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GROWTH CONDITIONS AND DECOMPOSABLE OPERATORS

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Throughout this paper T will denote a bounded linear operator which is defined on a Banach space $\mathscr{X}$ and whose spectrum lies on a rectifiable Jordan curve J.

The operators having some growth conditions on their resolvents have been the subject of discussion for a long time. Many sufficient conditions have been found to ensure that such operators have invariant subspaces [**2**; **3**; **7**; **8**; **12**; **13**; **14**; **21**; **27**; **28**; **29**], are S-operators [**14**], are quasidecomposable [**9**], are decomposable [**4**; **11**], are spectral [**7**; **10**; **15**; **17**], are similar to normal operators [**16**; **23**; **25**; **26**], or are normal [**15**; **18**; **22**]. In this line we are going to show that many such operators are decomposable. More precisely we will prove among other things, that if J is a smooth Jordan curve with no singular point and if

$$||(z - T)^{-1}|| \leqq \exp(\exp([\text{dist}(z, J)]^{-p}))$$

for $z \notin J$ and some $p \in (0, 1)$ then T is a strongly decomposable operator.

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1. Main theorems. Recall that since $\sigma(T)$ is a nowhere dense subset of the plane, the operator T has the single valued extension property [**7**], i.e., if $x(z)$ is an analytic function from an open subset of the plane into $\mathscr{X}$ with

$$(z - T)x(z) \equiv 0$$

then $x(z) \equiv 0$.

For a closed subset F of the plane and an operator S in some Banach space Y define

$$X_s(F) = \{x \in Y: \text{there exists an analytic function } f_x: \mathbf{C}\backslash F \rightarrow Y \text{ such that } (z - S)f_x(z) \equiv x\}.$$

It is shown in [**4**] that if S has the single valued extension property and $X_s(F)$ is closed, then $X_s(F)$ is a maximal spectral subspace of S, i.e., $X_s(F)$ is an invariant subspace of S and if M is another invariant subspace of S with the property that $\sigma(S|M) \subseteq \sigma(S|X_s(F))$ then $M \subseteq X_s(F)$. Moreover, $X_s(F)$ is a hyperinvariant subspace of S and $\sigma(S|X_s(F)) \subseteq \sigma(S) \cap F$. (See also [**5**, Lemma 5].)

For convenience, we allow singletons in the collection of closed arcs.

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LEMMA 1. *Let $X_T(F)$ be closed for any closed subarc F of J. Let F_1 and F_2 be two disjoint closed subsets of the plane. Then $X_T(F_1)$, $X_T(F_2)$ are closed and $X_T(F_1 \cup F_2) = X_T(F_1) \oplus X_T(F_2)$.*

Proof. Since every closed subset of J is the intersection of a countable set of closed subarcs of J, it follows that $X_T(F) = (X_T(F \cap J))$ is closed for all closed subsets F of the plane. Therefore $X_T(F_1 \cup F_2)$ is closed and thus by [**1**, Proposition I.2.3] we have

$$X_T(F_1 \cup F_2) = X_T(F_1) \oplus X_T(F_2).$$

LEMMA 2. *Let S be a bounded linear operator defined on some Banach space Y. Let F be a closed subset of* **C**. *Assume S has the single valued extension property and $X_S(F)$ is closed. Then $\sigma(S) = \sigma(S|X_S(F)) \cup \sigma(S^F)$ where S^F denotes the operator induced on the quotient $Y/X_S(F)$ by S. Moreover, $\sigma(S^F)$ cannot be the disjoint union of two non-empty closed sets E_1 and E_2 with $E_1 \subseteq F$.*

The first part of Lemma 2 is proved in [**1**, Lemma I.3.1] (or in [**6**, Proposition 1]); the second part follows from the Riesz decomposition theorem, [**1**, Lemma I.3.1 and Proposition I.3.2(1)], and the maximality of the spectral subspace $X_S(F)$. (See also Step II of the proof of Proposition 1 below.)

PROPOSITION 1. *Assume that for any closed subarc F of J*

(1) $X_T(F)$ is closed, and

(2) $\sigma(T^F) \subseteq \overline{J \backslash F}$ where T^F denotes the operator induced on $\mathscr{X}/X_T(F)$ by T.

Let F_1 and F_2 be two closed subarcs of J with the property that $F_1 \cap F_2$ contains no isolated point. Then $X_T(F_1 \cup F_2) = X_T(F_1) + X_T(F_2)$.

Proof. In view of Lemma 1 we may and shall assume without loss of generality that $F_1 \cup F_2$ is connected. By Lemma 1, $X_T(F)$ is closed for all closed subsets F of **C**. In particular $L = X_T(F_1 \cup F_2)$ is a closed invariant subspace of T and the operator $S = T|L$ is a bounded operator defined on L. Obviously, $X_S(F) = X_T(F \cap (F_1 \cup F_2))$ which is closed for all closed subsets F of **C**. We continue the proof of the proposition in three steps.

Step I. We show that if E is the disjoint union of two closed subarcs E_1 and E_2 of J then $\sigma(T^E) \subseteq \overline{J \backslash E}$. Let $A_j = T|X_T(E_j)$, B_j be the operator induced on $X_T(E)/X_T(E_j)$ by $T|X_T(E)$, C_j be the operator induced on $\mathscr{X}/X_T(E_j)$ by T, and let $D = T^E$. (To make the proof clearer note that if $\mathscr{X}$ is a Hilbert space then

$$T = \begin{bmatrix} A_j & * & * \\ 0 & B_j & * \\ 0 & 0 & D \end{bmatrix} \begin{matrix} X_T(E_j) \\ X_T(E)/X_T(E_j) \\ \mathscr{X}/X_T(E) \end{matrix}$$

for $j = 1, 2$.) Since $X_T(E_j)$ is a maximal spectral subspace of $T|X_T(E)$ [**1**, Proposition I.3.2(1)] and $X_T(E)$ is a maximal spectral subspace of T, it follows that $X_T(E)/X_T(E_j)$ is a maximal spectral subspace of C_j [**1**, Proposition I.3.2(3)] and thus $\sigma(B_j) \cup \sigma(D) = \sigma(C_j) \subseteq \overline{J \backslash E_j}$, $j = 1, 2$ (see Lemma 2 and the paragraph preceding Step I). Hence $\sigma(D) \subseteq \overline{J \backslash E}$ because $E_1 \cap E_2 = \emptyset$.

Step II. We prove that $\mathscr{X} = X_T(F_1) + X_T(F_2)$ if $F_1 \cup F_2 \supseteq \sigma(T)$. Let $F = F_1 \cap F_2$, $A = T|X_T(F)$, and let $B = T^F$. It follows from Condition (2) and Step *I* that $\sigma(B) \subseteq \overline{J\backslash F}$ and therefore $\sigma(B)$ is the disjoint union of two closed sets $E_j \subseteq \sigma(B) \cap F_j, j = 1, 2$. Thus by the Riesz decomposition theorem

$$\mathscr{X}/X_T(F) = X_B(E_1) \oplus X_B(E_2).$$

Let $M_j = \phi^{-1}(X_B(E_j))$ where ϕ is the canonical mapping from $\mathscr{X}$ onto $\mathscr{X}/X_T(F)$. Obviously M_j is closed and thus $X_T(F)$ is a maximal spectral subspace of $T|M_j$ [**1**, Proposition I.3.2 (1)]. Hence $\sigma(T|M_j) = \sigma(A) \cup E_j \subseteq F_j$ which implies that $M_j \subseteq X_T(F_j)$, $j = 1, 2$. Now it is an easy matter to show that every element $x \in \mathscr{X}$ is of the (not necessarily unique) form $x = y + u + v$ where $y \in X_T(F)$, $u \in X_T(F_1)$ and $v \in X_T(F_2)$. Thus $\mathscr{X} \subseteq X_T(F_1) + X_T(F_2)$ which completes the proof of Step II.

Step III. In view of Step II, the proof of the proposition is complete as soon as we prove $S(= T|L)$ satisfies the Conditions (1) and (2) of the proposition. Condition (1) is proved in the paragraph preceding Step I. Now we prove Condition (2) for S. Let $M = X_S(F)$, $N = L/M$, $A = S|M(= T|M)$, $B(= S^F)$ be the operator induced on N by S, and let C be the operator induced on $\mathscr{X}/L$ by T, where F is a closed subarc of J. By a proof similar to the proof of Step I we see that $\sigma(A) \subseteq F \cap (F_1 \cup F_2)$, $\sigma(S) = \sigma(A) \cup \sigma(B) \subseteq F_1 \cup F_2$ and $\sigma(B) \cup \sigma(C) = \sigma(D)$ where D is the operator induced on $\mathscr{X}/M$ by T. In light of Condition (2) and Step I we have $\sigma(D) \subseteq \overline{J\backslash(F \cap (F_1 \cup F_2))}$ and thus $\sigma(B) \subseteq \overline{(J\backslash F)} \cup (\{a, b\} \cap F)$ where a, b are the endpoints of $F_1 \cup F_2$ (assume $F_1 \cup F_2 \neq J$). But Lemma 2 implies that if a (respectively b) is an element of $\sigma(B)$ then a (respectively b) cannot be an interior point of F. Thus $\sigma(B) \subseteq \overline{J\backslash F}$ and hence the proof of the proposition is complete.

By induction we can prove the following corollary.

COROLLARY 1. *Let T be as in Proposition* 1. *Let F_j $j = 1, 2, \ldots, n$, be n closed arcs on J with the property that $F_i \cap F_j$ contains no isolated point for all, i, j. Then $X_T(\cup F_j) = \sum X_T(F_j)$.*

It is interesting to note that Proposition 1 is no longer true if $F_1 \cap F_2$ has an isolated point. In [**19**] we have constructed a bounded operator T on a Hilbert space $\mathscr{X}$ with the following properties:

(1) $\sigma(T)$ is a countable subset of $\{e^{i\theta}: -\pi/2 \leqq \theta \leqq \pi/2\}$,

(2) $||(z - T)^{-1}|| \leqq g(|z| - 1)^2$ for $|z| \neq 1$ and some $g > 0$,

(3) T is decomposable (in fact in view of [**4**, Theorem 5.3.2] T is an $\mathscr{U}$-unitary operator),

(4) $X_T(\{e^{i\theta}: -\pi/2 \leqq \theta \leqq 0\}) + X_T(\{e^{i\theta}: 0 \leqq \theta \leqq \pi/2\})$ is not closed.

For convenience we accept the following definition of a decomposable operator [**4**, p. 57]:

Definition. An operator T is called decomposable if for every finite open covering $G_i (i = 1, 2, \ldots, n)$ of $\sigma(T)$ there exists a set of maximal spectral subspaces Y_i of T such that

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(1) $\sigma(T|Y_i) \subseteq \bar{G}_i$, $i = 1, 2, \ldots, n$, and
(2) $\mathscr{X} = Y_1 + Y_2 + \ldots + Y_n$.
Moreover, T is called strongly decomposable if its restriction to an arbitrary maximal spectral subspace is again decomposable [**1**].

Now we prove the following key theorem.

THEOREM 1. *Let T be as in Proposition 1. Then T is decomposable.*

Proof. Let G_i, $i = 1, 2, \ldots, n$, be an arbitrary open covering of $\sigma(T)$. Since $\sigma(T)$ is compact and every open subset of J is a disjoint union of a (countable) set of open arcs, we may and shall assume without loss of generality that for each i the set $G_i \cap J$ is a finite union of a set of open arcs (a_{ij}, b_{ij}), $j = 1, 2, \ldots, n_i$, for some positive integer n_i. Also, assume that whenever necessary we have shortened the arc interval (a_{ij}, b_{ij}) on one or both sides to ensure that

$$(a_{ij}, b_{ij}) \cap (a_{kl}, b_{kl})$$

contains no isolated point for all i, j, k, l. (This is possible without violating the requirement that $\sigma(T) \subseteq \bigcup (a_{ij}, b_{ij})$.)

Now let $F_{ij} = \overline{(a_{ij}, b_{ij})}$ and $Y_i = X_T(\bar{G}_i)$. Then Y_i is closed, $\sigma(T|Y_i) \subseteq \bar{G}_i$, and $\sum Y_i = \sum X_T(F_{ij}) = X_T(\bigcup F_{ij}) = X_T(\sigma(T)) = \mathscr{X}$ (see Lemma 1 and Corollary 1). Since each Y_i is a maximal spectral subspace of T, the proof of the theorem is complete.

The proof of the next lemma is essentially the same as the proof of its special cases given in [**12**; **14**; **20**] with minor differences. We give a proof for completeness.

LEMMA 3. *Let J be oriented. Suppose that for each point $a \in J$ there exists a pair of open piecewise smooth Jordan arcs L_a, L_a^*, and a pair of non-zero functions f_a f_a^* with the following properties*:

(α) *$L_a \cap J = L_a^* \cap J = \{a\}$, and L_a lies on the positive side of L_a^** (see figure).

(β) *For each $b \in J$, $b \neq a$, there exists a piecewise smooth Jordan curve J_{ab} (respectively J_{ab}^*) such that $L_a \cup L_b^* \subseteq J_{ab}$ (respectively $L_a^* \cup L_b \subseteq J_{ab}^*$), the arc interval (a, b) (respectively (b, a)) on J lies inside J_{ab} (respectively J_{ab}^*), and f_a (respectively f_a^*) is analytic inside J_{ab} (respectively J_{ab}^*) and has a continuous extension to the boundary.*

(γ) *$\|f_a(z)(z - T)^{-1}\| + \|f_a^*(z)(z - T)^{-1}\| \leqq M$ for $z \in (L_a \cup L_a^*) \backslash \{a\}$ where M is a positive constant independent of z.*

Then for any closed arc *F on J we have*

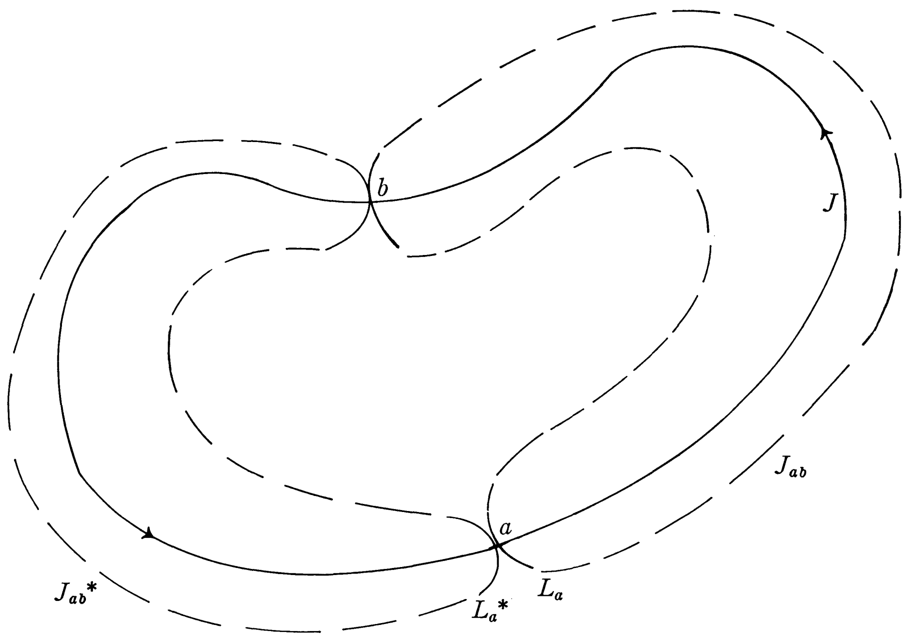
(i) *$X_T(F)$ is closed,*

(ii) *$X_T(F) \neq \{0\}$ if $F^0 \cap \sigma(T) \neq \emptyset$*

where F^0 is the open arc whose closure is F.

Note. The functions f_a, f_a^* need not be defined on an unbounded domain (cf. [**14**, Formula (2.2.13)]).

Proof of Lemma 3. Let $[a, b]$ be an arbitrary closed subarc of J in the complement of a given closed arc $F \subseteq J$. Assume without loss of generality that $J_{ab} = J_{ba}^*$. Let x_n be an arbitrary Cauchy sequence in $X_T(F)$ with $\lim x_n = x$.



By imitating the proof of [7, Lemma XVI.5.4] we are able to show that $y(z) = \lim (z - a)(z - b)f_a(z)f_b^*(z)x_n(z)$ is analytic inside J_{ab} and

$$(z - T)\frac{y(z)}{(z - a)(z - b)f_a(z)f_b^*(z)} \equiv x$$

for all z inside J_{ab}, where $x_n(z)$ is the analytic function satisfying $(z - T)x_n(z) \equiv x_n$ for $z \notin F$. This shows that $x \in X_T(J\backslash(a, b))$ for all open arcs (a, b) in the complement of F and thus $x \in X_T(F)$. Hence $X_T(F)$ is closed.

Now we show that $X_T(F) \neq \{0\}$ if $F^0 \cap \sigma(T) \neq \emptyset$. Let $F = [a, b]$ and

$$A = \int_{J_{ab}} f_a(z)f_b^*(z)(z - T)^{-1}dz.$$

By applying the techniques of Theorems 1 and 1′ of [**24**] we can show that $Ax \neq 0$ for some $x \in \mathscr{X}$ and

$$(\lambda - T)\int_{J_{ab}} \frac{f_a(z)f_b^*(z)}{\lambda - z}(z - T)^{-1}xdz \equiv Ax$$

for all λ outside J_{ab}. This shows that $Ax \in X_T(F)$ and thus $X_T(F) \neq \{0\}$. The proof of the lemma is complete.

THEOREM 2. *Let T be as in Lemma 3. Then T is strongly decomposable.*

Proof. In view of Lemmas 1 and 3, $X_T(F)$ is a closed invariant subspace of T for all closed subsets F of $\mathbf{C}$. Therefore $\sigma(T|X_T(F)) \subseteq J$ and thus $T|X_T(F)$ also satisfies the hypotheses of Lemma 3. Hence it suffices to show that any operator satisfying these hypotheses is decomposable.

In light of Theorem 1 and Lemma 3 we need only to show that $\sigma(T^F) \subseteq \overline{J\backslash F}$ for all closed subarcs F of J, where T^F as usual denotes the operator induced on $\mathscr{X}/X_T(F)$ by T. Let $M = X_T(F)$, $A = T|M$, and let $C = T^F$. Here again since M is a maximal spectral subspace of T, we have $\sigma(A) \cup \sigma(C) = \sigma(T) \subseteq J$ and thus $\sigma(C) \subseteq J$. Also since $||(z - C)^{-1}|| \leqq ||(z - T)^{-1}||$, the operator C satisfies the conditions of Lemma 3. Now let $N = X_C(F)$. Then $\phi^{-1}(N)$ is a closed invariant subspace of T, where ϕ is the canonical mapping from $\mathscr{X}$ onto $\mathscr{X}/M$. Since M is a maximal spectral subspace of $\phi^{-1}(N)$ [**1**, Proposition I.3.2], we have $\sigma(T|\phi^{-1}(N)) = \sigma(A) \cup \sigma(C|X_C(F)) \subseteq F$. Thus $\phi^{-1}(N) = M$ and $X_C(F) = \{0\}$. Hence $\sigma(C) \cap F^0 = \emptyset$ and the proof of the theorem is complete.

COROLLARY 2. *Let J be a smooth Jordan curve with no singular point. Assume there exist a positive number ϵ and a non-increasing function $M(t):(0, \epsilon) \rightarrow (0, \infty)$ such that*

$$\int_0^{\epsilon} \ln \ln M(t)dt < \infty$$

and $||(z - T)^{-1}|| \leqq M(\text{dist}(z, J))$ for $z \notin J$. Then T is strongly decomposable.

Note. As an example, $M(t) = \exp(\exp t^{-p})$, $0 < p < 1$.

Proof of Corollary 2. In view of [**14**, Lemma 2.2.1 and Theorem 5] the operator T satisfies the conditions of Lemma 3 and hence, by Theorem 2, T is strongly decomposable.

Remark. Corollary 2 is a generalization of [**4**, Theorems 5.3.6 and 5.4.3]. (See also [**4**, pp. 155, 159, 186].) The case $M(t) = t^{-n}$ and $J = R$ is essentially due to H. Tillmann [**27**, § 2].

COROLLARY 3. *Let $\mathscr{X}$ be a Hilbert space and let J be a C^2 Jordan curve. Let A be a bounded linear operator in $\mathscr{X}$ satisfying $||(z - A)^{-1}|| \leqq K[\text{dist}(z, J)]^{-n}$ for $z \notin J$, where K, n are positive constants. Assume $T = A + K$ where $K \in C_p$* (the Shatten p-class). *Then T is strongly decomposable.* (Note that $\sigma(T) \subseteq J$.)

Proof. In view of [**2**, proof of Theorem 3.5; **9**, Theorem III.1.1] (see also [**12**] in case A is normal) for each $a \in J$ and each closed bounded line segment L with a as endpoint which is not tangent to J and satisfies $L \cap J = \{a\}$, there is a constant M such that $||(z - T)^{-1}|| \leqq \exp\{M|z - a|^{-q}\}$ for $z \in L\backslash\{a\}$, where q is a positive constant independent of a. Let J be oriented. Let $0 < \beta < \pi/(2q)$ and let $\gamma = \gamma(a) \in [-\pi, \pi)$ be the angle between the x-axis and the tangent to the positive direction of J. Let

$$\begin{aligned} L_a &= \{z: |\arg(z-a) - \gamma| = \beta\}, \\ L_a^* &= \{z: |\arg(z-a) - \gamma - \pi| = \beta\}, \\ f_a(z) &= \exp\{-e^{is\gamma}(z-a)^{-s}\}, \text{ and} \\ f_a^*(z) &= \exp\{-e^{is(\gamma+\pi)}(z-a)^{-s}\} \end{aligned}$$

where $q < s < \pi/(2\beta)$. By [**23**, Example 2] the functions f_a, f_a^* satisfy the conditions of Lemma 3 and thus, in view of Theorem 2, T is strongly decomposable.

Remark. As an example, in Corollary 3 the operator A can be a spectral operator of finite type whose spectrum lies on J [**7**, p. 2162].

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