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ELEMENTARY ABELIAN OPERATOR GROUPS AND ADMISSIBLE FORMATIONS

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Abstract

Suppose the elementary abelian group A acts on the group G where A and G have relatively prime orders. If $C_G(a)$ belongs to some formation $\mathfrak{F}$ for all non-identity elements a in A, does it follow that G belongs to $\mathfrak{F}$? For many formations, the answer is shown to be yes provided that the rank of A is sufficiently large.

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Suppose A is an elementary abelian r-group of order r^n which operates on the finite r'-group G. A frequently used method to study this situation is to look at the subgroups $C_G(a)$ for the non-identity elements $a \in A$ and to ask whether the structure of these subgroups gives any information about the structure of G as a whole. In this paper, we are interested in the following type of question: If $C_G(a) \in \mathfrak{F}$, where $\mathfrak{F}$ is some "nice" class of groups, for all $a \in A^\#$, does it follow that $G \in \mathfrak{F}$? For this question to have any hope of receiving an affirmative answer, we usually have to exclude certain small values of n. Thus we are trying to prove theorems of the following sort: If $C_G(a) \in \mathfrak{F}$ for all $a \in A^\#$ and if $n \geqslant n_0$ (where n_0 depends only on $\mathfrak{F}$), then $G \in \mathfrak{F}$. The solvable signalizer function theorem [5] implies such a result with $\mathfrak{F}$ the class of all finite solvable groups and $n_0 = 3$. Another example may be found in [14] where $\mathfrak{F}$ is the class of finite nilpotent groups and $n_0 = 3$. We will say that a class $\mathfrak{F}$ is admissible provided that such a result is true.

The main thrust of this paper is to determine sufficient conditions for a formation and, in particular, for a subgroup-closed saturated formation to be admissible. If $\mathfrak{F}$ is a subgroup-closed saturated formation we find sufficient

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conditions for $\mathfrak{F}$ to be admissible in terms of the local formations determining $\mathfrak{F}$ (Theorems 3.12 and 3.17). The basis of our results is the following theorem:

Assume $\mathfrak{F}$ is an admissible subgroup-closed formation and define $\mathfrak{G}$ by

$$\mathfrak{G} = \{G \mid G/K \in \mathfrak{F}\}.$$

(Here K is some specified characteristic subgroup of G. Examples of some of the possibilities for K which are covered by this paper are $F(G)$, $Z(G)$, $O_\pi(G)$, and $O_{\pi,\pi}(G)$ where π is any set of primes.) Then $\mathfrak{G}$ is an admissible subgroup-closed formation.

Using this, we may construct many admissible formations. For example, if $K = Z(G)$ and $\mathfrak{F}$ consists of all nilpotent groups of class at most c, then $\mathfrak{G}$ consists of all nilpotent groups of class at most $(c + 1)$. In this way, an easy induction yields that

$$\{G \mid G \text{ is a finite nilpotent group of class} \leqslant c\}$$

is admissible. (For $c = 1$, this had been done in [8].)

The groups G we consider need not be solvable. Here, we make use of a simple but rather striking consequence of the classification of all finite simple groups. Namely, if $n \geqslant 2$, we show that any composition factor group of G is isomorphic to a composition factor group of $C_G(a)$ for some $a \in A^{\#}$ (Theorem 3.1). All of our theorems may be made independent of the classification by adding the hypothesis that each composition factor group of G is one of the known simple groups. (In this paper, simple does not necessarily imply non-abelian.)

Although most of our results deal with formations, we also prove that certain other classes are admissible. For example, if $\mathfrak{F}$ is the class of all finite cyclic groups or if $\mathfrak{F}$ consists of all finite groups G such that a Sylow p-subgroup of G may be generated by at most d elements (where p and d are fixed), then $\mathfrak{F}$ is admissible. Neither of these examples is a formation and the second is not subgroup-closed if $d > 1$. On the other hand, we give an example in 3.18 of a subgroup-closed saturated formation which is not admissible. Two subgroup-closed formations whose admissibility is still open are the following:

(1) $\{G \mid G'' = 1\}$. (More generally, $\{G \mid G^{(m)} = 1\}$.)

(2) $\{G \mid x^p = 1 \text{ for all } x \in G\}$ where p is an odd prime.

2. Notation and introductory results

All groups considered in this paper are finite. $G^{\#}$ denotes the set of non-identity elements of G while $F(G)$ and $\Phi(G)$ denote the Fitting and Frattini subgroup, respectively, of G. $L_n(G)$ is the n-th term of the lower central series of G, that is,

$L_1(G) = G$ and $L_{n+1}(G) = [L_n(G), G]$. If G is a solvable group, then $l(G)$ denotes its nilpotent length. If G is a nilpotent group, then $\mathrm{cl}(G)$ denotes its class. $\mathrm{Aut}(G)$ is the automorphism group of G while $m(G)$ is the smallest number of elements necessary to generate G. If x is a positive real number, then $[x]$ is the largest integer $\leqslant x$. If V is a vector space, then $d(V)$ is its dimension. ·

Throughout, π denotes a set of primes. If π is neither empty nor the set of all primes, then π is said to be non-trivial. As usual π' is the set of primes not belonging to π. If G is a group, then $K_\pi(G) = O_\pi(G)O_{\pi'}(G)$. Clearly $K_\pi(G) = K_{\pi'}(G)$. If π (or π') consists of a single prime p, then we write $K_p(G)$. As in [5], a group G is called π- separable if each composition factor group of G is either a π-group or a π'-group. The π-length, $l_\pi(G)$, of the π-separable group G is defined in [6, page 226]. In our examples, we repeatedly use the fact that in a solvable group G, $l_\pi(G) \leqslant [(l(G) + 1)/2]$.

The Greek letter Λ is reserved to denote a partition of the set of primes, that is, the members of Λ are non-empty sets of primes and each prime belongs to exactly one member of Λ. If each member of Λ is a singleton set, then we call Λ the discrete partition. The group G is Λ-separable if G is π-separable for each $\pi \in \Lambda$. The subgroup $K_\Lambda(G)$ is defined by

$$K_\Lambda(G) = \bigcap_{\pi \in \Lambda} K_\pi(G) = \prod_{\pi \in \Lambda} O_\pi(G).$$

If Λ consists of just 2 sets, one of which is π, then $K_\Lambda(G) = K_\pi(G)$. If Λ is the discrete partition, then $K_\Lambda(G) = F(G)$ and a group is Λ-separable if, and only if, it is solvable.

Following [9], a group G satisfies C_π if G has exactly one conjugacy class of Hall π-subgroups. If, in addition, each π-subgroup of G is contained in a Hall π-subgroup of G, then G satisfies D_π. If G has a normal Hall π-subgroup (equivalently, if $G/O_\pi(G)$ is a π'-group), they we say that G is π-closed.

Any class $\mathfrak{F}$ of groups is to be understood to be closed under isomorphisms (that is, if $G \in \mathfrak{F}$ and $G \cong H$, then $H \in \mathfrak{F}$). The empty class is denoted by ϕ while any other classes will be denoted by script letters. A class $\mathfrak{F}$ is subgroup-closed if $G \in \mathfrak{F}$ and $H \leqslant G$ always implies $H \in \mathfrak{F}$. A class $\mathfrak{F}$ is admissible if there is a positive integer n such that the following statement is always true.

Suppose A is an elementary abelian group which operates on the group G. If $(|A|, |G|) = 1$, $C_G(a) \in \mathfrak{F}$ for all $a \in A^{\#}$, and $m(A) \geqslant n$, then $G \in \mathfrak{F}$.

If $\mathfrak{F}$ is admissible, then the smallest positive integer which will work for n is denoted by $n(\mathfrak{F})$.

A formation $\mathfrak{F}$ is a class of groups which is closed under taking homomorphic images and subdirect products. The $\mathfrak{F}$-residual of the group G is denoted by $G_{\mathfrak{F}}$

and is the intersection of all normal subgroups whose factor groups belong to $\mathfrak{F}$. $\mathfrak{F}$ is saturated if a group G belongs to $\mathfrak{F}$ whenever $G/\Phi(G)$ belongs.

Now suppose that $\mathfrak{F}(p)$ is a formation for each prime p and let $\pi = \{p \mid \mathfrak{F}(p) \neq \varnothing\}$. Define $\mathfrak{L}$ by

$$\mathfrak{L} = \{G \mid G \text{ is a } \pi\text{-group and } G/O_{p'p}(G) \in \mathfrak{F}(p) \text{ for all } p \in \pi\}.$$

Gaschütz showed that $\mathfrak{L}$ is a saturated formation [10, VI. 7.5]. Conversely, Schmid [11] proved that every saturated formation can be obtained in this way. For the purposes of this paper, better results may sometimes be obtained by working with a slightly different formation. Namely, define $\mathfrak{K}$ by

$$\mathfrak{K} = \{G \mid G \text{ is a } \pi\text{-group and } G/K_p(G) \in \mathfrak{F}(p) \text{ for all } p \in \pi\}.$$

To distinguish between them, we will say that $\mathfrak{L}$ is locally defined by $\{\mathfrak{F}(p)\}$ while $\mathfrak{K}$ is $\mathfrak{K}$-generated by $\{\mathfrak{F}(p)\}$. It is shown in 2.7 that $\mathfrak{K}$ is also a saturated formation. Furthermore, given any saturated formation $\mathfrak{F}$, it is always possible to find formations $\mathfrak{G}(p)$ such that $\mathfrak{F}$ is both locally defined by $\{\mathfrak{G}(p)\}$ and also K-generated by $\{\mathfrak{G}(p)\}$. This does not mean that $\mathfrak{L}$ and $\mathfrak{K}$ are always the same. For example, if $\mathfrak{F}(p)$ is the formation of all p'-groups for each p, then $\mathfrak{K}$ is the class of all nilpotent groups while $\mathfrak{L}$ consists of all solvable groups G satisfying $l_p(G) \leqslant 1$ for each p. Thus the group S_3 belongs to $\mathfrak{L}$ but not to $\mathfrak{K}$. It is always true that $\mathfrak{K} \subseteq \mathfrak{L}$.

We now list some basic results needed later. Most of these are well-known, easily proved, and require no comment.

2.1. *The class of all π-separable groups is closed under taking subgroups, factor groups, direct products, and extensions.*

2.2. *If G is π-separable, then G satisfies both D_π and $D_{\pi'}$.*

Proof. This follows from the Feit-Thompson Theorem [3] and [6, Theorems 6.3.5 and 6.3.6].

2.3. (i) If $H \leqslant G$, *then* $K_\Lambda(G) \cap H \leqslant K_\Lambda(H)$.
(ii) *If* $H \trianglelefteq G$, *then* $K_\Lambda(G)H/H \leqslant K_\Lambda(G/H)$.
(iii) $K_\Lambda(G_1 \times G_2) = K_\Lambda(G_1) \times K_\Lambda(G_2)$.
(iv) $K_\Lambda(G) \geqslant \Phi(G)$ *and* $K_\Lambda(G/\Phi(G)) = K_\Lambda(G)/\Phi(G)$.
(v) $K_p(G) \leqslant O_{p'p}(G)$ *and* $O_p(G/K_p(G)) = O_{p'p}(G)/K_p(G)$.

2.4. *If G is Λ-separable, then $C_G(K_\Lambda(G)) \leqslant K_\Lambda(G)$.*

Proof. Let $K = K_\Lambda(G)$ and $C = C_G(K)$. If $C \neq Z(K)$, then $C/Z(K)$ contains a minimal normal subgroup $H/Z(K)$ of $G/Z(K)$. Since G is Λ-separable,

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$H/Z(K)$ is a π-group for some $\pi \in \Lambda$. By 2.2, H must have a Hall π-subgroup L. Then, since $[H, Z(K)] = 1$, $H = L \times O_{\pi'}(Z(K))$. It follows from this that $L \trianglelefteq G$. Then $L \leqslant O_\pi(G) \leqslant K_\Lambda(G)$. Hence $H \leqslant K$ and we have a contradiction.

2.5. *Suppose A is an abelian group which operates on the group G with* $(|A|, |G|) = 1$. *Then the following are true*:

(i) $G = [G, A]C_G(a)$.

(ii) *If H is an A-invariant normal subgroup of G, then* $C_{G/H}(a) = C_G(a)H/H$.

(iii) *If A is not cyclic, then* $G = \langle C_G(a) \mid a \in A^\# \rangle$.

(iv) *If G satisfies C_π, then there is an A-invariant Hall π-subgroup H in G and $C_H(a)$ is a Hall π-subgroup of $C_G(a)$ for all $a \in A^\#$.*

(v) *If G is simple, then $A/C_A(G)$ is cyclic.*

Proof. Section 6.2 of [6] contains (i), (ii), and (iii). The first part of (iv) is well-known and so let H be an A-invariant Hall π-subgroup of G. If $p \in \pi$, then H must contain an A-invariant Sylow p-subgroup S of G. Similarly, since $C_G(a)$ is A-invariant, there is an A-invariant Sylow p-subgroup P of $C_G(a)$. Then there is an $x \in C_G(A)$ such that $x^{-1}Px \leqslant S$ [6, Theorem 6.2.2]. But $C_G(A)$ is contained in $C_G(a)$ and so

$$x^{-1}Px \leqslant S \cap C_G(a) \leqslant H \cap C_G(a) = C_H(a).$$

Hence $C_H(a)$ is a π-subgroup of $C_G(a)$ and $C_H(a)$ contains a Sylow p-subgroup of $C_G(a)$ for all $p \in \pi$. This implies that $C_H(a)$ is a Hall π-subgroup of $C_G(a)$.

Now (v) depends upon the recently completed classification of all simple groups. For if $B = A/C_A(G)$, then $B \leqslant \mathrm{Aut}(G)$ and $(|B|, |G|) = 1$. If G is a sporadic group or an alternating group, then this forces $B = 1$ (see [1] and [4] for a description of $\mathrm{Aut}(G)$ when G is a sporadic group). If G is a Chevalley group, then it follows from [13] that B is isomorphic to a group of automorphisms of some finite field. Hence B is cyclic in this case.

2.6. *Assume that $\mathfrak{F}$ is a non-empty formation. Then*

(i) *If $H \trianglelefteq G$, then* $(G/H)_{\mathfrak{F}} = G_{\mathfrak{F}}H/H$.

(ii) *If $\mathfrak{F}$ is subgroup-closed and $H \leqslant G$, then* $H_{\mathfrak{F}} \leqslant H \cap G_{\mathfrak{F}}$.

(iii) *If $\mathfrak{G} = \{G \mid G/O_\pi(G) \in \mathfrak{F}\}$, then $\mathfrak{G}$ is a formation and $\mathfrak{G}$ is subgroup-closed if $\mathfrak{F}$ is.*

2.7. I. *Assume that $\mathfrak{F}(p)$ is a formation for each prime p. Let $\{\mathfrak{F}(p)\}$ locally define $\mathfrak{L}$ and K-generate $\mathfrak{K}$. Let $\pi = \{p \mid \mathfrak{F}(p) \neq \varnothing\}$.*
Then the following are true:

(i) *$\mathfrak{L}$ and $\mathfrak{K}$ are both saturated formations.*

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(ii) $\mathfrak{L} \supseteq \mathfrak{K}$.

(iii) *If $\mathfrak{F}(p)$ is subgroup-closed for each $p \in \pi$, then both $\mathfrak{L}$ and $\mathfrak{K}$ are subgroup-closed.*

(iv) *Define $\mathfrak{G}(p)$ by $\mathfrak{G}(p) = \varnothing$ if $p \notin \pi$ and $\mathfrak{G}(p) = \{G \mid G/O_p(G) \in \mathfrak{F}(p)\}$ if $p \in \pi$. Then $\mathfrak{L}$ is both locally defined and K-generated by $\{\mathfrak{G}(p)\}$.*

II. *If $\mathfrak{F}$ is a non-empty saturated formation, then there are formations $\mathfrak{L}(p)$, one for each prime p, such that $\mathfrak{F}$ is both locally defined and K-generated by $\{\mathfrak{L}(p)\}$.*

PROOF. Using 2.3 and [10, VI.7], we easily derive (i), (ii) (since $G/O_{p'p}(G)$ is a homomorphic image of $G/K_p(G)$), and (iii). Now $O_p(G/O_{p'p}(G)) = 1$ and so $G/O_{p'p}(G) \in \mathfrak{G}(p)$ if, and only if, $G/O_{p'p}(G) \in \mathfrak{F}(p)$. Hence $\{\mathfrak{G}(p)\}$ locally defines $\mathfrak{L}$. Since $O_p(G/K_p(G)) = O_{p'p}(G)/K_p(G)$, we see that $G/K_p(G) \in \mathfrak{G}(p)$ if, and only if, $G/O_{p'p}(G) \in \mathfrak{F}(p)$. This implies that $\{\mathfrak{G}(p)\}$ K-generates $\mathfrak{L}$. Thus I is proved.

Now any saturated formation is locally defined [12]. Using I(iv), we see that II follows.

The next result follows immediately from the definitions but is very useful in determining whether a class is admissible.

2.8. *Let I be a non-empty set and suppose that for each $i \in I$, $\mathfrak{F}_i$ is an admissible class of groups. Assume that $\{n(\mathfrak{F}_i) \mid i \in I\}$ has an upper bound. If $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$, then $\mathfrak{F}$ is admissible and $n(\mathfrak{F}) \leqslant \sup\{n(\mathfrak{F}_i) \mid i \in I\}$.*

PROOF. Suppose that A is an elementary abelian group which operates on the group G with $(|A|, |G|) = 1$. Assume that $C_G(a) \in \mathfrak{F}$ for all $a \in A^{\#}$. If $m(A) \geqslant n(\mathfrak{F}_i)$ for all $i \in I$, then the admissibility of $\mathfrak{F}_i$ implies that $G \in \mathfrak{F}_i$ for all $i \in I$. But then $G \in \mathfrak{F}$.

For our next results in this introductory section, we present two simple methods of producing new admissible classes from old ones.

2.9. *Suppose that $\mathfrak{A}$ is an admissible class of groups such that every group in $\mathfrak{A}$ satisfies C_π. Let $\mathfrak{B}$ be an admissible class of groups and define $\mathfrak{F}$ by*

$$\mathfrak{F} = \{G \mid G \in \mathfrak{A} \text{ and a Hall } \pi\text{-subgroup of } G \text{ belongs to } \mathfrak{B}\}.$$

Then $\mathfrak{F}$ is admissible and $n(\mathfrak{F}) \leqslant \operatorname{Max}\{n(\mathfrak{A}), n(\mathfrak{B})\}$.

PROOF. Assume A acts on G, $(|A|, |G|) = 1$, A is elementary abelian, $C_G(a) \in \mathfrak{F}$ for all $a \in A^{\#}$, and $m(A) \geqslant \operatorname{Max}\{n(\mathfrak{A}), n(\mathfrak{B})\}$. Since $C_G(a) \in \mathfrak{A}$ and $m(A) \geqslant n(\mathfrak{A})$, G must belong to $\mathfrak{A}$. Then G satisfies C_π. By 2.5(iv), G has an A-invariant

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Hall π-subgroup H and $C_H(a)$ is a Hall π-subgroup of $C_G(a)$. Since $C_G(a)$ satisfies C_π (since $C_G(a) \in \mathfrak{A}$), $C_H(a)$ must belong to $\mathfrak{B}$ for all $a \in A^\#$. Since $m(A) \geqslant n(\mathfrak{B})$, $H \in \mathfrak{B}$. But then $G \in \mathfrak{F}$.

REMARK. If $\pi = \{p\}$, then $\mathfrak{A}$ may be the class of all groups.

2.10. *Suppose that $T(G)$ is a characteristic subgroup of G for each group G. Assume that the following hold*:

(i) *If σ is an isomorphism of G onto H, then*

$$T(H) = (T(G))^\sigma.$$

(ii) *If $H \leqslant G$, then $T(G) \cap H \leqslant T(H)$.*

Assume that $\mathfrak{A}$ is an admissible subgroup-closed class of groups and define $\mathfrak{B}$ by

$$\mathfrak{B} = \{G \mid T(G) \in \mathfrak{A}\}.$$

Then $\mathfrak{B}$ is admissible and $n(\mathfrak{B}) \leqslant n(\mathfrak{A})$.

PROOF. First, note that there are many choices for $T(G)$ that satisfy (i) and (ii). For example, $T(G)$ could be any of the following: $Z(G)$, $F(G)$, $O_\pi(G)$, $K_\Lambda(G)$, $O_{\pi'\pi}(G)$.

Now suppose A acts on G, $(|A|, |G|) = 1$, A is elementary abelian, $C_G(a) \in \mathfrak{B}$ for all $a \in A^\#$, and $m(A) \geqslant n(\mathfrak{A})$. If $H = T(G)$, then $C_H(a) = H \cap C_G(a) \leqslant T(C_G(a))$. Since $C_G(a) \in \mathfrak{B}$ and $\mathfrak{A}$ is subgroup-closed, $C_H(a) \in \mathfrak{A}$ for all $a \in A^\#$. Since $m(A) \geqslant n(a)$, we see that $H \in \mathfrak{A}$ and $G \in \mathfrak{B}$.

To illustrate 2.10, consider groups whose center is the identity. Now if $\mathfrak{A}$ is the identity class, (that is, $G \in \mathfrak{A}$ if, and only if, $|G| = 1$), then $\mathfrak{A}$ is subgroup-closed and it follows from 2.5(iii) that $n(\mathfrak{A}) = 2$. If

$$\mathfrak{B} = \{G \mid Z(G) = 1\},$$

then 2.10 implies that $\mathfrak{B}$ is admissible and $n(\mathfrak{B}) \leqslant 2$.

Virtually all of the admissible classes to be considered later are subgroup-closed. The example just given is an exception. (If $H \leqslant G$ and $Z(G) = 1$, it does not follow that $Z(H) = 1$.) Another exception is given in 2.12 below. This depends upon the following easy result about modules for elementary abelian groups.

2.11. *Let A be an elementary abelian r-group and let F be a field of characteristic $\neq r$. Let s be the degree of λ over F where λ is a primitive r-th root of unity in the algebraic closure of F. Assume that V is an FA-module and that $d(C_V(a)) \leqslant n$ for all $a \in A^\#$. If $m(A) \geqslant 1 + (n+1)/s$, then $d(V) \leqslant n$.*

PROOF. If U is any irreducible FA-module and $C_A(U) \neq A$, then our assumptions imply that $d(U) = s$. If $a \in A^{\#}$, then $C_V(a)$ is the direct sum of irreducible FA-submodules. It follows that if $n < s$, then $C_V(a) = C_V(A)$ for all $a \in A^{\#}$. But then each element of A acts fixed point-freely on $V/C_V(A)$. This is impossible if $V \neq C_V(A)$ since A is not cyclic ($m(A) > 1$). Hence if $n < s$, $V = C_V(A) = C_V(a)$ and so $d(V) \leqslant n$.

Assume now that $n \geqslant s$ and proceed by induction on n. If $C_A(V) \neq 1$, then $V \leqslant C_V(a)$ for some $a \in A^{\#}$ and the result is trivial. Thus assume $C_A(V) = 1$. V must contain a non-trivial irreducible FA-submodule U. Then $d(U) = s$ and $|A/C_A(U)| = r$. If $B = C_A(U)$, then $m(B) = m(A) - 1 \geqslant (n+1)/s$. Also $C_V(b) \supseteq U$ for all $b \in B^{\#}$. Hence

$$d(C_{V/U}(b)) = d(C_V(b)) - d(U) \leqslant n - s.$$

By induction then, $d(V/U) \leqslant n - s$ and so $d(V) \leqslant n$.

2.12. *Let p be a prime, let d be a non-negative integer, and let $\mathfrak{F}$ be the class of all groups G such that $m(P) \leqslant d$ where P is a Sylow p-subgroup of G. Then $\mathfrak{F}$ is admissible and*

$$n(\mathfrak{F}) = \begin{cases} d+2 & \text{if } p \neq 2, \\ \left[\dfrac{d}{2}\right] + 2 & \text{if } p = 2. \end{cases}$$

PROOF. If $d = 0$, then the requirement on P is that $P = 1$. Since $n(\{\text{identity}\}) = 2$, the result follows from 2.9 in this case. Assume now that $d \geqslant 1$ and that $m = d + 2$ if p is odd and $m = [d/2] + 2$ if $p = 2$.

Assume that A acts on G, $(|A|, |G|) = 1$, A is an elementary abelian r-group, $C_G(a) \in \mathfrak{F}$ for all $a \in A^{\#}$, and $m(A) \geqslant m$. Then by 2.5(iv), there is an A-invariant Sylow p-subgroup P and $C_P(a)$ is a Sylow p-subgroup of $C_G(a)$. Hence $m(C_P(a)) \leqslant d$ for all $a \in A^{\#}$.

Let V be $P/\Phi(P)$ written additively. Then V is a $GF(p)A$-module and $d(C_V(a)) \leqslant d$ for all $a \in A^{\#}$. Let s be the degree of a primitive r-th root of unit over $GF(p)$. Then $s \geqslant 1$ and so

$$1 + \frac{d+1}{s} \leqslant 1 + (d+1) = m \leqslant m(A).$$

if $p \neq 2$. Hence, if $p \neq 2$, it follows from 2.11 that $|P/\Phi(P)| \leqslant p^d$ and so $G \in \mathfrak{F}$. If $p = 2$, then s must be at least 2. In that case

$$1 + \frac{d+1}{s} \leqslant 1 + \frac{d+1}{2} \leqslant \left[\frac{d}{2}\right] + 2 \leqslant m(A).$$

Hence, again by 2.11, $|P/\Phi(P)| \leqslant p^d$ and so $G \in \mathfrak{F}$.

So far, we have proved that $\mathfrak{F}$ is admissible and that $n(\mathfrak{F}) \leqslant m$. To prove that $n(\mathfrak{F})$ cannot be any smaller, we construct some examples.

If $p \neq 2$, let r be any prime dividing $p - 1$ ($r = 2$, for example). Let G be an elementary abelian group of order p^{d+1}. Then $\mathrm{Aut}(G)$ contains an elementary abelian r-group A of order r^{d+1}. Then, if $a \in A^{\#}$, $C_G(a) < G$ and so $m(C_G(a)) \leqslant d$. Thus $C_G(a) \in \mathfrak{F}$ for all $a \in A^{\#}$ but $G \notin \mathfrak{F}$.

If $p = 2$, let $n = [d/2]$ and let G be an elementary abelian group of order 2^{2n+2}. Then $\mathrm{Aut}(G)$ contains an elementary abelian 3-group A of order 3^{n+1}. Since for all $a \in A^{\#}$, a must act faithfully on $G/C_G(a)$, we must have $|C_G(a)| \leqslant 2^{2n}$. Then $C_G(a) \in \mathfrak{F}$ for all $a \in A^{\#}$ but $G \notin \mathfrak{F}$.

REMARK. If $d > 1$, then $\mathfrak{F}$, the class in the above result, is not subgroup-closed. Nor is $\mathfrak{F}$ a formation since $\mathfrak{F}$ is not closed under direct products.

3. The main results

The next result is a direct consequence of the classification of all simple groups. The theorem could be made independent of the classification by adding the assumption that the composition factor group in question is a known simple group.

3.1. THEOREM. *Suppose that A is an abelian but not cyclic group which operates on the group G with $(|A|, |G|) = 1$. Then any composition factor group of G occurs as a composition factor group of some $C_G(a)$ with $a \in A^{\#}$.*

PROOF. Replacing A by one of its subgroups, if necessary, we may assume that A is elementary abelian and $m(A) = 2$. Let M be a minimal A-invariant normal subgroup of G. By induction, any composition factor group of G/M occurs as a composition factor group of $C_{G/M}(a)$ for some $a \in A^{\#}$. Now $C_{G/M}(a) \cong C_G(a)/C_M(a)$ and so the theorem will be proved once we show that any composition factor group of M occurs as a composition factor group of $C_M(a)$ for some $a \in A^{\#}$. Thus it suffices to prove the theorem when $M = G$.

Hence G is a minimal normal subgroup of GA. Thus, if G is abelian, all composition factor groups of G and of any subgroup of G are the same. Since $G = \langle C_a(a) \mid a \in A^{\#} \rangle$, there is an $a \in A^{\#}$ such that $C_G(a) \neq 1$. The theorem now follows.

Assume, therefore, that G is non-abelian. Then

$$G = S_1 \times \cdots \times S_n$$

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where $S_1, \dots, S_n$ are isomorphic, simple, non-abelian groups and A must permute $\{S_1, \dots, S_n\}$ transitively. Since A is abelian, we must have

$$N_A(S_1) = N_A(S_2) = \cdots = N_A(S_n).$$

Thus if $n > 1$, there is an $a \in A^{\#}$ such that

$$\langle a \rangle \cap N_A(S_i) = 1$$

for all i. But then $C_G(a)$ is the direct product of $(n/|\langle a \rangle|)$ copies of S_1 and so certainly the theorem is true in this case.

Finally, assume $n = 1$. Then G is a simple group and so by 2.5(v), $A/C_A(G)$ is cyclic. Then $C_A(G) \neq 1$ and so $C_G(a) = G$ for some $a \in A^{\#}$. Thus the theorem is proved.

3.2. Corollary. *$n(\mathfrak{F}) \leq 2$ if $\mathfrak{F}$ is any of the following classes*: $\{G \mid G$ *is solvable*$\}$, $\{G \mid |G| = 1\}$, $\{G \mid G$ *is a π-group*$\}$, $\{G \mid G$ *is Γ-separable*$\}$.

Proof. For each of these classes, $G \in \mathfrak{F}$ if, and only if, each composition factor group of G belongs to $\mathfrak{F}$. The corollary now follows.

More generally, suppose $\mathfrak{S}$ is a class of simple groups (and here simple does not necessarily imply non-abelian) and let $\mathfrak{F}$ consist of those groups G such that each composition factor group of G belongs to $\mathfrak{S}$. Then $\mathfrak{F}$ is admissible and $n(\mathfrak{F}) \leq 2$.

The next result is the main theorem of this paper.

3.3. Theorem. *Let $\mathfrak{F}$ be an admissible subgroup-closed formation. Let $\mathfrak{G} = \{G \mid G/K_\Lambda(G) \in \mathfrak{F}\}$. Then $\mathfrak{G}$ is an admissible, saturated, subgroup-closed formation and $n(\mathfrak{G}) \leq n(\mathfrak{F}) + 1$.*

Proof. Using 2.3, we easily conclude that $\mathfrak{G}$ is a subgroup-closed saturated formation. Assume now that A acts on G, $(|A|, |G|) = 1$, A is elementary abelian, $C_G(a) \in \mathfrak{G}$ for all $a \in A^{\#}$, and $m(A) \geq n(\mathfrak{F}) + 1$. We need to prove that $G \in \mathfrak{G}$. Suppose that G is a minimal counterexample. Let $M = G_{\mathfrak{G}}$ and $K = G_{\mathfrak{F}}$. Then M and K must be non-identity characteristic subgroups of G. Let π be some member of Λ such that π contains at least one prime dividing $|M|$. We now proceed in a series of steps.

1. (i) $C_A(G) = 1$.
(ii) *If H is any non-identity A-invariant normal subgroup of G, then $H \geq M$.*
(iii) *If H is any A-invariant proper subgroup of G, then $H \in \mathfrak{G}$.*
(iv) *Either $K_\Lambda(G) = 1$ or $K_\Lambda(G) = O_\pi(G) \geq M$.*

PROOF. If $C_A(G) \neq 1$, then for some $a \in A^{\#}$, $G = C_G(a) \in \mathfrak{G}$. If $1 < H \lhd GA$ and $H \leqslant G$, then the minimality of G implies that $G/H \in \mathfrak{G}$. But then $H \geqslant M$. The minimality of G together with the fact that $\mathfrak{G}$ is subgroup-closed imply (iii). Finally, suppose $K_\Lambda(G) \neq 1$. Then $K_\Lambda(G) \geqslant M$ from (ii). Since $K_\Lambda(G)$ is the direct product of a π-group and a π'-group, it follows from (ii) that $K_\Lambda(G)$ is either a π-group or a π'-group. Since M is not a π'-group, (iv) follows.

2. $C_G(M) = Z(M)$.

PROOF. Suppose $C_G(M) \neq Z(M)$. then $C_G(M)$ is a non-identity A-invariant normal subgroup of G. Then $C_G(M) \geqslant M$ and so M is abelian. Then $M \leqslant F(G) \leqslant K_\Lambda(G)$ and so $K_\Lambda(G) \neq 1$. This implies that $M \leqslant O_\pi(G) = K_\Lambda(G)$. Since $G/M \in \mathfrak{G}$ and $K = G_{\mathfrak{F}}$, we must have $K/M \leqslant K_\Lambda(G/M)$. However, $G \notin \mathfrak{F}$ and so $K \nleqslant K_\Lambda(G)$. It follows from this that K/M is not a π-group. Since K must be π-separable (since $K/M \leqslant K_\Lambda(G/M)$), 2.2 implies that K satisfies $D_{\pi'}$. Then there must be an A-invariant Hall π'-subgroup S in K. Then $SM/M = O_{\pi'}(K/M)$. It follows from this that $SM \trianglelefteq G$. Then $G = MN_G(S)$. Since $S \neq 1$, $N_G(S) \neq G$. Since M is abelian, $M \cap N_G(S)$ is an A-invariant normal subgroup of G. Then we must have $N_G(S) \cap M = 1$. Then $C_G(M) \cap N_G(S)$ is an A-invariant normal subgroup of G which does not contain M. Hence $C_G(M) \cap N_G(S) = 1$. It now follows that $C_G(M) = M$.

3. *Let* $B = C_A(G/M)$. *Then*
(i) *If* $a \in A - B$, *then* $C_G(a)K_\Lambda(G)/K_\Lambda(G) \in \mathfrak{F}$.
(ii) $m(B) \geqslant 2$.

PROOF. Let $a \in A - B$, $H = C_G(a)M$ and $L = H_{\mathfrak{F}}$. Since $a \notin B$, $H < G$. Then, from (1(iii)), $H \in \mathfrak{G}$. Hence $L \leqslant K_\Lambda(H)$. Now if $L = 1$, then $H \in \mathfrak{F}$, and, since $\mathfrak{F}$ is subgroup-closed, $C_G(a) \in \mathfrak{F}$ and (i) follows. Assume now that $L \neq 1$. Then $K_\Lambda(H) \neq 1$ and

$$[M, K_\Lambda(H)] \leqslant M \cap K_\Lambda(H) \leqslant K_\Lambda(M) \leqslant K_\Lambda(G).$$

Now if $M \cap K_\Lambda(H) = 1$, then $K_\Lambda(H) \leqslant C_G(M) \leqslant M$ which contradicts $K_\Lambda(H) \neq 1$. We now see that $K_\Lambda(G) \neq 1$. But then $K_\Lambda(G) = O_\pi(G) \geqslant M$. Now $[O_{\pi'}(H), M] = 1$ and so $K_\Lambda(H) = O_\pi(H)$. Thus L is a π-group. Since $L = H_{\mathfrak{F}} \leqslant G_{\mathfrak{F}} = K$ and $K/M \leqslant K_\Lambda(G/M)$ (since $G/M \in \mathfrak{F}$), we must have $LM/M \leqslant O_\pi(G/M)$. It follows from this that $L \leqslant O_\pi(G)$. This implies that $C_G(a)K_\Lambda(G)/K_\Lambda(G) \in \mathfrak{F}$.

Now suppose $m(B) < 2$. Then A contains a subgroup B_0 such that $A = B \times B_0$ and $m(B_0) \geqslant m(A) - 1 \geqslant n(\mathfrak{F})$. Since, by (i), $C_G(b)K_\Lambda(G)/K_\Lambda(G) \in \mathfrak{F}$ for all $b \in B_0^{\#}$, this would imply that $G/K_\Lambda(G) \in \mathfrak{F}$ and so $G \in \mathfrak{G}$.

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4. *Let* $C = C_G(B)$ *and* $D = C_{\mathfrak{F}}$. *Then*
(i) $G = CM$.
(ii) $C \in \mathfrak{G}$.
(iii) *If* $K_\Lambda(G) \neq 1$, *then* $D \leq O_\pi(G)$.
(iv) $K_\Lambda(G) = 1$.

PROOF. B centralizes G/M and so $G = CM$. $C_A(G) = 1$ and so $C < G$. Then, by (1(iii)), $C \in \mathfrak{G}$. Therefore, $D \leq K_\Lambda(C)$. Suppose now that $K_\Lambda(G) \neq 1$. Then $K_\Lambda(G) = O_\pi(G) \geq M$. Now $D = O_\pi(D) \times O_{\pi'}(D)$, $M = \langle C_M(b) \mid b \in B^\# \rangle$, and, since $C_G(b) \geq C$, $D \leq (C_G(b))_{\mathfrak{F}}$ for $b \in B^\#$. Since $C_G(b) \in \mathfrak{G}$, we have $O_{\pi'}(D) \leq O_{\pi'}(C_G(b))$. This implies that

$$[O_{\pi'}(D), C_M(b)] \leq [O_{\pi'}(C_G(b)), O_\pi(C_G(b))] = 1.$$

Hence $O_{\pi'}(D)$ centralizes M. Since $C_G(M) \leq M \leq O_\pi(G)$, $O_{\pi'}(D) = 1$. Then D is a π-group. But $D < G_{\mathfrak{F}} = K$ and $K/M \leq K_\Lambda(G/M)$. It now follows that $D \leq O_\pi(G)$.

If $K_\Lambda(G) \neq 1$, we have just shown that $CO_\pi(G)/O_\pi(G)$ belongs to $\mathfrak{F}$. But $G = CM = CO_\pi(G)$ and so $G/O_\pi(G) \in \mathfrak{F}$. Since this would mean that $G \in \mathfrak{G}$, we must have $K_\Lambda(G) = 1$.

5. $M = S_1 \times \cdots \times S_n$ *where* $S_1, \ldots, S_n$ *are isomorphic, simple, non-abelian groups which are permuted transitively by* CA. *Also*, $C_G(M) = 1$.

PROOF. M is a minimal normal subgroup of $GA = MCA$. If $M' = 1$, then $M \leq K_\Lambda(G)$. Since $K_\Lambda(G) = 1$, $M' \neq 1$. (5) now follows. ($C_G(M) = Z(M) = 1$.)

6. *Let* $B_1 = N_B(S_1)$. *Then*
(i) $B_1 = N_B(S_k)$ *for all* k, $1 \leq k \leq n$.
(ii) $C_B(S_k) = 1$ *for all* k, $1 \leq k \leq n$.
(iii) $m(B_1) \leq 1$.
(iv) *If* $b \in B - B_1$, *then* $K_\Lambda(C_M(b)) = [D, C_M(b)] = 1$.

PROOF. CA permutes $\{S_1, \ldots, S_n\}$ transitively and $B \leq Z(CA)$. It follows from this that $B_1 = N_B(S_K)$ for all k and that $C_B(S_1) = C_B(S_2) = \cdots = C_B(S_n)$. Since $C_B(M)$ centralizes both M and G/M, $C_B(M) \leq C_A(G) = 1$. Thus (i) and (ii) are proved. Using 2.5(v), we obtain (iii).

Now suppose $b \in B - B_1$. Then $\langle b \rangle$ permutes $\{S_1, \ldots, S_n\}$ semi-regularly. Then $C_M(b)$ is the direct product of $(n/|\langle b \rangle|)$ copies of S_1. Now S_1 is not Λ-separable since $K_\Lambda(G) = 1$. It now follows that $K_\Lambda(C_M(b)) = 1$. But $C_G(b) \geq C$ and so $D = C_{\mathfrak{F}} \leq (C_G(b))_{\mathfrak{F}} \leq K_\Lambda(C_G(b))$ since $C_G(b) \in \mathfrak{G}$. Then $[D, C_M(b)] \leq K_\Lambda(C_G(b)) \cap C_M(b) \leq K_\Lambda(C_M(b)) = 1$.

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7. $D = 1$.

PROOF. Suppose $D \neq 1$. Then $[D, M] \neq 1$. Then, without loss of generality, we may assume that $[D, S_1] \neq 1$. Since $m(B) \geqslant 2 > m(B_1)$ there exists an element $b \in B - B_1$. Then, without loss of generality, we may assume that $\langle b \rangle$ transitively permutes $\{S_1, S_2, \ldots, S_r\}$. Then there is a diagonal subgroup S of $S_1 \times S_2 \times \cdots \times S_r$ which is a direct factor of $C_M(b)$. Then, from (6(iv)), $[D, S] = 1$. It follows from this that D normalizes $S_1 \times S_2 \times \cdots \times S_r$. Now if $\bar{D}$ and $\bar{b}$ denote the permutations induced on $\{S_1, \ldots, S_r\}$, then $\langle \bar{b} \rangle$ is an abelian regular group and since $[D, b] = 1$, it follows that $\bar{D} \leqslant \langle \bar{b} \rangle$. But $(|D|, |\langle b \rangle|) = 1$. Hence D must normalize S_k for $1 \leqslant k \leqslant r$. Then, since D centralizes S and D normalizes S_1, we must have $[D, S_1] = 1$.

8. *Contradiction.*

PROOF. A must contain a subgroup A_1 such that $A = A_1 \times B_1$. Then $m(A_1) = m(A) - m(B_1) \geqslant m(A) - 1 \geqslant n(\mathfrak{F})$. Now $G \notin \mathfrak{F}$. then there must be an $a \in A_1^{\#}$ such that $C_G(a) \notin \mathfrak{F}$. But if $a \notin B$, then it follows from (3(i)) (since $K_\Lambda(G) = 1$) that $C_G(a) \in \mathfrak{F}$. Hence $a \in B \cap A_1$. Then $C_G(a) \geqslant C$ and so $C_G(a) = C_M(a)C$. Since $C_{\mathfrak{F}} = D = 1$, we have $C_G(a)/C_M(a) \in \mathfrak{F}$. Since $C_G(a) \in \mathfrak{G}$ by hypothesis, we must have

$$(C_G(a))_{\mathfrak{F}} \leqslant C_M(a) \cap K_\Lambda(C_G(a)) \leqslant K_\Lambda(C_M(a)).$$

Since $K_\Lambda(C_M(a)) = 1$ by (6(iv)), we have $C_G(a) \in \mathfrak{F}$ and so the proof is complete.

The next result, which was proved in [8] and independently by Jones [11], now follows immediately.

3.4. COROLLARY. *For each non-negative integer k, let $\mathfrak{N}_k$ denote the class of all solvable groups of nilpotent length $\leqslant k$. Then $\mathfrak{N}_k$ is admissible and $n(\mathfrak{N}_k) = k + 2$.*

PROOF. $\mathfrak{N}_0$ is the class of identity groups and so $n(\mathfrak{N}_0) = 2$. Now let Λ be the discrete partition. Then $K_\Lambda(G) = F(G)$, and, for $k \geqslant 1$, $G \in \mathfrak{N}_k$ if, and only if, $G/F(G) \in \mathfrak{N}_{k-1}$. Using the theorem and induction on k, we obtain $n(\mathfrak{N}_k) \leqslant k + 2$ for all k. Examples in [8] show that $n(\mathfrak{N}_k)$ cannot be less than $k + 2$.

REMARKS. 1. $\mathfrak{N}_1$ is the class of all nilpotent groups and so the corollary includes Ward's result [14].

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2. By analogy with nilpotent length, one could define a Λ-length for any Λ-separable group (that is $K_0 = 1$ and $K_{n+1}(G)/K_n(G) = K_\Lambda(G/K_n(G))$. Then the same argument as in 3.4 may be used to prove that $n(\{G \mid K_k(G) = G\}) \leqslant k + 2$.

The special case $\pi = \{p\}$ of the next result appears in [15].

3.5. COROLLARY. *If π is non-trivial and if $\mathcal{C}$ is the class of all π-closed groups, then $\mathcal{C}$ is admissible and $n(\mathcal{C}) = 3$.*

PROOF. Let $\Lambda = \{\pi, \pi'\}$ and let $\mathfrak{F}$ be the class of all π'-groups. Then $n(\mathfrak{F}) = 2$ by 3.2 and $\mathcal{C} = \{G \mid G/K_\Lambda(G) \in \mathfrak{F}\}$. The theorem now yields $n(\mathcal{C}) \leqslant 3$. Example 1 in §4 shows that $n(\mathcal{C})$ cannot be any smaller.

3.6. COROLLARY. *If $\prec$ is a total ordering of the set of all primes and if $\mathcal{C}$ is the class of all groups which have a Sylow tower of type $\prec$, then $n(\mathcal{C}) = 3$.*

PROOF. Sylow tower of type $\prec$ is defined in [10, VI.6.13] where it is shown that $G \in \mathcal{C}$ if, and only if, G is π_i-closed for all $i = 1, 2, \ldots$ and $\pi_1, \pi_2, \ldots$ are sets of primes depending on $\prec$. Thus $\mathcal{C}$ is the intersection of classes of the type in the previous corollary. Using that result and 2.8, we easily obtain the result.

3.7. THEOREM. *Let $\mathfrak{F}$ be an admissible, subgroup-closed formation and let*

$$\mathcal{G} = \{G \mid G/Z(G) \in \mathfrak{F}\}.$$

Then $\mathcal{G}$ is an admissible, subgroup-closed formation and $n(\mathcal{G}) \leqslant n(\mathfrak{F}) + 1$.

PROOF. It is easily verified that $\mathcal{G}$ is a subgroup-closed formation. Now suppose A acts on G, $(|A|, |G|) = 1$, A is elementary abelian, $C_G(a) \in \mathcal{G}$ for all $a \in A^{\#}$, and $m(A) \geqslant n(\mathfrak{F}) + 1$. Assume that G is a minimal example such that $G \notin \mathcal{G}$. Let M be the $\mathcal{G}$-residual of G. Then $M \neq 1$.

1. (i) *If H is a non-identity A-invariant normal subgroup of G, then $H \geqslant M$.*
(ii) $G_{\mathfrak{F}}/M \leqslant Z(G/M)$.

PROOF. In (i), $G/H \in \mathcal{G}$ by the minimality of G. This implies $H \geqslant M$. Now $G \notin \mathcal{G}$ and so $G \notin \mathfrak{F}$. Then $G_{\mathfrak{F}} \geqslant M$ by (i). Now $G/M \in \mathcal{G}$ and so $G_{\mathfrak{F}}/M \leqslant Z(G/M)$.

2. $Z(G) = 1$.

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PROOF. Suppose $Z(G) \neq 1$. Then $M \leqslant Z(G)$. Then M is an irreducible A-module and so $A/C_A(M)$ must be cyclic. Then $m(C_A(M)) \geqslant n(\mathfrak{F})$. Let $a \in (C_A(M))^{\#}$, $C = C_G(a)$, and $D = C_{\mathfrak{F}}$. Then $C \in \mathfrak{G}$ and so $D \leqslant Z(C)$. Also $D \leqslant G_{\mathfrak{F}}$ and so $[G, D] \leqslant M$. Then $[G, D, \langle a \rangle] \leqslant [M, \langle a \rangle] = 1$. Also $[D, \langle a \rangle, G] = 1$. Hence $[G, \langle a \rangle, D] = 1$. Since $G = [G, \langle a \rangle]C$ and since $[D, C] = 1$, we obtain $[D, G] = 1$. This implies that $C_{G/Z(G)}(a) \in \mathfrak{F}$ for all $a \in C_A(M)$. This in turn implies that $G/Z(G) \in \mathfrak{F}$, contrary to $G \notin \mathfrak{G}$. Hence $Z(G) = 1$.

3. *If* $B \leqslant A$ *and* $m(B) \geqslant 2$, *then* $C_G(B) \in \mathfrak{F}$.

PROOF. Let $C = C_G(B)$ and $D = C_{\mathfrak{F}}$. Let $b \in B^{\#}$. Then $C_G(b) \geqslant C$. Since $C_G(b) \in \mathfrak{G}$,

$$D \leqslant (C_G(b))_{\mathfrak{F}} \leqslant Z(C_G(b)).$$

Thus $[D, C_G(b)] = 1$ for all $b \in B^{\#}$. Since $m(B) \geqslant 2$, $G = \langle C_G(b) \mid b \in B^{\#} \rangle$. Hence $[D, G] = 1$. Since $Z(G) = 1$, we obtain $D = 1$ and so $C \in \mathfrak{F}$.

4. $C_G(a) \in \mathfrak{F}$ *for all* $a \in A^{\#}$.

PROOF. Let $a \in A^{\#}$ and $H = C_G(a)$. A has a subgroup A_1 such that $A = \langle a \rangle \times A_1$. If $b \in A_1^{\#}$, we have $C_H(b) = C_G(\langle a, b \rangle) \in \mathfrak{F}$ by (3). Since $m(A_1) \geqslant n(\mathfrak{F})$ and since $C_H(b) \in \mathfrak{F}$ for all $b \in A_1^{\#}$, H must belong to $\mathfrak{F}$.

5. *Contradiction.*

PROOF. Since $C_G(a) \in \mathfrak{F}$ for all $a \in A^{\#}$ and since $m(A) > n(\mathfrak{F})$, $G \in \mathfrak{F}$. But then $G/Z(G) \in \mathfrak{F}$ and so $G \in \mathfrak{G}$.

3.8. COROLLARY. *For* $k \geqslant 0$, *define* $\mathcal{C}_k$ *by*

$$\mathcal{C}_k = \{G \mid G \textit{ nilpotent and } \mathrm{cl}(G) \leqslant k\}.$$

Then $\mathcal{C}_k$ *is admissible and* $n(\mathcal{C}_k) = k + 2$.

PROOF. For $k = 1$, this was proved in [7]. Now $\mathcal{C}_0$ is the identity class and so $n(\mathcal{C}_0) = 2$. If $k \geqslant 1$, then

$$\mathcal{C}_k = \{G \mid G/Z(G) \in \mathcal{C}_{k-1}\}.$$

It now follows by induction on k and by the theorem that $n(\mathcal{C}_k) \leqslant k + 2$. Example 2 in §4 shows that $n(\mathcal{C}_k)$ is no smaller.

3.9. COROLLARY. *Let* $\mathcal{Q}$ *denote the class of all cyclic groups. Then* $\mathcal{Q}$ *is admissible and* $n(\mathcal{Q}) = 3$.

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PROOF. $G \in \mathfrak{A}$ if, and only if, G is abelian and $m(P) \leqslant 1$ for each Sylow subgroup P in G. Putting 3.8 together with 2.12 and 2.8 yields $n(\mathfrak{A}) \leqslant 3$. Example 3 in §4 shows that $n(\mathfrak{A})$ is at least 3.

The next result was first proved by Ward [15].

3.10 COROLLARY. *If* $\mathfrak{A} = \{G \mid G' \text{ is nilpotent}\}$, *then* $n(\mathfrak{A}) = 4$.

PROOF. $\mathfrak{A} = \{G \mid G/F(G) \text{ is abelian}\}$ and so $n(\mathfrak{A}) \leqslant 4$ using 3.8 together with 3.1 (with Λ being the discrete partition). The last example in [16] shows that $n(\mathfrak{A}) \geqslant 4$.

The next result is useful in proving the admissibility of the class of supersolvable groups.

3.11. COROLLARY. *Let n be a positive integer and let* $\mathfrak{A} = \{G \mid G' = 1 \text{ and } x^n = 1 \text{ for all } x \in G\}$. *Then $\mathfrak{A}$ is admissible and* $n(\mathfrak{A}) \leqslant 3$.

PROOF. Suppose A acts on G, $(|A|, |G|) = 1$, A is elementary abelian, $C_G(a) \in \mathfrak{A}$ for all $a \in A^{\#}$, and $m(A) \geqslant 3$. Then G is abelian by 3.8. Since G is generated by the subgroups $C_G(a)$ with $a \in A^{\#}$, we see that $x^n = 1$ for all $x \in G$.

3.12. THEOREM. *Let $\mathfrak{F}$ be K-generated by $\{\mathfrak{F}(p)\}$ where each non-empty $\mathfrak{F}(p)$ is an admissible subgroup-closed formation. Assume further that $\{n(\mathfrak{F}(p)) \mid \mathfrak{F}(p) \neq \varnothing\}$ has an upper bound. Then $\mathfrak{F}$ is admissible and*

$$n(\mathfrak{F}) \leqslant 1 + \sup\{n(\mathfrak{F}(p)) \mid \mathfrak{F}(p) \neq \varnothing\}.$$

PROOF. Let $\pi = \{p \mid \mathfrak{F}(p) \neq \varnothing\}$ and let $\mathfrak{P}$ be the class of all π-groups. For $p \in \pi$, define $\mathfrak{G}(p)$ by

$$\mathfrak{G}(p) = \{G \mid G/K_p(G) \in \mathfrak{F}(p)\}.$$

Then $n(\mathfrak{G}(p)) \leqslant n(\mathfrak{F}(p)) + 1$ by 3.1. Since $n(\mathfrak{P}) \leqslant 2$, and since $\mathfrak{F} = \mathfrak{P} \cap \bigcap_{p \in \pi} \mathfrak{G}(p)$, the result now follows from 2.8.

The next result was first proved in [8].

3.13. COROLLARY. *If $\mathfrak{S}$ is the class of all supersolvable groups, then $\mathfrak{S}$ is admissible and* $n(\mathfrak{S}) = 4$.

PROOF. For each prime p, let $\mathfrak{F}(p)$ be the class of all groups G such that G is p-closed and a Hall p'-subgroup of G is abelian of exponent dividing $(p - 1)$. It follows from 3.5, 2.9, and 3.10, that $n(\mathfrak{F}(p)) \leqslant 3$. Now it is straight forward to verify that $\{\mathfrak{F}(p)\}$ K-generates $\mathfrak{S}$. Hence $n(\mathfrak{S}) \leqslant 4$. The reverse inequality follows from an example in [8].

We now prove the necessary machinery to handle formations defined in terms of π-length.

3.14. LEMMA. *Let $\mathfrak{F}$ be an admissible, subgroup-closed formation. Let $\mathfrak{G} = \{G \mid G/O_\pi(G) \in \mathfrak{F}\}$. Then $\mathfrak{G}$ is an admissible, subgroup-closed formation and $n(\mathfrak{G}) \leqslant n(\mathfrak{F}) + 1$.*

PROOF. It is straightforward to verify that $\mathfrak{G}$ is a subgroup-closed formation. Assume now that A acts on G, $(|A|, |G|) = 1$, A is elementary abelian, $C_G(a) \in \mathfrak{G}$ for all $a \in A^\#$, and $m(A) \geqslant n(\mathfrak{F}) + 1$. Assume that G is a minimal example such that $G \notin \mathfrak{G}$. Let M be the $\mathfrak{G}$-residual of G. Then $M \neq 1$ and every non-identity A-invariant normal subgroup of G must contain M. In particular, since G cannot belong to $\mathfrak{F}$, $G_{\mathfrak{F}} \geqslant M$. Since $G/M \in \mathfrak{G}$, $G_{\mathfrak{F}}/M \leqslant O_\pi(G/M)$.

Now $K_\pi(C_G(a)) \geqslant O_\pi(C_G(a))$ and so $C_G(a)/K_\pi(C_G(a)) \in \mathfrak{F}$ for all $a \in A^\#$. Then 3.3 implies that $G/K_\pi(G) \in \mathfrak{F}$. Hence $G_{\mathfrak{F}} \leqslant K_\pi(G)$ and so $K_\pi(G) \neq 1$. then $K_\pi(G) \geqslant M$. Now $G \notin \mathfrak{G}$ and so $G/O_\pi(G) \notin \mathfrak{F}$. It follows that $K_\pi(G)$ is not a π-group. Since $O_\pi(G)$ and $O_{\pi'}(G)$ cannot both be different from the identity, we must have $O_\pi(G) = 1$ and $O_{\pi'}(G) = K_\pi(G)$. Then $G_{\mathfrak{F}}$ is a π'-group.

If $a \in A^\#$, then $C_G(a)/O_\pi(C_G(a)) \in \mathfrak{F}$. Hence $(C_G(a))_{\mathfrak{F}} \leqslant O_\pi(C_G(a))$. Since $\mathfrak{F}$ is subgroup-closed, $(C_G(a))_{\mathfrak{F}} \leqslant G_{\mathfrak{F}} \leqslant O_{\pi'}(G)$. We now obtain $C_G(a) \in \mathfrak{F}$ for all $a \in A^\#$. But since $m(A) > n(\mathfrak{F})$, this implies that $G \in \mathfrak{F}$ and the proof is complete.

3.15. THEOREM. *Let $\mathfrak{F}$ be an admissible, subgroup-closed formation. Let $\mathfrak{G} = \{G \mid G/O_{\pi'\pi}(G) \in \mathfrak{F}\}$. Then $\mathfrak{G}$ is an admissible, subgroup-closed, saturated formation and $n(\mathfrak{G}) \leqslant n(\mathfrak{F}) + 2$.*

PROOF. Let $\mathfrak{H} = \{G \mid G/O_\pi(G) \in \mathfrak{F}\}$. Then $\mathfrak{G} = \{G \mid G/O_{\pi'}(G) \in \mathfrak{H}\}$ and so 2 applications of 3.14 yield the result.

3.16. COROLLARY. *Suppose π is non-trivial and let $\mathfrak{P}_k$ denote the class of all π-separable groups of π-length $\leqslant k$. Then $\mathfrak{P}_k$ is admissible and $n(\mathfrak{P}_k) = 2k + 2$.*

PROOF. $\mathfrak{P}_0$ is the class of all π'-groups and so $n(\mathfrak{P}_0) = 2$. Now if $k \geqslant 1$, then $G \in \mathfrak{P}_k$ if, and only if, $G/O_{\pi'\pi}(G) \in \mathfrak{P}_{k-1}$. Using the theorem and induction on k, we obtain $n(\mathfrak{P}_k) \leqslant 2k + 2$ for all k. Example 4 in §4 demonstrates that $n(\mathfrak{P}_k)$ cannot be smaller than $2k + 2$.

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3.17. THEOREM. *Let $\mathfrak{F}$ be locally defined by $\{\mathfrak{F}(p)\}$ where each non-empty $\mathfrak{F}(p)$ is an admissible, subgroup-closed formation. Assume further that $\{n(\mathfrak{F}(p)) \mid \mathfrak{F}(p) \neq \varnothing\}$ has an upper bound. Then $\mathfrak{F}$ is admissible and $n(\mathfrak{F}) \leqslant 2 + \sup\{n(\mathfrak{F}(p) \mid \mathfrak{F}(p)) \neq \varnothing\}$.*

PROOF. The proof of 3.17 is identical with the proof of 3.12 except that $K_p(G)$ is replaced by $O_{p'p}(G)$ and 3.16 is used instead of 3.1.

Note that although each saturated formation is both locally defined and K-generated, we may not get as good a bound for $n(\mathfrak{F})$ using 3.17 as compared with 3.12. For example, suppose $\mathfrak{F}(p)$ is the class of all abelian groups of exponent dividing $p-1$. Then $n(\mathfrak{F}(p)) = 3$ if $p > 2$ and $\{\mathfrak{F}(p)\}$ locally defines $\mathfrak{S}$, the class of all supersolvable groups. Thus using 3.17 would yield $n(\mathfrak{S}) \leqslant 5$ which is weaker than $n(\mathfrak{S}) \leqslant 4$, the result we obtained using 3.12.

To show the necessity of requiring that $\{n(\mathfrak{F}(p))\}$ has an upper bound and also to exhibit a subgroup-closed, saturated formation which is not admissible, we have the following result.

3.18. THEOREM. *Suppose $f(p)$ is a positive integer for each prime p. Define $\mathfrak{F}$ by*

$$\mathfrak{F} = \{G \mid G \text{ is solvable and } l_p(G) \leqslant f(p) \text{ for all } p\}.$$

Then $\mathfrak{F}$ is a subgroup-closed saturated formation. $\mathfrak{F}$ is admissible if, and only if, $\{f(p) \mid p \text{ a prime}\}$ has an upper bound.

PROOF. $\mathfrak{F}$ is certainly a subgroup-closed saturated formation and the only question is whether or not $\mathfrak{F}$ is admissible. Now if $f(p) \leqslant N$ for all p, then $n(\mathfrak{F}) \leqslant 2N + 2$ by 3.16 and 2.8.

Now suppose that $\{f(p)\}$ has no upper bound. Let n be any positive integer. Then there must be primes p and q such that

$$\frac{n-1}{2} < f(p) < f(q).$$

Next let $m = 2f(p) + 1$ and let r be any prime distinct from p and q. If Q is an elementary abelian r-group of order r^m, then it follows from [7] that there is a $\{p, q\}$-group G such that A acts in a fixed-point-free manner on G, $l(G) = m$, and $l_p(G) = [\frac{1}{2}(m+1)] = f(p) + 1$. Hence $G \notin \mathfrak{F}$. Assume that $a \in A^{\#}$ and $C = C_G(a)$. Then $A/\langle a \rangle$ acts without fixed points on C. Hence $l(C) \leqslant m - 1$ by [2]. It follows from this that

$$l_p(C) \leqslant [m/2] = f(p) \quad \text{and} \quad l_q(C) \leqslant [m/2] = f(p) \leqslant f(q).$$

Therefore $C_G(a) \in \mathfrak{F}$ for all $a \in A^{\#}$. Since $G \notin \mathfrak{F}$ and $m(A) = m > n$, we see that $n(\mathfrak{F})$ cannot be $\leqslant n$. Since n was arbitrary, $\mathfrak{F}$ cannot be admissible.

4. Examples

1. Let π be non-trivial. Then there are primes p and q with $p \in \pi$ and $q \in \pi'$. If r is any prime distinct from p and q, then [7] implies that there is a $\{p, q\}$-group G such that $O_p(G) = 1$, $l(G) = 2$, and G admits a fixed-point-free operator group A which is an elementary abelian r-group of order r^2. Then G is not π-closed but $C_G(a)$ has a fixed-point free operator group $A/\langle a \rangle$ of prime order. Thus $C_G(a)$ is nilpotent [6, 10.2.1] and so $C_G(a)$ is π-closed for all $a \in A^{\#}$. This example justifies the equality in 3.5.

2. Let p and q be primes with $p \equiv 1 \pmod{q}$. Let k be any positive integer and let V be an elementary abelian p-group with $m(V) = k + 1$. Let $H = \mathrm{Aut}(V)$, let P be a Sylow p-subgroup of H, and let $G = VP$. If $N = N_H(P)$, then N/P is the direct product of $(k + 1)$ copies of a cyclic group of order $p - 1$. Then N must contain a subgroup A such that A is an elementary abelian q-group and $m(A) = k + 1$. Since A normalizes P, A will operate on G. I assert that $\mathrm{cl}(C_G(a)) \leqslant k$ for all $a \in A^{\#}$ while $\mathrm{cl}(G) > k$. This justifies the equality in 3.38.

Now if $1 \leqslant n \leqslant k + 1$, then it is easy to verify that

$$\big|[V, \underbrace{P, P, \ldots, P}_{n}]\big| = p^{k+1-n}.$$

It follows from this that $L_{k+1}(G) \neq 1$. Hence $\mathrm{cl}(G) > k$. (Actually, $\mathrm{cl}(G) = k + 1$ but we don't need this.) Suppose $a \in A^{\#}$, $C = C_G(a)$, $Q = C_P(a)$, and $U = C_V(a)$. Then $C = UQ$ and $L_{k+1}(Q) = 1$ [10, III.16.3] and so

$$L_{k+1}(C) = [U, \underbrace{Q, Q, \ldots, Q}_{k}].$$

If $L_{k+1}(C) \neq 1$, then we would have

$$U > [U, Q] > [U, Q, Q] > \cdots > [U, \underbrace{Q, \ldots, Q}_{k}] > 1.$$

This would imply that

$$|U| \geqslant p^{k+1} = |V|.$$

Since $1 \neq a \in \mathrm{Aut}(V)$, this is impossible. Hence $L_{k+1}(C) = 1$ and so $\mathrm{cl}(C_G(a)) \leqslant k$ for all $a \in A^{\#}$.

3. Let G be an elementary abelian group of order 9 with basis $\{x, y\}$. Let A be an elementary abelian group of order 4 with generators $\{a, b\}$. Have A operate on G by $x^a = x^{-1}$, $y^a = y$, $x^b = x$, $y^b = y^{-1}$. Then $C_G(c)$ is cyclic for all $c \in A^{\#}$ but G is not cyclic. Thus we have the example needed for 3.9.

4. If π is non-trivial, then there exist primes p and q with $p \in \pi$ and q in π'. Let r be any prime distinct from p and q, let k be any positive integer, and let A be an elementary abelian r-group A with $m(A) = 2k + 1$. It follows from [7] that there is a $\{p, q\}$-group G such that A operates in a fixed-point-free manner on G, $O_q(G) = 1$, and $l(G) = 2k + 1$. Then $l(G) = l_\pi(G) + l_{\pi'}(G)$ and $l_\pi(G) \geqslant l_{\pi'}(G)$. Hence we must have $l_\pi(G) = k + 1$. However, if $a \in A^{\#}$ and $C = C_G(a)$, then $A/\langle a \rangle$ acts in a fixed-point-free manner on C. It follows from [2] that $l(C) \leqslant 2k$. But then

$$l_\pi(C) = l_p(C) \leqslant \left[\frac{2k+1}{2}\right] = k.$$

Hence $l_\pi(C_G(a)) \leqslant k$ for all $a \in A^{\#}$ but $l_\pi(G) > k$. This justifies the equality in 3.16.

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