A NOTE ON GREEN'S THEOREM

B. D. CRAVEN

(received 31 October 1963)

Green's theorem, for line integrals in the plane, is well known, but proofs of it are often complicated. Verblunsky [1] and Potts [2] have given elegant proofs, which depend on a lemma on the decomposition of the interior of a closed rectifiable Jordan curve into a finite collection of subregions of arbitrarily small diameter. The following proof, for the case of Riemann integration, avoids this requirement by making a construction closely analogous to Goursat's proof of Cauchy's theorem. The integrability of $Q_x - P_y$ is assumed, where $P(x, y)$ and $Q(x, y)$ are the functions involved, but not the integrability of the individual partial derivatives Q_x and P_y; this latter assumption being made by other authors. However, P and Q are assumed differentiable, at points interior to the curve.

THEOREM. *Let C be a closed rectifiable Jordan curve, enclosing a plane region R. Let the functions $P(x, y)$ and $Q(x, y)$ be differentiable at all points of R, and continuous on $W = C+R$. Let $Q_x - P_y$ be Riemann-integrable on R. Then*

$$\int_C (Pdx+Qdy) = \iint_R (Q_x - P_y)\,dx\,dy. \tag{1}$$

PROOF. Let C have finite positive length L. Then there is a square A of area L^2, with sides parallel to the axes, which contains W.

Choose any positive ε. In what follows, a neighbourhood of a point shall denote a square neighbourhood, with the point as its centre, and with sides parallel to the axes. Then, from the hypotheses, every point (x_0, y_0) of R has a neighbourhood $N(x_0, y_0)$ such that, for every point (x, y) which lies in both N and W,

$$|P(x, y) - P(x_0, y_0)| < \tfrac{1}{2}\varepsilon/L \tag{2}$$

$$|Q(x, y) - Q(x_0, y_0)| < \tfrac{1}{2}\varepsilon/L \tag{3}$$

$$P(x, y) = P^* + P_x^*(x-x_0) + P_y^*(y-y_0) + \xi \tag{4}$$

$$Q(x, y) = Q^* + Q_x^*(x-x_0) + Q_y^*(y-y_0) + \eta \tag{5}$$

where P^*, P_x^*, P_y^*, Q^*, Q_x^*, Q_y^* denote the values of P, P_x, P_y, Q, Q_x, Q_y at (x_0, y_0), and

(6) $$|\xi| < \varepsilon r/L^2$$

(7) $$|\eta| < \varepsilon r/L^2$$

where

$$r^2 = (x-x_0)^2+(y-y_0)^2.$$

Divide A into four squares of side $\frac{1}{2}L$, by lines parallel to the axes. Repeat this procedure for each of the four squares, and so on indefinitely. Denote by F the family of closed squares, with sides tending to zero, so obtained. Denote by $F(\delta)$ the subset of F consisting of squares each of side δ, for $\delta = \frac{1}{2}L, \frac{1}{4}L, \cdots$. In this notation, Lemma 2 of Potts [2] states that the number of squares of $F(\delta)$ necessary to cover C is less than $4(L/\delta)+4$. This follows, since an arc of C of length less than δ can have points in common with at most four such squares.

There exists, for some δ, a finite collection F_1 of squares A_i of F, disjoint except for common boundaries, such that every point of W lies in some A_i, and such that if A_i lies wholly interior to R, and (x_0, y_0) is its centre point, then every point (x, y) of A_i satisfies (2) to (7), whereas if A_i contains points of C, then A_i belongs to $F(\delta)$, and (2) and (3) hold for any two points (x_0, y_0) and (x, y) which lie in both A_i and W. For if not, some region of W requires an infinite collection of squares. Then successive subdivision of this region produces a nested sequence of squares, to each of which the same statement applies. Since W is compact, the nested sequence defines a limit point in W, at which P or Q is discontinuous or not differentiable, contrary to hypothesis.

Let F' denote any finite collection of squares of F, obtained by further subdividing the squares of F_1. The relation between F' and F_1 will be written $F' < F_1$. Then if F_1 has the property stated in the previous paragraph, the same statement applies to any $F' < F_1$.

Denote by R^* the union of those squares of F_1 which contain points of W. Let

$$\begin{aligned}\phi(x, y) &= Q_x(x, y)-P_y(x, y) \quad \text{for} \quad (x, y) \text{ in } W\\ &= 0 \quad \text{elsewhere.}\end{aligned}$$

Then

$$\iint_R (Q_x-P_y)\,dx\,dy = \iint_{R^*} \phi\,dx\,dy.$$

Since by hypothesis, this Riemann integral exists, there exists, for some $\delta < \varepsilon/(L+1)$, a finite collection F_2 of squares of $F(\delta)$, with $F_2 < F_1$, such that

(8) $$\left|\iint_{R^*} \phi\,dx\,dy - \sum(Q_x^*-P_y^*)|A_i|\right| < \varepsilon$$

where Q_x^* and P_y^* refer to the centre point of A_i, and $|A_i|$ is the area of A_i. The summation includes all squares A_i' of F_2 which lie wholly within R, and some (possibly all or none) of those squares A_i'' which include points of C. Let B denote an upper bound to ϕ, implied by its Riemann-integrability. Then

(9) $$\left|\sum(Q_x^* - P_y^*)\, |A_i''|\right| < B \sum |A_i''|$$

By Pott's Lemma,

(10) $$\sum |A_i''| \leqq \delta^2[4(L/\delta)+4] < 4\varepsilon$$

supposing $\varepsilon < 1$. Therefore

(11) $$\left|\iint_R (Q_x - P_y)\, dx\, dy - \sum (Q_x^* - P_y^*)|A_i''|\right| < (1+4B)\varepsilon.$$

Denote by ρ_i' the boundary of A_i', and by ρ_i'' the boundary of that part of A_i'' which lies in W. Then if $|\rho_i''|$ denotes the length of ρ_i'',

(12) $$\sum |\rho_i''| < L + (4\delta)[4(L/\delta)+4] < 17(L+1)$$

again applying Potts' Lemma. Now

(13) $$\int_C (P\,dx + Q\,dy) = \sum \int_{\rho_i'} (P\,dx + Q\,dy) + \sum \int_{\rho_i''} (P\,dx + Q\,dy)$$

with all paths traversed in the positive direction. From (4) and (5),

(14) $$\int_{\rho_i'} (P\,dx + Q\,dy) = (Q_x^* - P_y^*)\, |A_i| + \int_{\rho_i'} (\xi\, dx + \eta\, dy),$$

since for a square,

$$-\int_{\rho_i'} y\,dx = |A_i'| \quad \text{and} \quad \int_{\rho_i'} x\,dy = |A_i'|.$$

By (6) and (7),

(15) $$\sum \left|\int_{\rho_i'} (\xi\,dx + \eta\,dy)\right| \leqq \sum (\varepsilon/L^2) \cdot 4\sqrt{2}\, |A_i'| < 6\varepsilon.$$

And from (2) and (3), and (12),

(16) $$\sum \left|\int_{\rho_i''} (P\,dx + Q\,dy)\right| < (\varepsilon/L) \cdot 17(L+1) = B'\varepsilon, \text{ say.}$$

Combining (11), (13), (14), (15), and (16),

$$\left|\iint_R (Q_x - P_y)\, dx\, dy - \int_C (P\,dx + Q\,dy)\right| < (7+4B+B')\varepsilon.$$

Since ε is arbitrary, (1) is proved.

https://doi.org/10.1017/S1446788700024058 Published online by Cambridge University Press

Acknowledgement

Thanks are due to Professor E. R. Love, whose suggestion it was that Goursat's method might be applicable to Green's Theorem, and to the referees for several suggestions.

References

[1] Verblunsky, J., On Green's formula, J. Lond. Math. Soc., 24 (1949), 146—148.
[2] Potts, D. H., A note on Green's Theorem, J. Lond. Math. Soc., 26 (1951), 302—304.

Department of Mathematics,
University of Melbourne.