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RIGID SUBSETS IN EUCLIDEAN AND HILBERT SPACES

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Abstract

A subset Y of a metric space (X,ρ) is called rigid if all the distances $\rho(y_1, y_2)$ between points $y_1, y_2 \in Y$ in Y are mutually different. The main purpose of this paper is to prove the existence of dense rigid subsets of cardinality c in Euclidean spaces E_n and in the separable Hilbert space l_2. Some applications to abstract point set geometries are given and the connection with the theory of dimension is discussed.

Introduction

The concept of rigidity occurs in different branches of mathematics in different contexts. In topology it expresses the lack of non-trivial continuous mappings of a topological space into itself and it has a similar meaning in algebra. It appears natural to define rigidity of a metric space (X, ρ) requiring the nonexistence of a nonidentical isometry $f: X \to X$ of X onto itself. (See Janos (1972)). We adopt here a definition of rigidity which implies this condition, requiring that all the nonzero distances $\rho(x_1, x_2)$ in X are mutually unequal which means that the distance function ρ provides a one-to-one mapping $\{x_1, x_2\} \to (0, \infty)$ from the unordered pairs $\{x_1, x_2\}$ of points of X into the interval $(0, \infty)$. If (X, ρ) is a metric space, we introduce the concept of a rigid subset $T \subset X$ applying the above definition to the subspace (Y, ρ).

1. Rigid subsets in metric spaces

DEFINITION 1.1. *We say that a subset $Y \subset X$ of a metric space (X, ρ) is rigid if $\rho(y_1, y_2) = \rho(y_3, y_4)$ and $y_1 \neq y_2$ implies $\{y_1, y_2\} = \{y_3, y_4\}$ for all $y_1, y_2, y_3, y_4 \in Y$. In particular any subset Y of cardinality $|Y| \leqq 2$ less or equal 2 is rigid according to this definition.*

REMARK 1.1. One may also define this property requiring that, given $a > 0$ arbitrarily, there exists *at most one* solution $\{y_1, y_2\}$ in Y of the equation $\rho(y_1, y_2) = a$. The equivalence of both definitions is obvious.

LEMMA 1.1. *The family* $\mathscr{R}$ *of all rigid subsets of a metric space* (X,ρ) *partially ordered by set-inclusion has a maximal element (is inductive).*

PROOF. We check that the family is inductive; i.e., given any linearly ordered subfamily $\mathscr{R}_1 \subset \mathscr{R}$ we show that $\cup \mathscr{R}_1 \in \mathscr{R}$. Given $x_1, x_2, x_3, x_4 \in \cup \mathscr{R}_1$ with $x_1 \neq x_2$ and such that $\rho(x_1,x_2) = \rho(x_3,x_4)$ there is $Y \in \mathscr{R}_1$ such that $x_1, x_2, x_3, x_4 \in Y$ implying that $\{x_1,x_2\} = \{x_3,x_4\}$ showing that $\cup \mathscr{R}_1$ is rigid.

DEFINITION 1.2. *Given* $x \in X$ *in a metric space* (X,ρ) *and* $r \geqq 0$ *we denote by* $S(x,r)$ *the sphere about* x *and of radius* $r : S(x,r) = \{y \mid \rho(x,y) = r\}$. *If* $Y \subset X$ *we denote by* $\mathscr{D}(Y)$ *the set of all nonzero distances in* $Y : \mathscr{D}(Y) = \{\rho(y_1,y_2) \mid y_1, y_2 \in Y$ *and* $y_1 \neq y_2\}$. *If* $x_1, x_2 \in X$ *and* $x_1 \neq x_2$ *we denote by* $[x_1,x_2]$ *their symetral:* $[x_1,x_2] = \{y \mid \rho(x_1,y) = \rho(x_2,y)\}$. *For any subset* $Y \subset X$ *we denote by* $S(Y)$ *and* $[Y]$ *the subsets of* X *defined by:*

$$S(Y) = \bigcup \{S(y,r) \mid y \in Y \text{ and } r \in \mathscr{D}(Y)\}$$

$$[Y] = \bigcup \{[y_1,y_2] \mid y_1, y_2 \in Y \text{ and } y_1 \neq y_2\}.$$

LEMMA 1.2. *Let* $Y \subset X$ *be a rigid subset of a metric space* (X,ρ) *and let* $x \in X$ *be a point in* X *such that* $x \notin Y$. *Then the subset* $Y \cup \{x\}$ *is rigid if and only if* $x \notin S(Y) \cup [Y]$.

PROOF. Assuming $x \in S(Y) \cup [Y]$ we must show that $Y \cup \{x\}$ is no longer rigid. If $x \in S(Y)$ then there exists $y \in Y$ and $d \in \mathscr{D}(Y)$ such that $x \in S(y,d)$, but this implies that the equation $\rho(x_1,x_2) = d$ has at least two different solutions in $Y \cup \{x\}$. The one is namely $\{y_1,y_2\} \subset Y$ for which $\rho(y_1,y_2) = d$ and the other is $\{x,y\}$. (They are distinct since $x \notin Y$). If we assume $x \in [Y]$ then there exist $y_1, y_2 \in Y$ such that $\rho(x,y_1) = \rho(x,y_2)$ so that the above equation has again at least two distinct solutions. Thus $Y \cup \{x\}$ is not rigid in this case.

Conversely, assuming that $Y \cup \{x\}$ is not rigid, the adjunction of the point x to the rigid set violates this property in the sense that either there exists a distance $d = \rho(y_1,y_2)$ in Y such that $\rho(x,y) = \rho(y_1,y_2)$ for some $y, y_1\, y_2 \in Y$ and in this case we have $x \in S(y,d)$ or there exist $y_1,\ y_2 \in Y$ such that $\rho(x,y_1) = \rho(x,y_2)$ and in this case we have $x \in [y_1,y_2]$. So in both cases we have $x \in S(Y) \cup [Y]$ which completes our proof.

DEFINITION 1.3. *We say that a metric space* (X,ρ) *is geometric, or has property* P_1 *if and only if all the spheres* $S(x,r)(x \in X,\ r \geqq 0)$ *and all the symetrals* $[x,y](x,y \in X,\ x \neq y)$ *in* X *are nowhere dense in* X; *i.e., have no interior points in* X.

DEFINITION 1.4. *We say that a metric space* (X,ρ) *has the rigidity developing property, or property* P_2, *if and only if given any finite rigid subset* $Y \subset X$, *any*

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point $x \in X$ and any ε-neighbourhood $B(x,\varepsilon)$ about x, there exists a point $y \in B(x,\varepsilon)$ such that $Y \cup \{y\}$ is rigid.

REMARK 1.2. It is obvious that the finite metric space (X,ρ) cannot have the property P_1, since the singletons $\{x\}$ are open in this case and they can be written in the form $S(x,0)$. On the other hand, it can have the property P_2. This happens if and only if the whole space (X,ρ) is rigid.

LEMMA 1.3. *The property P_1 implies the property P_2.*

PROOF. Let Y be a finite rigid subset in (X,ρ) with property P_1, let $x \in X$ and let $B(x,\varepsilon)$ be an ε-neighbourhood of x. Forming the set $S(Y) \cup [Y]$ we observe that since it is a finite union of closed sets without interior points it has empty interior. Hence the open set $B(x,\varepsilon)$ is not contained in $S(Y) \cup [Y]$ and therefore there is $y \in B(x,\varepsilon)$ such that $y \notin S(Y) \cup [Y]$. If $y \in Y$ then $Y \cup \{y\} = Y$ is rigid since Y is rigid and if $y \notin Y$ then the rigidity of $Y \cup \{y\}$ follows from the Lemma 1.2.

LEMMA 1.4. *The property P_1 is hereditary with respect to dense subsets; i.e., given a metric space (X,ρ) having the property P_1 and a dense subset $Y \subset X$ then the subspace (Y,ρ) has again the property P_1.*

PROOF. We observe that if $U \subset X$ is open and $Y \subset X$ dense then the closure of $U \cap Y$ equals the closure of U. Given $y \in Y$ and $r \geqq 0$ the r-sphere about y in Y is the set $S(y,r) \cap Y$. Assuming that this set contains a nonempty open set in Y there exists a nonempty open set U in X such that $U \cap Y \subset S(y,r) \cap Y$. Denoting by $\bar{A}$ the closure of a subset A in X we have:

$$\bar{U} = \overline{U \cap Y} \subset \overline{S(y,r) \cap Y} \subset \overline{S(y,r)} = S(y,r)$$

since the set $S(y,r)$ is closed. But this implies $U \subset S(y,r)$ contrary to the assumption that (X,ρ) has property P_1. The same reasoning applies to the symetrals $[y_1, y_2] \cap Y$ in Y.

EXAMPLES. It is obvious that all Euclidean spaces $E_n (n = 1, 2, \cdots)$ with respect to the usual metric $\rho(x,y) = \sqrt{\sum_{k=1}^{n} (y_k - x_k)^2}$ have the property P_1 and hence also P_2. Using the last Lemma 1.4 we see that also the dense subspaces $\mathrm{Rat}(E_n) \subset E_n$ or $\mathrm{Irrat}\,(E_n) \subset E_n$ (the set of points with all co-ordinates rationals or irrationals) also enjoy these properties. But it is not so obvious that these properties hold also for infinite dimensional linear spaces.

LEMMA 1.5. *Any sphere $S(x,r) = \{y \mid \| y - x \| = r\}$ $(x \in B,\ r \geqq 0)$ in a normed linear space $(B, \| \cdot \|)$ has empty interior.*

PROOF. If $U \subset \{y \mid \| y - x \| = r\}$ were a nonempty open set in B, let $y \in U$ and consider the sequence $a_k = (1/k)x + (1 - 1/k)y (k = 1, 2, \cdots)$. Obviously is

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$a_k \to y$. Since $\| a_k - x \| = (1 - 1/k)\| y - x \| = (1 - 1/k)r$ it follows that for all k $a_k \notin S(x, r)$ contradicting to the assumption that U is open in B and containing y.

THEOREM 1.1. *All Hilbert spaces (separable or not) have the property* P_1.

PROOF. In view of the last lemma we have only to show that in any Hilbert space $(H, (\cdot, \cdot))$ the symetrals $[x_1, x_2]$ $(x_1 \neq x_2)$ have empty interiors. Using the translation $z \to z - \frac{1}{2}(x_1 + x_2)$ we can without loss of generality assume that our symetral has the form $[-v, v]$ with $v \in H$, $v \neq 0$. But then we have $[-v, v] = \{x \mid (x, v) = 0\}$. If there were a nonempty open set U in $[-v, v]$ with $x \in U$ then for some $\varepsilon > 0$ all the vectors $x + y$ with $\| y \| < \varepsilon$ would belong to $[-v, v]$, hence $(x + y, v) = (y, v) = 0$. Choosing y parallel to v we would reach the contradiction to our assumption $v \neq 0$, which completes our proof.

2. Dense rigid subsets

THEOREM 2.1. *If* (X, ρ) *is a separable metric space with the property* P_2 *then there is a dense rigid subset* Y *in* X.

PROOF. In view of the Remark 1.2 we only have to deal with infinite spaces. Let $a, b \in X$ be two distinct points in X and let $\{x_n\}$ be a dense sequence in X (with repetitions or not). We set up a process constructing consecutively larger and larger rigid sets using repeatedly the property P_2: choosing $\varepsilon = 1$ and using P_2 there is a point $y_1{}^1 \in B(x_1, 1)$ such that the set $\{a, b, y_1{}^1\}$ is rigid. Now choosing $\varepsilon = \frac{1}{2}$ and using P_2 again we construct rigid sets $\{a, b, y_1{}^1, y_1{}^2\}$ and $\{a, b, y_1{}^1, y_1{}^2, y_2{}^2\}$ with $y_1{}^2 \in B(x_1, \frac{1}{2})$ and $y_2{}^2 \in B(x_2, \frac{1}{2})$. Continuing this way we construct rigid sets of the form $\{a, b, y_1{}^1, y_1{}^2, y_2{}^2, \cdots, y_1{}^n, y_2{}^n, \cdots y_n{}^n\}$ with $y_1{}^n \in B(x_1, 1/n)$, $y_2{}^n \in B(x_2, 1/n) \cdots y_n{}^n \in B(x_n, 1/n)$. Defining Y as the union of these sets it is obvious that Y is rigid and also dense in X as claimed.

COROLLARY. *All Euclidean spaces* E_n *and the separable Hilbert space* l_2 *possess dense rigid subsets.*

PROOF. It follows from Theorems 1.1. and 2.1 and from the fact that $P_1 \to P_2$.

LEMMA 2.1. *If a metric space* (X, ρ) *possesses a dense rigid subset* $Y \subset X$ *then there is a maximal rigid subset* $M \subset X$ *in* X *containing* Y.

PROOF. It follows from the inductive property of the family of all rigid subsets.

3. Cardinality of maximal rigid subsets

It is clear that the cardinality of any rigid subset $Y \subset X$ of any metric space (X, ρ) cannot be larger than c, since the set of all distances $\mathscr{D}(Y)$ in Y is in one-to-one correspondence with the family of unordered pairs $\{y_1, y_2\} \subset Y$. Assuming the Continuum Hypothesis we will now show that in *complete* metric spaces with

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property P_1 every maximal rigid subset must have cardinality c. We will use the fact that a complete metric space has the Baire property (is of second category).

THEOREM 3.1. *Assuming Continuum Hypothesis every maximal rigid subset of a complete metric space* (X, ρ) *with property* P_1 *has cardinality* c.

PROOF. The Baire Category theorem implies that X cannot be countable since otherwise X would be a countable union of singletons $\{x\}$ which are closed and nowhere dense since $\{x\} = S(x, 0)$ (Property P_1). Let $M \subset X$ be a maximal rigid subset of X. Using Continuum Hypothesis there are only two possibilities for $|M|$, namely either M is countable or of cardinality c. Assume M is countable, $M = \{y_n\}_1^\infty$, then the sets $S(M)$ and $[M]$ are of the first category as countable unions of nowhere dense sets $S(y_n, r_m)$ and $[y_n, y_m]$ respectively. On the other hand given $x \in X$ such that $x \notin M$ the Lemma 1.2 says that $x \in S(M) \cup [M]$ since $M \cup \{x\}$ cannot be rigid (maximality of M). Thus we obtain a representation of X in the form: $X = M \cup S(M) \cup [M]$ implying that X is of first category contrary to Baire Theorem, which completes the proof.

COROLLARY. *Maximal rigid subsets of Euclidean spaces* E_n *and of Hilbert spaces have cardinality* c.

One may ask the question to what extent the size of maximal rigid subsets subsets $Y \subset X$ can be increased. For example: when is the cardinality of Y larger than the cardinality of its complement $X \setminus Y$? We will show that in Euclidean spaces and in Hilbert spaces the cardinality of the complement $Y^c = X \setminus Y$ of any rigid subset Y is always c.

LEMMA 3.1. *Let* (X, ρ) *be a metric space,* $Y \subset X$ *a rigid subset and* $f: X \to X$ *an isometric bijection of* X *onto itself such that neither* f *nor any of its powers* f^n *has a fixed point in* X. *Then the intersection* $Y \cap f(Y)$ *is either empty or a one point set.*

PROOF. It is obvious that an isometric image of a rigid subset is again a rigid subset. Let us suppose that the intersection $Y \cap f(Y)$ is not empty. Thus there is $a \in Y$ such that $b = f(a) \in Y$. If $y \in Y$ is any element in Y distinct from a we consider the pair $\{a, y\}$ and its image $\{f(a), f(y)\}$. Since $\rho(a, y) = \rho(f(a), f(y))$ we conclude that either $\{a, y\} = \{f(a), f(y)\}$ or $f(y) \notin Y$. But the first case would imply $a = f(y)$ and $y = f(a)$ which in turn would imply $f^2(y) = y$ and $f^2(a) = a$ contrary to the assumption that no power of f has a fixed point. Hence $f(y) \notin Y$ for all $y \in Y$, $y \neq a$ and we have in this case $Y \cap f(Y) = \{f(a)\}$ as claimed.

We are now ready to prove our theorem.

THEOREM 3.2. *In the Euclidean spaces* E_n *and in a Hilbert spaces (separable or not) the complements* Y^c *of rigid subsets* Y *have always cardinality* c.

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PROOF. In each of these spaces the translation $f: x \to x + a$ $(a \neq 0)$ has the properties required in Lemma 3.1. Let Y be a rigid subset. If $|Y| < c$ then of course we have $|Y^c| = c$. In case that $|Y| = c$ we use the result of Lemma 3.1. showing that either $f(Y) \cap Y = \theta \Rightarrow f(Y) \subset Y^c$ or $f(Y) \cap Y = \{b\} \Rightarrow f(Y) \setminus \{b\} \subset Y^c$. Since $|f(Y)| = |Y| = c$ this implies that $|Y^c| = c$ as claimed.

4. An application to abstract point set geometries

An abstract point set geometry is a system (Σ, β, A) where Σ is a nonempty set of points, β is a nonempty class of nonempty subsets of Σ called blocks and A is a list of axioms describing the *meeting* and *covering* done by blocks of β. (See Killgrove (1971).) We will give now a realization of one of these geometries where the set Σ will be the underlying set of any Euclidean space E_n or of an Hilbert space H, β will be a certain subfamily of the family of all rigid subsets and the list of axioms A will be: (using the notation adopted in Killgrove (1971))

M_2 If two distinct blocks meet their meet is a point.

C_3 For each pair of points x, y there is at most one block A containing both of them.

Let H stand for any Euclidean or Hilbert space, and let $Y \subset H$ be any nonempty rigid subset in H. Let G be the group of all translations of H. We define the family β by: $\beta = \{gY \mid g \in G\}$ and using Lemma 3.1 we observe that both axioms, M_2 and C_3 are satisfied. Our construction of a model for (Σ, β, A), where $A = \{M_2, C_3\}$ depends on the choice of the rigid subset Y. If we choose Y to have cardinality c then our model enjoys the following property: $|\Sigma| = c$, $|\beta| = c$ (since G has cardinality c), and finally each block $gY \in \beta$ has cardinality c.

5. Connection with the dimension theory

In Janos (1972) it is proved that a separable metric space (X, ρ) is zero-dimensional if and only if there exists a metric ρ^* on X which is topologically equivalent to ρ and such that (X, ρ^*) is rigid. The proof of this theorem is based on the following fact which we will need in the sequal:

LEMMA 5.1. *There is a metric ρ on the Cantor set $C \subset [0,1]$ with the following properties:*

(i) *ρ is topologically equivalent to the Euclidean metric in C,*

(ii) *(C, ρ) is rigid,*

(iii) *for $x, y, z \in C$, $0 \leqq x \leqq y \leqq z \leqq 1$ holds: $\rho(x,y) + \rho(y,z) = \rho(x,z)$.*

For the proof see Janos (1972)

We will use this lemma to prove that rigid subsets of the real line R form a universal model for separable zero-dimensional spaces in the sense that for any separable metrizable zero-dimensional space X there exists a rigid subset $Y \subset R$ of the real line R such that Y is homeomorphic to X. We need the following lemma:

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Lemma 5.2. *There exists a rigid subset $C^* \subset R$ on the real line which is homeomorphic to the Cantor set C.*

Proof. Using the metric ρ on C described in Lemma 5.1 we observe that the mapping $f: C \to R$ defined by $f(x) = \rho(0, x)$ for $x \in C$ is an isometry, since given $x, y \in C$, $x \leqq y$ we have, using the property (iii) of ρ: $\rho(0, x) + \rho(x, y) = \rho(0, y)$, thus $f(x) + \rho(x, y) = f(y)$ showing that $f(y) - f(x) = \rho(x, y)$. Defining C^* as $f(C)$ it follows that C^* is a rigid subset of R and homeomorphic to C.

Now we are ready to prove the theorem:

Theorem 5.1. *Given any separable metrisable zero-dimensional space X there is a rigid subset $Y \subset R$ of the real line which is homeomorphic to X.*

Proof. It follows immediately from Lemma 5.2 and from the fact that every separable metrisable zero-dimensional space can be topologically embedded in the Cantor set.

This theorem shows a close relationship between zero-dimensionality and rigidity. A natural question arises whether this relationship can be extended and generalized to characterize n-dimensional spaces. It is well known that a separable metrisable space X satisfies $\dim(X) \leqq n$ if and only if there exist $n + 1$ dense zero-dimensional subsets $Y_1, Y_2, \cdots, Y_{n+1}$ of X such that $\bigcup_1^{n+1} Y_k = X$. So in view of the results so far obtained it seems reasonable to ask: Given a separable metrisable space X with $\dim(X) \leqq n$. Does there exist a metric on X, compatible with the topology of X and such that there are $n + 1$ rigid subsets $Y_1, Y_2, \cdots, Y_{n+1}$ of X such that $\bigcup_1^{n+1} Y_k = X$?

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