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# Hardy Inequalities on the Real Line

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*Abstract.* We prove that some inequalities, which are considered to be generalizations of Hardy's inequality on the circle, can be modified and proved to be true for functions integrable on the real line. In fact we would like to show that some constructions that were used to prove the Littlewood conjecture can be used similarly to produce real Hardy-type inequalities. This discussion will lead to many questions concerning the relationship between Hardy-type inequalities on the circle and those on the real line.

## 1 Introduction

In 1948, G. H. Hardy and J. E. Littlewood [2] conjectured that a constant $K$ exists such that for any set $\{n_1 < n_2 < \cdots < n_N\} \subset \mathbb{Z}$, the following inequality holds

$$\Big\| \sum_{k=1}^{N} e^{in_k t} \Big\|_1 \geq K \log N.$$

Many attempts were made in the three decades that followed and all led to weaker results. In 1980 the conjecture was proved by S. V. Konjagin [4] and independently by O. C. McGehee, L. Pigno and B. Smith [5].

In [5], the Littlewood conjecture was proved as a special case of the following more general statement.

***Theorem 1.1*** (McGehee, Pigno, and Smith) *There is an absolute constant $c > 0$ such that for any function $f \in L^1([0,2\pi))$ whose spectrum is contained in the set $\{n_1 < n_2 < \cdots\} \subset \mathbb{Z}$, we have*

$$\sum_{k=1}^{\infty} \frac{|\hat{f}(n_k)|}{k} \leq c\|f\|_1. \tag{1.1}$$

Then many questions regarding the "best" generalization of Hardy's inequality were asked. The challenging conjecture is the one that suggests the existence of an absolute constant $C > 0$ such that

$$\sum_{n>0} \frac{|\hat{f}(n)|}{n} \leq C\|f\|_1 + C \sum_{n>0} \frac{|\hat{f}(-n)|}{n} \quad \forall f \in L^1\big([0,2\pi)\big).$$

See [3, 8]. The truth of this inequality is still an open problem.

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In an attempt to solve the above open problem, Klemes proved that a constant $C > 0$ exists such that

$$\sum_{j=1}^{\infty}\Bigl(4^{-j}\sum_{n=4^{j-1}}^{4^j-1}|\hat{f}(n)|^2\Bigr)^{1/2} \le C\|f\|_1 + C\sum_{j=1}^{\infty}\Bigl(4^{-j}\sum_{n=4^{j-1}}^{4^j-1}|\hat{f}(-n)|^2\Bigr)^{1/2}$$

for all functions $f$ integrable on the circle $[0, 2\pi)$, see [3].

This inequality generalizes Hardy's inequality that states that a constant $C > 0$ exists such that

$$\sum_{n=1}^{\infty}\frac{|\hat{f}(n)|}{n} \le C\|f\|_1$$

for all integrable functions $f$ on the circle for which $\hat{f}(n) = 0$ when $n < 0$. In fact, Klemes' proof can be modified to get the following result: "There exists an absolute constant $C > 0$ such that when $f \in L^1([0, 2\pi))$ and $\operatorname{spec}(f) \subset S := \{n_1 < n_2 < n_3, \dots\} \subset \mathbb{Z}$, we have

$$\sum_{j=1}^{\infty}\Bigl(4^{-j}\sum_{4^{j-1}\le k<4^j}|\hat{f}(n_k)|^2\Bigr)^{1/2} \le C\|f\|_1."$$

This inequality implies inequality (1.1).

It can be seen that proving such Hardy-type inequalities is equivalent to the construction of certain bounded functions. We refer the reader to [1, 3, 5, 8] for more Hardy-type inequalities and for more on how the required bounded function is constructed.

All of these articles treat functions integrable on the circle. In this article, we transform this study to functions integrable on $\mathbb{R}$. We prove that certain inequalities are true also for $f \in L^1(\mathbb{R})$. We shall follow the technique used in [3] with very little modifications.

For the rest of the paper, $L^1$ denotes the space of Lebesgue integrable functions on $\mathbb{R}$ and, for $f \in L^1$, $\hat{f}$ denotes the fourier transform of $f$. This is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx.$$

If $\hat{f} \in L^1$, then the inversion formula applies, and we have

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}d\xi.$$

When $f \in L^2$, we let $\hat{f}$ be its Plancherel–Fourier transform. This is defined to be the limit, in the $L^2$ sense,

$$\hat{f}(\xi) = \lim_{n\to\infty}\int_{-n}^{n} f(x)e^{-ix\xi}dx.$$

Also, for $f \in L^1 \cup L^2$ we define the spectrum of $f$ by

$$\operatorname{spec}(f) = \{\xi \in \mathbb{R} : \hat{f}(\xi) \neq 0\}.$$

If $f$ and $g$ are in $L^1$ then it is clear that $\operatorname{spec}(f+g) \subset \operatorname{spec}(f) \cup \operatorname{spec}(g)$.

When $f$ and $g$ are in $L^2$, we have $fg \in L^1$ and $\operatorname{spec}(fg) \subset \operatorname{spec}(f) + \operatorname{spec}(g)$. This follows from the fact that $(fg)^\wedge = \hat{f} * \hat{g}$ when $f$ and $g$ are in $L^2$.

## 2 A Preliminary Discussion

Let $f \in L^1$ be such that $\hat{f}$ is of compact support. For $j \geq 1$ we define the following sequence of functions: If $\hat{f}(\xi) = 0$ almost everywhere in $[4^{j-1}, 4^j)$, put $f_j(x) = 0$, otherwise put

$$f_j(x) = \frac{1}{\sqrt{2\pi}} 4^{-j/2} \left( \int_{4^{j-1}}^{4^j} |\hat{f}(\xi)|^2 d\xi \right)^{-1/2} \int_{4^{j-1}}^{4^j} \hat{f}(\xi) e^{i\xi x} d\xi.$$

The following lemmas give the basic properties of the sequence $\{f_j\}$.

***Lemma 2.1*** *Let $f_j$ be as above, then $\|f_j\|_2 \leq 4^{-j/2}$.*

**Proof** Define

$$g_j(\xi) = \begin{cases} \hat{f}(\xi), & 4^{j-1} \leq \xi < 4^j, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

then $g_j \in L^2$ and, hence, it is the Fourier transform of some function, say $h_j \in L^2$. By Plancherel's theorem, for almost every $x$,

$$h_j(x) = \frac{1}{2\pi} \int_{\mathbb{R}} g_j(\xi) e^{i\xi x} d\xi = \frac{1}{2\pi} \int_{4^{j-1}}^{4^j} \hat{f}(\xi) e^{i\xi x} d\xi.$$

Moreover,

$$\|h_j\|_2 = \frac{1}{\sqrt{2\pi}} \|g_j\|_2 = \frac{1}{\sqrt{2\pi}} \left( \int_{4^{j-1}}^{4^j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

Consequently, when $f_j \not\equiv 0$,

$$\|f_j\|_2 = \frac{1}{\sqrt{2\pi}} 4^{-j/2} \left( \int_{4^{j-1}}^{4^j} |\hat{f}(\xi)|^2 d\xi \right)^{-1/2} 2\pi \|h_j\|_2 = 4^{-j/2}.$$

■

***Lemma 2.2*** *If $f_j \not\equiv 0$ and $g_j$ is as above,*

$$\hat{f}_j(\xi) = \sqrt{2\pi} 4^{-j/2} \left( \int_{4^{j-1}}^{4^j} |\hat{f}(\xi)|^2 d\xi \right)^{-1/2} g_j(\xi).$$

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**Proof** This follows from the fact that, when $f_j \not\equiv 0$,

$$\hat{f}_j(\xi) = \frac{4^{-j/2}}{\sqrt{2\pi}} \left( \int_{4^{j-1}}^{4^j} |\hat{f}(\xi)|^2 d\xi \right)^{-1/2} \times 2\pi \hat{h}_j(\xi)$$
$$= \sqrt{2\pi} 4^{-j/2} \left( \int_{4^{j-1}}^{4^j} |\hat{f}(\xi)|^2 d\xi \right)^{-1/2} g_j(\xi). \qquad \blacksquare$$

***Lemma 2.3*** *For $f_j$ as above, $\|f_j\|_\infty \leq 1$.*

**Proof** If $f_j \not\equiv 0$, then, using the Cauchy–Schwarz inequality,

$$\|f_j\|_\infty \leq \frac{4^{-j/2}}{\sqrt{2\pi}} \left( \int_{4^{j-1}}^{4^j} |\hat{f}(\xi)|^2 d\xi \right)^{-1/2} \left( \int_{4^{j-1}}^{4^j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \left( \int_{4^{j-1}}^{4^j} d\xi \right)^{1/2}$$
$$\leq 1. \qquad \blacksquare$$

Now we construct a new sequence of functions as follows: Put $F_0 = 0$ and for $j \geq 0$ put

$$F_{j+1} = \frac{\epsilon}{2} f_{j+1} + (1 - \epsilon^2 |f_{j+1}|^2) F_j - \frac{\epsilon}{2} \overline{f}_{j+1} F_j^2,$$

where $0 < \epsilon < 1$ is to be specified later.

Since $\hat{f}$ is of compact support, $f_j \equiv 0$ for large $j$. Therefore, there exists an index $k$ such that $F_k = F_{k+1} = F_{k+2} = \cdots$ . Let $F = \frac{2}{\epsilon} F_k$.

The proof of the following lemma is a nice application of the maximum modulus principle, see [8].

***Lemma 2.4*** *Let $a, z \in \mathbb{C}$ be such that $|a|, |z| \leq 1$. If*

$$w = \frac{\epsilon}{2} z + (1 - \epsilon^2 |z|^2) a - \frac{\epsilon}{2} \overline{z} a^2, \ 0 < \epsilon \leq 1,$$

*then $|w| \leq 1$.*

Basic and important properties of $F_j$ are given in the following sequence of lemmas.

***Lemma 2.5*** *For each $j \geq 0$ we have $\|F_j\|_\infty \leq 1$.*

**Proof** This follows from Lemma 2.4 and the inductive definition of $F_{j+1}$ on taking $a = F_j$ and $z = f_{j+1}$. $\blacksquare$

***Lemma 2.6*** *For each $j \geq 0$, $\operatorname{spec}(F_j) \subset (-2 \cdot 4^{j+1}, 4^j)$.*

**Proof** We proceed by induction on $j$. The result is true for $F_0$ because $\operatorname{spec}(F_0) = \phi$. Suppose that $\operatorname{spec}(F_j) \subset (-2 \cdot 4^{j+1}, 4^j)$. Observe that $f_j$ is a scalar multiple of the inverse Fourier transform of $g_j$, where $g_j$ is as given in (2.1). Therefore, $\operatorname{spec}(f_j) \subset [4^{j-1}, 4^j)$. Now

$$\begin{aligned}
\operatorname{spec}(f_{j+1}) &\subset [4^j, 4^{j+1}) \\
\operatorname{spec}(|f_{j+1}|^2 F_j) &\subset \operatorname{spec}(f_{j+1}) + \operatorname{spec}(\overline{f}_{j+1}) + \operatorname{spec}(F_j) \\
&\subset [4^j, 4^{j+1}) + (-4^{j+1}, -4^j] + [-2\cdot 4^{j+1}, 4^j) \\
&\subset (-3\cdot 4^{j+1}, 4^{j+1}) \\
\operatorname{spec}(\overline{f}_{j+1} F_j^2) &\subset -\operatorname{spec}(f_{j+1}) + 2\operatorname{spec}(F_j) \\
&\subset (-4^{j+1}, -4^j] + [-4\cdot 4^{j+1}, 2\cdot 4^j) \\
&\subset (-5\cdot 4^{j+1}, 4^{j+1}).
\end{aligned}$$

Whence,

$$\begin{aligned}
\operatorname{spec}(F_{j+1}) &\subset [4^j, 4^{j+1}) \cup [-2\cdot 4^{j+1}, 4^j) \cup (-3\cdot 4^{j+1}, 4^{j+1}) \cup (-5\cdot 4^{j+1}, 4^{j+1}) \\
&\subset (-2\cdot 4^{j+2}, 4^{j+1}).
\end{aligned}$$

■

***Lemma 2.7*** *For $k > j \geq 0$ we have*

$$\left( \int_{|\xi| \geq 4^{j-1}} |\hat{F}_k(\xi)|^2 d\xi \right)^{1/2} \leq 16\epsilon\sqrt{2\pi}4^{-j/2}. \tag{2.2}$$

**Proof** Observe that

$$F_k = F_{j-3} + \sum_{\ell=j-3}^{k-1} (F_{\ell+1} - F_\ell) = F_{j-3} + \sum_{\ell=j-3}^{k-1} \left( \frac{\epsilon}{2} f_{\ell+1} - \epsilon^2 |f_{\ell+1}|^2 F_\ell - \frac{\epsilon}{2} \overline{f}_{\ell+1} F_\ell^2 \right),$$

where by convention $F_\ell = 0$ when $\ell < 0$.

When $|\xi| \geq 4^{j-1}$,

$$\hat{F}_k(\xi) = \sum_{\ell=j-3}^{k-1} \left( \frac{\epsilon}{2} f_{\ell+1} - \epsilon^2 |f_{\ell+1}|^2 F_\ell - \frac{\epsilon}{2} \overline{f}_{\ell+1} F_\ell^2 \right)^{\wedge}(\xi).$$

Take the $L^2$ norm of both sides, use the triangle inequality, and recall that

$$\|f_{\ell+1}\|_2 \leq 4^{-(\ell+1)/2}, \quad \|f_{\ell+1}\|_\infty \leq 1, \quad \text{and} \quad \|F_\ell\|_\infty \leq 1,$$

to get

$$\begin{aligned}
&\left(\int_{|\xi|\geq 4^{j-1}} |\hat{F}_k(\xi)|^2 d\xi\right)^{1/2}\\
&\leq \sum_{\ell=j-3}^{k-1} \left\{ \frac{\epsilon}{2}\|\hat{f}_{\ell+1}\|_2 + \epsilon^2 \left\| \left(|f_{\ell+1}|^2 F_\ell\right)^{\wedge} \right\|_2 + \frac{\epsilon}{2}\left\| \left(\overline{f}_{\ell+1} F_\ell^2\right)^{\wedge}\right\|_2 \right\}\\
&= \sqrt{2\pi} \sum_{\ell=j-3}^{k-1} \left\{ \frac{\epsilon}{2}\|f_{\ell+1}\|_2 + \epsilon^2 \left\| |f_{\ell+1}|^2 F_\ell \right\|_2 + \frac{\epsilon}{2}\left\| \overline{f}_{\ell+1} F_\ell^2 \right\|_2 \right\}\\
&\leq \sqrt{2\pi} \sum_{\ell=j-3}^{k-1} \left( \frac{\epsilon}{2}\|f_{\ell+1}\|_2 + \epsilon^2 \|f_{\ell+1}\|_2 + \frac{\epsilon}{2}\|f_{\ell+1}\|_2 \right)\\
&\leq \sqrt{2\pi} \sum_{\ell=j-3}^{k-1} (\epsilon+\epsilon^2) 4^{-(\ell+1)/2}\\
&\leq 16\epsilon\sqrt{2\pi}4^{-j/2}. \qquad \blacksquare
\end{aligned}$$

***Lemma 2.8*** *Let $j \geq 1$ and $k > j$. Then*

$$\left(\int_{4^{j-1}}^{4^j} \left|\hat{F}_k(\xi) - \frac{\epsilon}{2}\hat{f}_j(\xi)\right|^2 d\xi\right)^{1/2} \leq 18\epsilon^2 4^{-j/2}.$$

**Proof** Observe that

$$F_k - \frac{\epsilon}{2} f_j = \left(F_{j-1} + \sum_{\ell=j}^{k-1} \frac{\epsilon}{2} f_{\ell+1}\right) + \left(\sum_{\ell=j-1}^{k-1} -\epsilon^2 |f_{\ell+1}|^2 F_\ell - \frac{\epsilon}{2}\overline{f}_{\ell+1} F_\ell^2\right),$$

and that

$$\left(F_{j-1} + \sum_{\ell=j}^{k-1} \frac{\epsilon}{2} f_{\ell+1}\right)^{\wedge}(\xi) = 0 \text{ for } 4^{j-1} \leq \xi < 4^j.$$

This is true because $\mathrm{spec}(F_{j-1}) \subset (-2\cdot 4^j, 4^{j-1})$ and $\mathrm{spec}(f_k) \subset [4^{k-1}, 4^k)$ and $k > j$. Note that $\mathrm{spec}(\overline{f}_{\ell+1} F_\ell) \subset (-\infty, 0)$ and therefore, for $4^{j-1} \leq \xi < 4^j$, we may write

$$(\overline{f}_{\ell+1} F_\ell^2)^{\wedge}(\xi) = (\overline{f}_{\ell+1} F_\ell F_{\ell,j})^{\wedge}(\xi),$$

where $F_{\ell,j}$ is the truncation defined by

$$F_{\ell,j}(x) = \int_{\xi \geq 4^{j-1}} \hat{F}_\ell(\xi) e^{i\xi x} d\xi.$$

Now we have the estimates

$$\| |f_{\ell+1}|^2 F_\ell \|_1 \leq \|f_{\ell+1}\|_2^2 \leq 4^{-(\ell+1)},$$

and

$$\|\overline{f}_{\ell+1}F_\ell F_{\ell,j}\|_1 \le \|f_{\ell+1}\|_2\|F_{\ell,j}\|_2$$
$$\le 4^{-(\ell+1)/2}\frac{1}{\sqrt{2\pi}}\left(\int_{\xi\ge 4^{j-1}}|\hat{F}_\ell(\xi)|^2 d\xi\right)^{1/2}$$
$$\le 4^{-(\ell+1)/2}\times\frac{1}{\sqrt{2\pi}}16\epsilon\sqrt{2\pi}4^{-j/2},$$

where, in the last line, we have used (2.2). Whence, for $4^{j-1}\le\xi<4^j$,

$$|\hat{F}_k(\xi)-\frac{\epsilon}{2}\hat{f}_j(\xi)| \le \|\sum_{\ell=j-1}^{k-1}-\epsilon^2|f_{\ell+1}|^2F_\ell-\frac{\epsilon}{2}\overline{f}_{\ell+1}F_\ell^2\|_1 \tag{2.3}$$
$$\le\sum_{\ell=j-1}^{k-1}\left(\epsilon^2 4^{-(\ell+1)}+\frac{\epsilon}{2}4^{-(\ell+1)/2}\times 16\epsilon 4^{-j/2}\right)$$
$$\le 18\epsilon^2 4^{-j}.$$

Hence

$$\left(\int_{4^{j-1}}^{4^j}\left|\hat{F}_k(\xi)-\frac{\epsilon}{2}\hat{f}_j(\xi)\right|^2 d\xi\right)^{1/2}\le 4^{j/2}\times 18\epsilon^2 4^{-j}=18\epsilon^2 4^{-j/2}. \qquad\blacksquare$$

Now, recall that $F=\frac{2}{\epsilon}F_k$ for some $k\ge 1$. Thence, we have the following lemma.

***Lemma 2.9*** *For a certain value of $\epsilon$, there is an absolute constant $c>0$ such that*

$$\|F\|_\infty\le c, \tag{2.4}$$

$$\left(\int_{4^{j-1}}^{4^j}|\hat{F}(-\xi)|^2 d\xi\right)^{1/2}\le c4^{-j/2} \tag{2.5}$$

$$|\hat{F}(\xi)-\hat{f}_j(\xi)|\le\frac{1}{2}4^{-j}\ \textit{when}\ 4^{j-1}\le\xi<4^j. \tag{2.6}$$

**Proof** Let $\epsilon=\frac{1}{72\sqrt{2\pi}}$. Since $F=\frac{2}{\epsilon}F_k$, we get $\|F\|_\infty\le 2/\epsilon=144\sqrt{2\pi}$. Equation (2.2) implies that

$$\left(\int_{4^{j-1}}^{4^j}|\hat{F}(-\xi)|^2 d\xi\right)^{1/2}\le 32\times 4^{-j/2}.$$

Therefore, $c=144\sqrt{2\pi}$. Moreover, (2.6) follows immediately from (2.3). ∎

As a final remark about $F$, observe that

$$F_k=F_0+\sum_{\ell=0}^{k-1}(F_{\ell+1}-F_\ell)=\sum_{\ell=0}^{k-1}\left(\frac{\epsilon}{2}f_{\ell+1}-\epsilon^2|f_{\ell+1}|^2F_\ell-\frac{\epsilon}{2}\overline{f}_{\ell+1}F_\ell^2\right).$$

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Hence

$$\begin{aligned}\|F_k\|_2 &\le \sum_{\ell=0}^{k-1}\Big(\frac{\epsilon}{2}\|f_{\ell+1}\|_2+\epsilon^2\|f_{\ell+1}\|_2+\frac{\epsilon}{2}\|f_{\ell+1}\|_2\Big)\\ &\le \sum_{\ell=0}^{\infty}(\epsilon+\epsilon^2)4^{-(\ell+1)/2}\le 16\epsilon.\end{aligned}$$

Therefore, $\|F\|_2=\frac{2}{\epsilon}\|F_k\|_2\le 32$.

Before proceeding further, we remind the reader of the following basic lemma.

***Lemma 2.10*** *If $f,g\in L^2$, then*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx=\frac{1}{2\pi}\int_{\mathbb{R}}\hat f(\xi)\overline{\hat g(\xi)}d\xi.$$

Observe that if $f\in L^1$ is such that $\hat f$ is compactly supported, then $\hat f\in L^2$. But the Plancherel theorem, then, guarantees that $f\in L^2$.

Now, if $F$ is as above, then $F\in L^2$, hence

$$\int_{\mathbb{R}} f(x)\overline{F(x)}dx=\frac{1}{2\pi}\int_{\mathbb{R}}\hat f(\xi)\overline{\hat F(\xi)}d\xi \tag{2.7}$$

by Lemma 2.10.

## 3 Main Results

Now we are ready to prove the following.

***Theorem 3.1*** *There exists an absolute constant $C>0$ such that for all $f\in L^1(\mathbb{R})$,*

$$\sum_{j=1}^{\infty}\Big(4^{-j}\int_{4^{j-1}}^{4^j}|\hat f(\xi)|^2d\xi\Big)^{1/2}\le C\|f\|_1+C\sum_{j=1}^{\infty}\Big(4^{-j}\int_{4^{j-1}}^{4^j}|\hat f(-\xi)|^2d\xi\Big)^{1/2}.$$

**Proof** We prove the result, first, for $f\in L^1$ whose Fourier transform $\hat f$ is of compact support. Thus, let $f\in L^1$ be such that $\hat f$ is compactly supported and let $f_j$ and $F$ be as above. Recall (2.4) and observe that (2.7) holds because $\hat f$ is of compact support. Therefore,

$$\begin{aligned}c\|f\|_1 &\ge \Big|\int_{\mathbb{R}} f(x)\overline{F(x)}dx\Big| = \frac{1}{2\pi}\Big|\int_{\mathbb{R}}\hat f(\xi)\overline{\hat F(\xi)}d\xi\Big|\\ &\ge \frac{1}{2\pi}\Big|\int_1^{\infty}\hat f(\xi)\overline{\hat F(\xi)}d\xi\Big|-\frac{1}{2\pi}\Big|\int_{-1}^{1}\hat f(\xi)\overline{\hat F(\xi)}d\xi\Big|-\frac{1}{2\pi}\Big|\int_{-\infty}^{-1}\hat f(\xi)\overline{\hat F(\xi)}d\xi\Big|.\end{aligned}$$

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Whence,

$$\begin{aligned}\left|\int_1^\infty \hat f(\xi)\overline{\hat F(\xi)}d\xi\right| &\le c\|f\|_1 + \int_{-1}^1 |\hat f(\xi)||\hat F(\xi)|d\xi + \int_{-\infty}^{-1} |\hat f(\xi)||\hat F(\xi)|d\xi\\
&\le c\|f\|_1 + \left(\int_{-1}^1 |\hat f(\xi)|^2 d\xi\right)^{\frac12}\left(\int_{-1}^1 |\hat F(\xi)|^2 d\xi\right)^{\frac12}\\
&\quad + \sum_{j=1}^\infty \int_{4^{j-1}}^{4^j} |\hat f(-\xi)||\hat F(-\xi)|d\xi\\
&\le c\|f\|_1 + \sqrt{2}\|f\|_1\|\hat F\|_2 + \sum_{j=1}^\infty \left\{\left(\int_{4^{j-1}}^{4^j} |\hat f(-\xi)|^2 d\xi\right)^{1/2}\right.\\
&\quad \left.\times\left(\int_{4^{j-1}}^{4^j} |\hat F(-\xi)|^2 d\xi\right)^{1/2}\right\}\\
&\le c\|f\|_1 + 2\sqrt{\pi}\|F\|_2\|f\|_1 + c\sum_{j=1}^\infty \left(4^{-j}\int_{4^{j-1}}^{4^j} |\hat f(-\xi)|^2 d\xi\right)^{1/2},\end{aligned}$$

where we have used (2.5) in the last line. But $\|F\|_2 \le 32$, hence

$$\left|\int_1^\infty \hat f(\xi)\overline{\hat F(\xi)}d\xi\right| \le 2c\|f\|_1 + c\sum_{j=1}^\infty \left(4^{-j}\int_{4^{j-1}}^{4^j} |\hat f(-\xi)|^2 d\xi\right)^{1/2}, \tag{3.1}$$

where we have used the fact that $c = 144\sqrt{2\pi} > 64\sqrt{\pi}$.

Now,

$$\begin{aligned}\left|\int_1^\infty \hat f(\xi)\overline{\hat F(\xi)}d\xi\right| &\ge \Re\left(\int_1^\infty \hat f(\xi)\overline{\hat F(\xi)}d\xi\right)\\
&= \sum_{j=1}^\infty \Re\left(\int_{4^{j-1}}^{4^j} \hat f(\xi)\overline{\hat F(\xi)}d\xi\right).\end{aligned}$$

But we have, for $4^{j-1} \le \xi < 4^j$,

$$\left|\overline{\hat F(\xi)} - \overline{\hat f_j(\xi)}\right| \le \frac12 4^{-j},$$

hence,

$$\left|\overline{\hat F(\xi)}\hat f(\xi) - \overline{\hat f_j(\xi)}\hat f(\xi)\right| \le \frac12 4^{-j}|\hat f(\xi)|,$$

which implies

$$\Re\left(\overline{\hat F(\xi)}\hat f(\xi) - \overline{\hat f_j(\xi)}\hat f(\xi)\right) \le \frac12 4^{-j}|\hat f(\xi)|.$$

Consequently, for $4^{j-1} \le \xi < 4^j$,

$$\Re\big(\overline{\hat{F}(\xi)}\hat{f}(\xi)\big) \ge \Re\big(\hat{f}(\xi)\overline{\hat{f}_j(\xi)}\big) - \frac{1}{2}4^{-j}|\hat{f}(\xi)|$$
$$= \sqrt{2\pi}4^{-j/2}\Big(\int_{4^{j-1}}^{4^j}|\hat{f}(\tau)|^2 d\tau\Big)^{-1/2}|\hat{f}(\xi)|^2 - \frac{1}{2}4^{-j}|\hat{f}(\xi)|.$$

Integrate both sides and then use the Cauchy–Schwarz inequality to get

$$\int_{4^{j-1}}^{4^j}\Re\big(\overline{\hat{F}(\xi)}\hat{f}(\xi)d\xi\big)$$
$$\ge \sqrt{2\pi}4^{-j/2}\Big(\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2} - \frac{4^{-j}}{2}\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|d\xi$$
$$\ge \sqrt{2\pi}4^{-j/2}\Big(\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2} - \frac{4^{-j}}{2}\Big(\int_{4^{j-1}}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2}\Big(\int_{4^{j-1}}^{4^j}d\xi\Big)^{1/2}$$
$$= \Big(\sqrt{2\pi} - \frac{\sqrt{3}}{4}\Big)\Big(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2}$$
$$\ge \Big(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2}.$$

Therefore, (3.1) becomes

$$\sum_{j=1}^{\infty}\Big(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2} \le C\|f\|_1 + C\sum_{j=1}^{\infty}\Big(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(-\xi)|^2 d\xi\Big)^{1/2},$$

where $C = 2c$. This completes the proof of the case when $\hat{f}$ is compactly supported.

For the general case, let $f \in L^1$ and apply the result on $f * K_\lambda$, where $K_\lambda$ is the Fejer kernel of order $\lambda$, to get the result. ■

Now, some corollaries are available. The first corollary is the real form of Hardy's inequality.

***Corollary 3.2*** *Let $f \in L^1$ be such that $\hat{f}(\xi) = 0$ for all $\xi < 0$, then*

$$\int_0^\infty \frac{|\hat{f}(\xi)|}{\xi}d\xi \le C'\|f\|_1,$$

*where $C'$ is an absolute constant not depending on $f$.*

**Proof** Observe that

$$\int_1^\infty \frac{|\hat{f}(\xi)|}{\xi}d\xi = \sum_{j=1}^{\infty}\int_{4^{j-1}}^{4^j}\frac{|\hat{f}(\xi)|}{\xi}d\xi \le \sum_{j=1}^{\infty}\Big(\int_{4^{j-1}}^{4^j}\frac{1}{\xi^2}d\xi\Big)^{1/2}\Big(\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2}$$
$$= \sqrt{3}\sum_{j=1}^{\infty}\Big(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2} \le C'\|f\|_1,$$

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where $C' = \sqrt{3}C$. Here, we have applied the result of the above theorem, noting that $\hat{f}(\xi) = 0$ when $\xi < 0$.

Now let $\alpha > 0$ be any given number and define $g(x) = f(\alpha x)$. Then

$$\|g\|_1 = \tfrac{1}{\alpha}\|f\|_1 \quad \text{and} \quad \hat{g}(\xi) = \tfrac{1}{\alpha}\hat{f}(\xi/\alpha).$$

Moreover, $\hat{g}(\xi) = 0$ when $\xi < 0$. Hence,

$$\int_1^\infty \frac{|\hat{g}(\xi)|}{\xi} d\xi \leq C'\|g\|_1 \Rightarrow \frac{1}{\alpha}\int_1^\infty \frac{|\hat{f}(\xi/\alpha)|}{\xi} d\xi \leq \frac{1}{\alpha}C'\|f\|_1.$$

That is,

$$\int_1^\infty \frac{|\hat{f}(\xi/\alpha)|}{\xi} d\xi \leq C'\|f\|_1, \ \forall\alpha > 0.$$

But this is equivalent to saying

$$\int_{1/\alpha}^\infty \frac{|\hat{f}(\xi)|}{\xi} d\xi \leq C'\|f\|_1, \ \forall\alpha > 0.$$

Since the right side of this inequality does not depend on $\alpha$, we may let $\alpha \to \infty$ to get

$$\int_0^\infty \frac{|\hat{f}(\xi)|}{\xi} d\xi \leq C'\|f\|_1$$

as required. ■

Now having proved the real Hardy's inequality, it is very natural to ask about the best generalization of Hardy's inequality for arbitrary $f \in L^1(\mathbb{R})$. The following theorem is a nice generalization.

***Theorem 3.3*** *Let $\epsilon > 0$ be given. Then, a constant $C_2 > 0$ exists such that for all $f \in L^1(\mathbb{R})$ we have*

$$\int_1^\infty \frac{|\hat{f}(\xi)|}{\xi} d\xi \leq C_2\|f\|_1 + C_2\int_1^\infty \frac{|\hat{f}(-\xi)|}{\xi}(\log\xi)^{1+\epsilon}.$$

**Proof** From the proofs of Theorem 3.1 and Corollary 3.2, we see that, for $f \in L^1(\mathbb{R})$,

$$\int_1^\infty \frac{|\hat{f}(\xi)|}{\xi} \leq C\|f\|_1 + C\sum_{j=1}^\infty \int_{4^{j-1}}^{4^j} |\hat{f}(-\xi)|\,|\hat{F}(-\xi)|\,d\xi, \tag{3.2}$$

where $C > 0$ is some constant independent of $f$.

As a matter of notation, denote

$$\sum_{j=1}^\infty \int_{4^{j-1}}^{4^j} |\hat{f}(-\xi)|\,|\hat{F}(-\xi)|\,d\xi$$

by $G(f)$. Let $\{\lambda_j\}$ be a sequence of positive numbers to be specified later and let $E_j = \{\xi \in [4^{j-1}, 4^j] : |\hat{F}(-\xi)| \le \lambda_j\}$ and let $E_j^c = [4^{j-1}, 4^j] \backslash E_j$.

Then

$$
\begin{aligned}
G(f) &= \sum_{j=1}^{\infty} \Big[ \int_{E_j} |\hat{f}(-\xi)||\hat{F}(-\xi)| d\xi + \int_{E_j^c} |\hat{f}(-\xi)||\hat{F}(-\xi)|\, d\xi \Big] \\
&\le \sum_{j=1}^{\infty} \Big[ \lambda_j \int_{E_j} |\hat{f}(-\xi)| d\xi + \int_{E_j^c} |\hat{f}(-\xi)||\hat{F}(-\xi)| \frac{|\hat{F}(-\xi)|}{\lambda_j} d\xi \Big] \\
&\le \sum_{j=1}^{\infty} \Big[ \lambda_j \int_{E_j} |\hat{f}(-\xi)| d\xi + \frac{\|f\|_1}{\lambda_j} \int_{E_j^c} |\hat{F}(-\xi)|^2 d\xi \Big]
\end{aligned}
$$

where, in the last line, we have used the fact that $|\hat{f}(-\xi)| \le \|f\|_1$. Now, let $\lambda_j = 4^{-j}\alpha_j$ where $\{\alpha_j\}$ is a sequence of positive numbers to be specified later. Thus

$$
\begin{aligned}
G(f) &\le \sum_{j=1}^{\infty} \alpha_j \int_{4^{j-1}}^{4^j} \frac{|\hat{f}(-\xi)|}{4^j} d\xi + \|f\|_1 \sum_{j=1}^{\infty} \frac{4^j}{\alpha_j} \int_{4^{j-1}}^{4^j} |\hat{F}(-\xi)|^2\, d\xi \\
&\le \sum_{j=1}^{\infty} \alpha_j \int_{4^{j-1}}^{4^j} \frac{|\hat{f}(-\xi)|}{\xi} d\xi + \|f\|_1 \sum_{j=1}^{\infty} \frac{4^j}{\alpha_j} \times c^2 4^{-j} \quad \text{(see (2.5))} \\
&= \sum_{j=1}^{\infty} \alpha_j \int_{4^{j-1}}^{4^j} \frac{|\hat{f}(-\xi)|}{\xi} d\xi + c^2 \|f\|_1 \sum_{j=1}^{\infty} \frac{1}{\alpha_j}.
\end{aligned}
$$

Let

$$
\alpha_j = \begin{cases} 1, & j = 1, \\ \big((j-1)\log 4\big)^{1+\epsilon}, & j \ge 2. \end{cases}
$$

Consequently

$$
\begin{aligned}
G(f) \le \int_1^4 \frac{\|\hat{f}(-\xi)|}{\xi} d\xi + \sum_{j=2}^{\infty} \int_{4^{j-1}}^{4^j} \frac{|\hat{f}(-\xi)|}{\xi} \big((j-1)\log 4\big)^{1+\epsilon} + \\
+ c^2 \|f\|_1 \Big( 1 + \sum_{j=2}^{\infty} \frac{1}{\big((j-1)\log 4\big)^{1+\epsilon}} \Big).
\end{aligned}
$$

Now denote

$$
\Big( 1 + \sum_{j=2}^{\infty} \frac{1}{\big((j-1)\log 4\big)^{1+\epsilon}} \Big)
$$

by $c(\epsilon)$, which is a real number. Moreover, if $\xi \in [4^{j-1}, 4^j]$, then

$$
\big((j-1)\log 4\big)^{1+\epsilon} \le (\log \xi)^{1+\epsilon}.
$$

Hence

$$G(f) \leq \|f\|_1 \log 4 + \sum_{j=2}^{\infty} \int_{4^{j-1}}^{4^j} \frac{|\hat{f}(-\xi)|}{\xi} \left(\log \xi\right)^{1+\epsilon} \, d\xi + c^2 c(\epsilon)\|f\|_1$$

$$\leq C'\|f\|_1 + \int_1^{\infty} \frac{|\hat{f}(-\xi)|}{\xi} (\log \xi)^{1+\epsilon} \, d\xi.$$

By substituting this inequality in (3.2), the result of the theorem follows. ■

***Remark*** (i) We may modify our choice of $\alpha_j$ above to get

$$\int_1^{\infty} \frac{|\hat{f}(\xi)|}{\xi} d\xi \leq C_2\|f\|_1 + C_2 \int_1^{\infty} \frac{|\hat{f}(-\xi)|}{\xi} \log \xi \left(\log \log \xi\right)^{1+\epsilon},$$

which is stronger than the inequality in Theorem 3.3. In fact we can modify the inequality to get as many log terms as we want.

(ii) In the author's Ph. D. thesis [6], it was proved that if $\epsilon > 0$ is given, then a constant $C > 0$ depending on $\epsilon$ exists such that

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq C\|f\|_1 + C \sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|}{n} (\log n)^{1+\epsilon}$$

for all $f \in L^1(\mathbb{T})$. The proof of this result is very similar to that of Theorem 3.3.

(iii) In [7], it is shown that for functions lying in an infinite dimensional subspace of $L^1(\mathbb{R})$,

$$\int_0^{\infty} \frac{|\hat{f}(\xi)|^2}{\xi} d\xi \leq 2\pi\|f\|_1^2 + \int_0^{\infty} \frac{|\hat{f}(-\xi)|^2}{\xi} d\xi.$$

## References


[1] J. J. F. Fournier, *Some remarks on the recent proofs of the Littlewood conjecture.* In: Second Edmonton conference on approximation theory (Edmonton, Alta., 1982), CMS Conf. Proc., 3, American Mathematical Society, Providence, RI, 1983, pp. 157–170.

[2] G. H. Hardy and J. E. Littlewood, *A new proof of a theorem on rearrangements.* J. London Math. Soc. **23**(1948), 163–168. doi:10.1112/jlms/s1-23.3.163

[3] I. Klemes, *A note on Hardy's inequality.* Canad. Math. Bull. **36**(1993), no. 4, 442–448.

[4] S. V. Konjagin, *On the Littlewood problem.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **45**(1981), no. 2, 243–265, 463.

[5] O. C. McGehee, L. Pigno, and B. Smith, *Hardy's inequality and the $L^1$ norm of exponential sums.* Ann. of Math. **113**(1981), no. 3, 613–618. doi:10.2307/2007000

[6] M. Sababheh, *Constructions of bounded functions related to two-sided Hardy inequalities.* Ph. D. thesis, McGill University, 2006.

[7] ———, *Hardy-type inequalities on the real line.* J. Inequal. Pure Appl. Math. **9**(2008), no. 3, Article 72, 6 pp.

[8] B. Smith, *Two trigonometric designs: one-sided Riesz products and Littlewood products.* In: General Inequalities, 3, Internat. Schriftenreihe Numer. Math., 64, Birkhäuser, Basel, 1983, pp. 141–148.



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