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Nagoya Math. J.
Vol. 53 (1974), 109-113

ON A CERTAIN ALGEBRA ASSOCIATED WITH A POLARIZED ALGEBRAIC VARIETY

Dedicated to Professor Minoru Kurita on the occation of his 60th birthday.

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In the present note we associate a certain algebra of finite rank over $\boldsymbol{Q}$ to each non-singular polarized algebraic variety defined over $\boldsymbol{C}$. For a surface the algebra is a Jordan algebra with identity, and for an abelian variety A the algebra is canonically isomorphic to the Jordan algebra of symmetric elements in $\mathrm{End}_{\boldsymbol{Q}}(A)$[1] with respect to the involution induced by the polarization. This algebra may be important for a polarized algebraic variety as much as $\mathrm{End}_{\boldsymbol{Q}}(A)$ for an abelian variety A.

§ 1. Composition

1.1. Let V be a compact nonsingular algebraic variety of dimension n with a Hodge structure ω, a fundamental (1.1)-form on V, and let $H^{(\ell,\ell)}(V,\boldsymbol{C})$ be the space of harmonic (ℓ,ℓ)-forms on V with respect to the Hodge structure $\omega, (0 \leq \ell \leq n)$. Regarding $H^{(\ell,\ell)}(V,\boldsymbol{C})$ as a subgroup of the 2ℓ-th cohomology group $H^{2\ell}(V,\boldsymbol{C})$, we denote

$$\mathfrak{H}^{(\ell,\ell)}(V,\boldsymbol{Q}) = H^{(\ell,\ell)}(V,\boldsymbol{C}) \cap H^{2\ell}(V,\boldsymbol{Q})$$

and

$$\mathfrak{H}(V,\boldsymbol{Q}) = \bigoplus_{\ell=1}^{n} \mathfrak{H}^{(\ell,\ell)}(V,\boldsymbol{Q}).$$

Then $\mathfrak{H}(V,\boldsymbol{Q})$ is considered as a commutative $\boldsymbol{Q}$-algebra with the product given by

(1) $\xi\cdot\eta =$ the harmonic form cohomologous to the closed form $\xi \wedge \eta$.

Received August 29, 1973.

[1] $\mathrm{End}_{\boldsymbol{Q}}(A) = \mathrm{End}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, where $\mathrm{End}(A)$ means the ring of endomorphisms.

For each φ in $\mathfrak{H}^{(1,1)}(V, Q)$ we mean by L_φ the operator

$$L_\varphi \xi = \varphi \cdot \xi \tag{2}$$

and denote

$$L = L_\omega, \Lambda = i(\omega) , \tag{3}$$

where $i(\omega)\xi$ mean the inner product of the fundamental form ω with ξ with respect to the Hodge structure. These operators may be considered as operators on $\mathfrak{H}(V, Q)$ and they satisfy the relations

$$[L, \Lambda] = H = \sum_{\ell=0}^{n} (2\ell - n)\pi^{(\ell,\ell)} , \tag{4}$$

$$[H, L_\varphi] = 2L_\varphi, [H, L] = 2L , \tag{5}$$

$$[H, \Lambda] = -2\Lambda , \tag{6}$$

where $\pi^{(\ell,\ell)}$ is the projection $\mathfrak{H}(V, Q) \to \mathfrak{H}^{(\ell,\ell)}(V, Q)$.

1.2. We define a binary composition $\circ$ in $\mathfrak{H}^{(1,1)}(V, Q)$ as follows

$$\varphi \circ \phi = \tfrac{1}{2}\{\Lambda\varphi \cdot \phi + \Lambda\phi \cdot \varphi - \Lambda(\varphi \cdot \phi)\} \qquad (\varphi, \phi \in \mathfrak{H}^{(1,1)}(V, Q)) . \tag{7}$$

The composition is obviously commutative, i.e.,

$$\varphi \circ \phi = \phi \circ \varphi . \tag{8}$$

LEMMA 1.

$$\Lambda L\varphi = (n-2)\varphi + \Lambda\varphi \cdot \omega \qquad (\varphi \in \mathfrak{H}^{(1,1)}(V, Q)) \tag{9}$$

$$\Lambda\omega = n = \dim V . \tag{10}$$

Proof. From (4) we have

$$\Lambda L\varphi = (-H + L\Lambda)\varphi = (n-2)\varphi + \Lambda\varphi \cdot \omega ,$$

and

$$\Lambda\omega = \Lambda L1 = (-H + L\Lambda)1 = -H1 = n .$$

PROPOSITION 1.

$$\varphi \circ \omega = \varphi , \tag{11}$$

Proof. From (8) we have

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$$\begin{aligned}\varphi \circ \omega &= \tfrac{1}{2}\{\Lambda\omega\cdot\varphi + \Lambda\varphi\cdot\omega - \Lambda(\omega\cdot\varphi)\}\\ &= \tfrac{1}{2}\{n\varphi + \Lambda\varphi\cdot\omega - \Lambda L\varphi\}\\ &= \tfrac{1}{2}\{n\varphi + \Lambda\varphi\cdot\omega - (n-2)\varphi - \Lambda\varphi\cdot\omega\}\\ &= \varphi\ .\end{aligned}$$

Let us give another expression of the composition $\circ$:

PROPOSITION 2.

$$\varphi \circ \phi = \tfrac{1}{2}[[L_\varphi, \Lambda], L_\phi]1 \tag{12}$$

Proof. From (9), (10), (11) it follows that

$$\begin{aligned}\tfrac{1}{2}[[L_\varphi, \Lambda], L_\phi]1 &= \tfrac{1}{2}\{L_\varphi \Lambda L_\phi + L_\phi \Lambda L_\varphi - \Lambda L_\varphi L_\phi - L_\varphi L_\phi \Lambda\}1\\ &= \tfrac{1}{2}\{L_\varphi \Lambda\phi + L_\phi \Lambda\varphi - \Lambda(\varphi\cdot\phi)\}\\ &= \tfrac{1}{2}\{\Lambda\phi\cdot\varphi + \Lambda\varphi\cdot\phi - \Lambda(\varphi\cdot\phi)\} = \varphi\circ\phi\ .\end{aligned}$$

PROPOSITION 3. *Let ρ_φ be the linear endomorphism of $\mathfrak{H}^{(1,1)}(V, \boldsymbol{Q})$ given by*

$$\rho_\varphi \phi = \varphi \circ \phi\ . \tag{13}$$

Then

$$\rho_\varphi = \tfrac{1}{2}[L_\varphi, \Lambda] + \tfrac{1}{2}\Lambda\varphi\cdot \mathrm{id}\ , \tag{14}$$

as linear endomorphism on $\mathfrak{H}^{(1,1)}(V, \boldsymbol{Q})$.

Proof. From the definitions it follows that

$$\begin{aligned}(\tfrac{1}{2}[L_\varphi, \Lambda] + \tfrac{1}{2}\Lambda\varphi\cdot\mathrm{id})\phi &= \tfrac{1}{2}\{[[L_\varphi, \Lambda], L_\phi]1 + L_\phi[L_\varphi, \Lambda]1 + \Lambda\varphi\cdot\phi\}\\ &= \varphi\circ\phi - \tfrac{1}{2}\Lambda\varphi\cdot\phi + \tfrac{1}{2}\Lambda\varphi\cdot\phi = \varphi\circ\phi = \rho_\varphi\phi\ .\end{aligned}$$

PROPOSITION 4. *The following two equalities are equivalent:*

$$[[L_\varphi, \Lambda], [L_{\Lambda\varphi^2}, \Lambda]]\phi = 0 \tag{15}$$

and

$$\varphi\circ((\varphi\circ\varphi)\circ\phi) = (\varphi\circ\varphi)\circ(\varphi\circ\phi) \qquad (\varphi, \phi \in \mathfrak{H}^{(1,1)}(V, \boldsymbol{Q}))\ . \tag{16}$$

Proof. (16) is equivalent to

$$[\rho_\varphi, \rho_{\varphi\circ\varphi}]\phi = 0\ .$$

On the other hand

$$\rho_\varphi \phi = \{\tfrac{1}{2}[L_\varphi, \Lambda] + \tfrac{1}{2}\Lambda\varphi \cdot \mathrm{id}\}\phi ,$$

and thus (16) is equivalent to

$$[[L_\varphi, \Lambda], [L_{\varphi\circ\varphi}, \Lambda]]\phi = 0 .$$

Since $\varphi \circ \varphi = \frac{1}{2}\{2\Lambda\varphi \cdot \varphi - \Lambda\varphi^2\}$, this equality is equivalent to (15).

1.3. A Jordan algebra is a commutative algebra satisfying (16), hence Proposition 4 may be stated as follows:

PROPOSITION 5. *The algebra* $(\mathfrak{H}^{(1,1)}(V, \boldsymbol{Q}), \circ)$ *is a Jordan algebra with identity* ω, *if and only if*

$$[[L_\varphi, \Lambda], [L_{\Lambda\varphi^2}, \Lambda]]\phi = 0 \qquad (\varphi, \phi) \in \mathfrak{H}^{(1,1)}(V, \boldsymbol{Q})) .$$

PROPOSITION 6. *If* $\dim V = 2$, *then* $(\mathfrak{H}^{(1,1)}(V, \boldsymbol{Q}), \circ)$ *is a Jordan algebra with identity* ω.

Proof. Since $\dim V = 2$, there exists a rational valued bilinear form $\beta_{\varphi,\phi}$ on $\mathfrak{H}^{(1,1)}(V, \boldsymbol{Q})$ such that $\varphi \cdot \phi = \beta_{\varphi,\phi}\omega^2$, and thus

$$L_{\Lambda\varphi^2} = \beta_{\varphi,\varphi} L_{\Lambda\omega^2} = \beta_{\varphi,\varphi} L_{\Lambda L_\omega} = (2n-2)\beta_{\varphi,\varphi} L_\omega .$$

Hence

$$\begin{aligned}[][[L_\varphi, \Lambda], [L_{\Lambda\varphi^2}, \Lambda]] &= (2n-2)\beta_{\varphi,\varphi}[[L_\varphi, \Lambda], H] \\ &= (2n-2)\beta_{\varphi,\varphi}\{[[L_\varphi, H], \Lambda] + [L_\varphi, [\Lambda, H]]\} = 0 .\end{aligned}$$

§ 2. Abelian variety case.

2.1. We shall show another important example, an abelian variety, for which $(\mathfrak{H}^{(1,1)}(V, \boldsymbol{Q}), \circ)$ is a Jordan algebra.

Let A be an abelian variety of dimension n defined over $\boldsymbol{C}$, which is expressed as a quotient

$$A = \boldsymbol{C}^n / \sum$$

with a lattice $\sum$ of rank $2n$. After a suitable choice of the coordinates $z_1, \cdots, z_n$ on $\boldsymbol{C}^n$, we may assume that

$$\omega = \sqrt{-1} dz_\wedge d^t\bar{z} ,$$

where $dz = (dz_1, \cdots, dz_n)$.

We denote by $\mathrm{End}_{\boldsymbol{Q}}(A)$ the $\boldsymbol{Q}$-algebra of $n \times n$-matrices A such that

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$$\sum A \subset \sum ,$$

and we mean by $\mathfrak{S}_Q(A, \omega)$ the subspace of $\mathrm{End}_Q(A)$ consisting of symmetric matrices. Then two vector spaces $\mathfrak{S}_Q(A, \omega)$ and $\mathfrak{H}_Q^{(1,1)}(A, \boldsymbol{Q})$ are canonically isomorphic in the following correspondence

$$A \leftrightarrow \sqrt{-1}(dzA)^t_\Lambda d\bar{z} .$$

The space $\mathfrak{S}_Q(A)$ is a Jordan algebra with the composition

$$\alpha \circ \beta = \tfrac{1}{2}(\alpha\beta + \beta\alpha) .$$

PROPOSITION 7. *If A is an abelian variety, then $(\mathfrak{H}^{(1,1)}(V, \boldsymbol{Q}), \circ)$ is a Jordan algebra canonically isomorphic to the Jordan algebra $(\mathfrak{S}_Q(A, \omega), \circ)$.*

Proof. Since $\Lambda = -\sqrt{-1}\sum_{i=1}^n i(dz_{i\Lambda} d\bar{z}_i)$, using the above notations, we have

$$\begin{aligned} \Lambda\varphi &= -\sqrt{-1}\Big(\sum_{i=1}^n i(dz_{i\Lambda} d\bar{z}_i)\Big)\Big(\sqrt{-1}\sum_{i,j=1}^n a_{ij}(\varphi)dz_{i\Lambda}d\bar{z}_j\Big) \\ &= \mathrm{tr}\, A_\varphi , \end{aligned}$$

$$\begin{aligned} \Lambda(\varphi\cdot\phi) &= \sqrt{-1}\Big(\sum_{i=1}^n i(dz_{i\Lambda} d\bar{z}_i)\Big)\Big(\sum_{i,j,p,q}^n a_{i,p}(\varphi)a_{j,q}(\phi)dz_{i\Lambda}d\bar{z}_{p\Lambda}dz_{j\Lambda}d\bar{z}_q\Big) \\ &= \mathrm{tr}\, A_\varphi\phi + \mathrm{tr}\, A_\phi\varphi - \sqrt{-1}(dz(A_\varphi A_\phi + A_\phi A_\varphi))^t_\Lambda d\bar{z} \\ &= \Lambda\phi\cdot\varphi + \Lambda\varphi\cdot\phi - \sqrt{-1}(dz(A_\varphi A_\phi + A_\phi A_\varphi))^t_\Lambda d\bar{z} . \end{aligned}$$

This shows that

$$A_{\varphi\circ\phi} = \tfrac{1}{2}(A_\varphi A_\phi + A_\phi A_\varphi) .$$

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