particular, an inverse subsemigroup S of an inverse semigroup T is said to be a convex subsemigroup of T, if S constitutes a convex subset with respect to the natural partial order on T.

Let X be a semilattice, G a group which acts on the left on X by order automorphisms, and Y a subsemilattice of X. Let

$$P(G, X, Y) = \{(\alpha, a) \in Y \times G | \alpha \wedge a\beta, a^{-1}(\alpha \wedge \beta) \in Y \textit{ for all } \beta \in Y\},$$

and define a multiplication on $P(G, X, Y)$ by

$$(\alpha, a)(\beta, b) = (\alpha \wedge a\beta, ab).$$

Then $P(G, X, Y)$ is an E-unitary inverse semigroup (McAlister (1974a, b)). A semigroup $P(G, X, Y)$ which is obtained in this way is called a P-semigroup. It follows easily from Theorem 2.6 of McAlister (1974b) that every E-unitary inverse semigroup can be represented by a P-semigroup.

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Let $P(G', X', Y')$ be a P-semigroup. If G is a subgroup of G', X a subsemilattice of X' which is left invariant under the action of G, and Y a subsemilattice of $X \cap Y'$, then we may consider $P(G, X, Y)$ as an inverse subsemigroup of $P(G', X', Y')$. One easily checks that $P(G, X, Y)$ is a convex inverse subsemigroup of $P(G', X', Y')$ if and only if Y is a convex subsemilattice of Y'.

Every inverse semigroup is an idempotent separating homomorphic image of a P-semigroup $P(G, X, Y)$ which in its turn can be embedded in the P-semigroup $P(G, X, X)$ which is a semi-direct product of the group G and the semilattice X (McAlister (1974a, b), O'Carroll (1976)). We shall establish a division theorem like the one considered here, where the embedding of Y in X and the action of G on X satisfy some nice conditions. We first have the following.

THEOREM 1. *Let S be an E-unitary inverse semigroup. Then the following are equivalent.*

(i) *$S \cong P(G, X, Y)$, where Y is a convex subsemilattice of the semilattice X, and for every $g \in G$, there exists $(\gamma, g) \in P(G, X, Y)$.*

(ii) *$S \cong P(G, L, Y)$, where Y is an ideal of the semilattice L.*

(iii) *S is isomorphic to a subsemigroup $P(G, L, Y)$ of $P(G', L', L')$, where G' acts transitively on L', where Y is an ideal of an open interval $L =]o, \varepsilon[$ of L', and where G is a subgroup of G' consisting of elements that fix both o and ε.*

PROOF. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious, and it suffices to show that (i) $\Rightarrow$ (iii). Therefore let $S \cong P(G, L, Y)$, where G, X and Y are as in (i). Let $\alpha, \beta \in Y$ and $k \in G$. Let (γ, k) be an element of $P(G, X, Y)$. Then $(\alpha, 1)(\gamma, k) = (\alpha \wedge \gamma, k) \in P(G, X, Y)$, and so $\alpha \wedge \gamma \wedge k\delta \in Y$ for every $\delta \in Y$. Therefore

$$\alpha \geqslant \alpha \wedge k\beta \geqslant \alpha \wedge \gamma \wedge k\beta \in Y$$

implies that $\alpha \wedge k\beta \in Y$ since Y is a convex subsemilattice of X. If α, β are any elements of Y and g, h any elements of G, then $h\alpha$ and $g\beta$ belong to GY, and by the foregoing we have

$$g\alpha \wedge h\beta = g(\alpha \wedge g^{-1}h\beta) \in gY \subseteq GY.$$

We conclude that $L = GY$ is a subsemilattice of X which contains Y as and ideal, and it is easy to see that $P(G, X, Y) = P(G, L, Y)$. We established (i) $\Rightarrow$ (ii). Let us adjoin a zero o and an identity ε to L, and let $L \cup \{o, \varepsilon\} = L^{o,\varepsilon}$. The action of G of L can be extended in a natural way to an action of G on $L^{o,\varepsilon}$: every element of G fixes o and ε. Then $P(G, X, Y) = P(G, L, Y)$ is a convex inverse subsemigroup of $P(G, L^{o,\varepsilon}, L^{o,\varepsilon})$. The mapping

$$\theta\colon G \to \operatorname{Aut} L^{o,\varepsilon}, \qquad g \to \theta_g,$$

where for every $g \in G$

$$\theta_g: L^{o,\varepsilon} \to L^{o,\varepsilon}, \qquad \alpha \to g\alpha,$$

is a homomorphism of G into Aut $L^{o,\varepsilon}$. From Theorem 4 of Pastijn (1980) it follows that there exists a semilattice L' which contains $L^{o,\varepsilon}$ as an interval and which has a transitive automorphism group, and an isomorphism

$$\text{Aut } L^{o,\varepsilon} \to \text{Aut } L', \qquad \rho \to \tau_{L'},$$

of Aut $L^{o,\varepsilon}$ into Aut L', such that $\tau = \tau_{L'}|L^{o,\varepsilon}$ for every $\tau \in$ Aut $L^{o,\varepsilon}$. Let $G' = G \star$ Aut L' be the free product of G and Aut L'. The mapping

$$G \cup \text{Aut } L' \to \text{Aut } L', g \to (\theta_g)_{L'} \qquad g \in G,$$
$$\kappa \to \kappa, \qquad \kappa \in \text{Aut } L',$$

can be extended in a unique way to a homomorphism of $G' = G \star$ Aut L' onto Aut L', and by this we obtain an action of G' on L' by order automorphisms. The group G' acts in a transitive way on L' since Aut L' does. Also G is a subgroup of G', and every element of G fixes o and ε. Further, for every $g \in G$, the restriction to L of the action of g on L' coincides with the action of g on L as defined originally. Therefore (iii) holds,

Remark 1. Not every E-unitary inverse semigroup satisfies the equivalent conditions of Theorem 1. We refer to McAlister (1978) for further details concerning the E-unitary inverse semigroups that satisfy the equivalent conditions of Theorem 1. We retain from Theorem 2.6 of McAlister (1978) that S is an F-inverse semigroup if and only if $S \cong P(G, X, Y) = P(G', L, Y)$ is as in Theorem 1, with the additional condition that Y has an identity. Also, the free inverse semigroup FI_X on the set X satisfies the conditions of Theorem 1 (O'Carroll (1974)).

Remark 2. The semigroup $P(G', L', L')$ which is mentioned in Theorem 1(iii) has high symmetry since G' acts in a transitive way on L'. Evidently $P(G, L', L')$ is bisimple. The best known example of such a semigroup is $P(Z, Z, Z)$, where Z stands for both the additive group of integers and the chain of integers. With the notation of Theorem 1, we have shown that every E-unitary inverse semigroup which can be represented by $P(G, L, L)$, can be isomorphically embedded as a convex subsemigroup in a P-semigroup $P(G', L', L')$, where G' acts transitively on L': in fact $P(G, L, L)$ is the disjoint union of the open intervals $](o, g), (\varepsilon, g)[$, $g \in G$. Also the P-semigroup $P(G, X, Y) = P(G, L, Y)$ is embedded as a convex inverse subsemigroup in $P(G', L', L')$. We further remark that the convex inverse subsemigroup of $P(G', L', L')$ which is the disjoint union of the half open intervals $](o, g), (\varepsilon, g)]$, $g \in G$, is an E-unitary factorizable

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inverse semigroup (in the sense of Chen and Hsieh (1974)), and every E-unitary factorizable inverse semigroup can be so obtained.

THEOREM 2. *Every inverse semigroup S divides an inverse semigroup $P(G', L', L')$ where G' acts in a transitive way on L', in such a way that S is an idempotent separating homomorphic image of a convex inverse subsemigroup $P(G, L, Y)$ of $P(G', L', L')$ where Y is an ideal of the open interval $]o, \varepsilon[= L$ of L', and where G is a subgroup of G' which consists of elements that fix both o and ε.*

PROOF. Every inverse semigroup S is an idempotent separating homomorphic image of a P-semigroup $P(G, L, Y)$ where the semilattice Y is an ideal of the semilattice L (Reilly and Munn (1976) and Theorem 4.3 of McAlister (1978)). The proof now follows from Theorem 1.

A pseudo-inverse semigroup S is a regular semigroup where for every $e = e^2 \in S$, eSe is an inverse semigroup. Let $P(G, X, Y)$ be a P-semigroup, and let I, Λ be sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$-matrix, where for every $(i, \lambda) \in I \times \Lambda$, $p_{\lambda i}$ is an element of G whose action on X induces an automorphism of Y. Let

$$\begin{aligned}\mathfrak{M} &= \mathfrak{M}(P(G, X, L); P; I, \Lambda)\\ &= \{(\alpha, a)_{i\lambda} \mid (\alpha, a) \in P(G, X, L), i \in I, \lambda \in \Lambda\},\end{aligned}$$

and define a multiplication on $\mathfrak{M}$ by

$$(\alpha, a)_{i\lambda}(\beta, b)_{j\mu} = (\alpha \wedge ap_{\lambda j}\beta, ap_{\lambda j}b)_{i\mu}.$$

Then $\mathfrak{M}$ becomes a pseudo-inverse semigroup which is a rectangular band $I \times \Lambda$ of E-unitary inverse semigroups which are all isomorphic to $P(G, X, Y)$. On $\mathfrak{M}$ define a partial order $\leqslant$ by

$$(\alpha, a)_{i\lambda} \leqslant (\beta, b)_{j\mu} \quad \text{if and only if } i = j, \lambda = \mu, a = b \text{ and } \alpha \leqslant \beta.$$

It is easy to see that $\leqslant$ is compatible with the multiplication on $\mathfrak{M}$ (Pastijn (preprint a)).

THEOREM 3. *Every pseudo-inverse semigroup S divides a pseudo-inverse semigroup $\mathfrak{M}(P(G', L', L'); P; I, \Lambda)$ where G' acts in a transitive way on L', in such a way that S is a homomorphic image of a convex pseudo-inverse subsemigroup S of $\mathfrak{M}(P(G', L', L'); P; I, \Lambda)$ by a homomorphism that induces a congruence on S whose congruence classes containing idempotents constitute completely simple semigroups.*

PROOF. Let S be a pseudo-inverse semigroup. Then S divides a pseudo-inverse semigroup $\mathfrak{M}(P(\bar{G}, X, Y); \bar{P}; I, \Lambda)$ in such a way that S is a homomorphic image of a pseudo-inverse subsemigroup $\bar{S}$ of $\mathcal{M}(P(\bar{G}, X, Y); \bar{P}; I, \Lambda)$ by a homomorphism φ that induces a congruence on $\bar{S}$ whose classes containing idempotents constitute completely simple semigroups, and $\bar{S}$ can be chosen to be a $\leqslant$-ideal of $\mathcal{M}(P(\bar{G}, X, Y); \bar{P}; I, \Lambda)$ (Pastijn (preprint b)). Let us adjoin an identity ω to the semilattice X. We put $X^\omega = X \cup \{\omega\}$ and $Y^\omega = Y \cup \{\omega\}$. We can extend the action of $\bar{G}$ on X to an action of $\bar{G}$ on X^ω: every element of $\bar{G}$ fixes ω. Obviously $P(\bar{G}, X, Y)$ is an ideal of $P(\bar{G}, X^\omega, Y^\omega)$, and $P(\bar{G}, X^\omega, Y^\omega) \setminus P(\bar{G}, X, Y)$ is the group of units of $P(\bar{G}, X^\omega, Y^\omega)$. This group of units consists of the elements $(\omega, \bar{g})$, $\bar{g} \in \bar{G}$, where $\bar{g}$ induces an automorphism on Y. Thus, if $\bar{p}_{\lambda i}$ is any entry of the $\Lambda \times I$-matrix $\bar{P}$, then $(\omega, \bar{p}_{\lambda i})$ belongs to the group of units of $P(\bar{G}, X^\omega, Y^\omega)$. Whence $\mathcal{M}(P(\bar{G}, X^\omega, Y^\omega); \bar{P}; I, \Lambda)$ is an ideal extension of $\mathcal{M}(P(\bar{G}, X, Y); \bar{P}; I, \Lambda)$ (by a completely simple semigroup with zero), and so $\mathcal{M}(P(\bar{G}, X^\omega, Y^\omega); \bar{P}; I, \Lambda)$ contains $\bar{S}$ as a pseudo-inverse subsemigroup and as a $\leqslant$-ideal. By Reilly and Munn (1976) and Theorem 4.3 of McAlister (1978), there exists a P-semigroup $P(G, L, Y^\omega)$, where Y^ω is a principal ideal of L, and an idempotent separating homomorphism ψ of $P(G, L, Y^\omega)$ onto $P(\bar{G}, X^\omega, Y^\omega)$. For any $(\lambda, i) \in \Lambda \times I$, let $p_{\lambda i}$ be an element of G such that $(\omega, p_{\lambda i})\psi = (\omega, \bar{p}_{\lambda i})$. Let $P = (p_{\lambda i})$ be the $\Lambda \times I$-matrix with entries $p_{\lambda i}$, $(\lambda, i) \in \Lambda \times I$. Since $(\omega, p_{\lambda i})$ belongs to the group of units of $P(G, L, Y^\omega)$, it follows that $p_{\lambda i}$ induces an automorphism on Y^ω which coincides with the automorphism which is induced on Y^ω by $\bar{p}_{\lambda i}$. Therefore it makes sense to consider the semigroup $\mathcal{M}(P(G, L, Y^\omega); P; I, \Lambda)$, and one readily checks that

$$\theta\colon \mathcal{M}(P(G, L, Y^\omega); P; I, \Lambda) \to \big(P\big(\bar{G}, X^\omega, Y^\omega\big); \bar{P}; I, \Lambda\big),$$
$$(\alpha, a)_{i\lambda} \to ((\alpha, a)\psi)_{i\lambda},$$

is an idempotent separating homomorphism onto $\mathcal{M}(P(\bar{G}, X^\omega, Y^\omega); \bar{P}; I, \Lambda)$. We denote the pre-image $\bar{S}\theta^{-1}$ of $\bar{S}$ under this homomorphism by $\underline{S}$. Since θ is idempotent separating, $\theta|\underline{S}$ is an idempotent separating homomorphism of the pseudo-inverse subsemigroup $\underline{S}$ of $\mathcal{M}(P(G, L, Y^\omega); P; I, \Lambda)$ onto $\bar{S}$. Further, $\underline{S}$ is a $\leqslant$-ideal of $\mathcal{M}(P(G, L, Y^\omega); P; I, \Lambda)$. The composition $(\theta|\underline{S})\varphi$ is a homomorphism of $\underline{S}$ onto S which induces a congruence on $\underline{S}$ whose classes containing idempotents form completely simple semigroups. The P-semigroup $P(G, L, Y^\omega)$ can be embedded as a convex inverse subsemigroup in a P-semigroup $P(G', L', L')$ where G' acts transitively on L' in the way described by Theorem 1. It follows that $\mathcal{M}(P(G, L, Y^\omega); P; I, \Lambda)$ is embedded as a convex pseudo-inverse subsemigroup in $\mathcal{M}(P(G', L', L'); P; I, \Lambda)$. Consequently $\underline{S}$ is embedded as a convex pseudo-inverse subsemigroup in $\mathcal{M}(P(G', L', L'); P; I, \Lambda)$.

We illustrate the proof of Theorem 3 by the following diagram.

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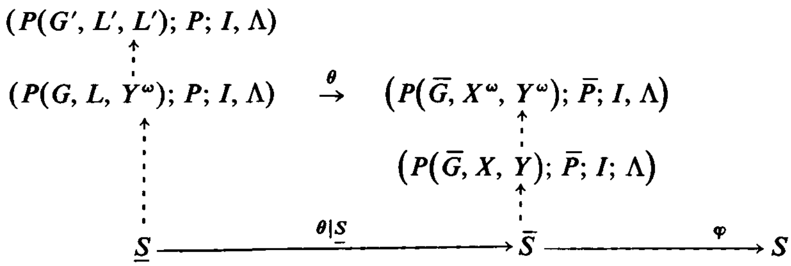

REMARK 3. We observe that the semigroup $\mathcal{M}(P(G', L', L'); P; I; \Lambda)$ which is mentioned in Theorem 3 has high symmetry. In particular, $\mathcal{M}(P(G', L', L'); P; I, \Lambda)$ is bisimple. By Theorem 4.20 of Pastijn (preprint a) there exist sets I', Λ', with $I \subseteq I'$ and $\Lambda \subseteq \Lambda'$, and a $\Lambda' \times I'$-matrix $P' = (p'_{\lambda i})$ with entries $p'_{\lambda i} \in G'$, where $p'_{\lambda i} = p_{\lambda i}$ for all $(\lambda, i) \in \Lambda \times I$, such that $\mathcal{M}(P(G', L', L'); P'; I'; \Lambda')$ is bisimple and idempotent generated. Thus $\underline{S}$ is embedded as a convex pseudo-inverse subsemigroup in $\mathcal{M}(P(G', L', L'); P'; I'; \Lambda')$, and so every pseudo-inverse semigroup S divides a pseudo-inverse semigroup of the form $\mathcal{M}(P(G', L', L'); P'; I', \Lambda')$, which is bisimple and idempotent generated.

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