PENCILS OF NULL POLARITIES

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1. Introduction. A theorem due to von Staudt states that a null polarity in complex projective space of three dimensions is determined by a self-polar skew pentagon. By allowing an element of the self-polar pentagon to vary in a suitable manner we can arrive at a family of ∞^1 null polarities, which we term a *pencil of null polarities.* Each polarity of the pencil distinguishes a linear complex as the class of self-polar lines. Thus, associated with the pencil is a family of ∞^1 linear complexes, which we term a *pencil of linear complexes.*

It is the purpose of this paper to continue an earlier investigation of pencils of polarities (**2**), by applying analogous techniques to the study of pencils of null polarities and pencils of linear complexes.

Since it develops that the lines common to all linear complexes of a pencil are the lines of a linear congruence, the central question has been: How many of the different types of linear congruences can be achieved in this manner? Happily, it can be reported that the classification of pencils of null polarities yields all of the three types of linear congruences (**4**, pp. 140-141).

We conclude with some remarks concerning such pencils in real projective space.

2. Basic notions and constructions. Our basic configuration (see Figure 1) shall be the skew pentagon $PQRST$, where we designate certain planes by Greek letters as follows:

$$TPQ = \pi,\ PQR = \eta,\ QRS = \rho,\ RST = \sigma,\ STP = \tau.$$

The pentagon will be called *complete* if no four vertices are coplanar, and *self-polar* with respect to a null polarity Π if Π makes the following correspondence:

$$P \rightarrow \pi,\ Q \rightarrow \eta,\ R \rightarrow \rho,\ S \rightarrow \sigma,\ T \rightarrow \tau.$$

The correlation Γ which distinguishes a single line l in such a manner that Γ maps each point X (not on l) into plane Xl, and each plane χ (not through l) into point $\chi \cdot l$, is clearly singular. We call Γ a *special null polarity* with *directrix* l. (In this connotation some authors might object to use of the word "polarity" since Γ is not $1-1$.) The points of l and the planes through l do not belong to the domain of definition of Γ.

VON STAUDT'S THEOREM. *If $PQRST$ is a complete pentagon, the correlation*

$$P \rightarrow \pi,\ Q \rightarrow \eta,\ R \rightarrow \rho,\ S \rightarrow \sigma,\ T \rightarrow \tau$$

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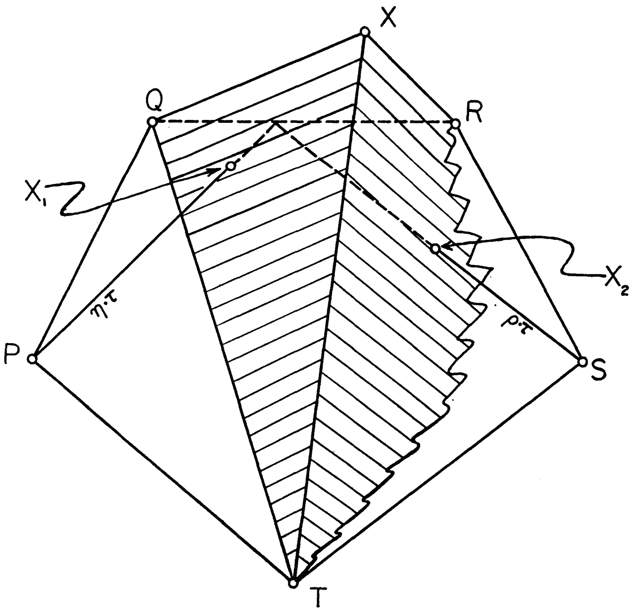

Figure 1

is a null polarity; that is, a complete self-polar pentagon determines a null polarity.

Since much of our work shall deal with the above correlation when the pentagon is not complete, we state the following: *If four vertices, say P, Q, R, and T are coplanar (with no three collinear), then $\pi = \eta$, and the correspondence*

$$P \to \pi,\ Q \to \eta,\ \rho \to R,\ S \to \sigma,\ \tau \to T$$

is a special null polarity with directrix RT. The verification of this fact is quite simple.

THEOREM 1. *Let X be a point in general position. The polar of X in the null polarity defined by the self-polar pentagon $PQRST$ is*

$$\chi = X(XQT\cdot\eta\cdot\tau)(XRT\cdot\rho\cdot\tau). \tag{1}$$

Proof. (See Figure 1.) The pole of every plane through QT is on $\eta\cdot\tau$. Therefore, the pole of XQT is the point $X_1 = XQT\cdot\eta\cdot\tau$. Similarly, the pole of XRT is the point $X_2 = XRT\cdot\rho\cdot\tau$. Thus, the three points X, X_1 and X_2 lie on χ, and determine it according to (1).

In the event that the null polarity is special, (1) reduces to the simple expression of χ as the join of X with the directrix.

The dual procedure yields the construction for the pole of an arbitrary plane.

3. Pencils of null polarities.

Definition. A pencil of null polarities is the set of null polarities defined by the self-polar pentagon $PQRST$, where P, Q, R, and S are all fixed, while T varies on a line t (not through any other vertices) in its fixed polar plane τ.

It follows from the definition that plane π and/or σ also vary in axal pencils about PQ and RS, respectively.

THEOREM 2. *The polar planes χ ;f any fixed point X, with respect to a pencil of null polarities, form a pencil of planes.*

Proof. Referring to Figure 1, we call $X_1 = XQT\cdot\eta\cdot\tau$ and $X_2 = XRT\cdot\rho\cdot\tau$. Then, by Theorem 1, $\chi = X\ X_1X_2$. Let T vary on t, and consider the projectivity which maps the points of $\eta\cdot\tau$ onto the points of $\rho\cdot\tau$ as follows:

$$X_1 \overset{P}{\underset{\wedge}{=}} QR\cdot\eta\cdot\tau \overset{S}{\underset{\wedge}{=}} X_2$$

This projectivity possesses an invariant point, which arises when T is on plane XQR. Thus, the projectivity is a perspectivity. If we call the centre of perspectivity 0, then χ is always on the line XO.

We classify null polarities into three types, according to whether the line t meets none, one or two of the fixed edges of the pentagon. Precisely,

(i) P, Q, R, and S non-coplanar, and t in general position (not meeting any of the fixed edges of the defining pentagon).

(ii) P, Q, R, and S non-coplanar, but t meets exactly one fixed edge of the pentagon. It follows from the definition of a pencil (τ being fixed) that the single edge which t meets must be QR.

(iii) The line t meets two of the fixed edges of the defining pentagon.

It is easy to see that any other degeneracy of the pentagon, or any other position of t yields a pencil equivalent to one of the above types. We shall refer to these as the *general*, *parabolic*, and *degenerate systems*, respectively. (The justification for these names will be seen in §4.)

The general system. T varies on the line t not through any of the fixed lines of the pentagon. Thus, there are two distinct positions of T, in the plane η and in the plane ρ, which yield special null polarities. Although it may appear that further special null polarities arise when T is coplanar with P, R, and S, or similarly when T is coplanar with P, Q, and S, this is not the case. For, in these two instances it is possible to choose an alternate position anywhere on $\pi\cdot\sigma$ for the fifth point T of the self-polar pentagon; thus, showing that the polarity is, in fact, not special. Therefore, *there are exactly two special null polarities in a general system.*

THEOREM 3. *If two null polarities belong to the same general system, their product is a general axal homography* (**1**, 385); *conversely, every general axal* *homography can be expressed as the product to two null polarities belonging to the same general system.*

Proof. The product of two such polarities leaves fixed the following elements: P, Q, R, S, η, ρ, τ, and t. Hence, $\tau \cdot QR$ is also fixed giving three invariant points on the line QR. It follows that QR is pointwise invariant, providing a point-axis for the homography. The tangential-axis is t, with the fixed points $t \cdot \eta$ and $t \cdot \rho$. Thus, the conditions for the general axal homography are established.

For the converse let the homography have QR as its point-axis and MN as its tangential-axis, with the collineation on the tangential axis defined by the projectivity $MNT_1 \barwedge MNT_2$. Then the given homography is the product of the two null polarities $PQRST_1$ and $PQRST_2$, where P and Q are arbitrary points on QRM and QRN, respectively.

The parabolic system. This pencil is defined by having the line t, the locus of T, meet the opposite edge QR. The position of $T = t \cdot QR$ yields a special null polarity, while all other positions of T yield non-special null polarities. Thus, *a parabolic system possesses exactly one special null polarity.*

THEOREM 4. *If two null polarities belong to the same parabolic system, their product is a biaxal homography* (**1**, pp. 385-386). *Conversely, every biaxal homography can be expressed as the product of two null polarities belonging to the same parabolic system.*

Proof. As in the proof of Theorem 3, the line QR is pointwise invariant under the product, since Q, R, and $t \cdot QR$ are three invariant points on the line. Further, the line PS is also pointwise invariant under the product, since P, S, and $t \cdot PS$ are three invariant points on that line. A biaxal homography is the projective transformation characterized by two such lines.

For a proof of the converse, let the homography be determined by the axes PS and QR, and a pair of corresponding points A and B (see Figure 2). Let T_1 be a point in general position, and call

$$\tau = PST_1 \quad \text{and} \quad t = \tau \cdot QRT_1$$

The null polarity $\Pi_1 = PQRST_1$ belongs to a parabolic system with T_1 varying on t. We shall determine another polarity of this system to satisfy the requirements of the theorem. We call

$$A_1 = AQT_1 \cdot \eta \cdot \tau, \quad A_2 = ART_1 \cdot \rho \cdot \tau \quad \text{and} \quad \alpha = A\,A_1A_2.$$

Then Π_1 maps $A \to \alpha$. Since τ is fixed for all members of the parabolic system, we may consider BA_1A_2 as the polar plane of B under one of the polarities in the system. To determine this polarity explicitly, we follow the converse construction calling $T_2 = BQA_1 \cdot t (= BRA_2 \cdot t)$. The desired polarity is then $\Pi_2 = PQRST_2$. The product $\Pi_1\Pi_2$ yields the original biaxal homography.

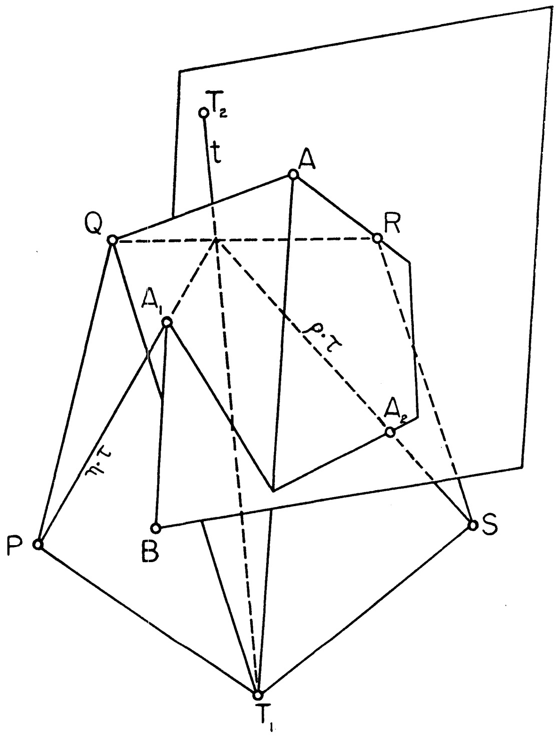

Figure 2

The degenerate system. To fix ideas let t meet edges PQ and QR. Thus, the points P, Q, R and T are always coplanar forcing every null polarity of the pencil to be special with directrix RT. The fixed plane τ is not defined in the degenerate system, sparing us the need of having S coplanar with the other four points, the consequence of which would be complete degeneracy with no null polarities defined. We may therefore conclude that *all the polarities of a degenerate system are special with their directrices forming a flat pencil of lines.*

4. Pencils of linear complexes. The self-polar lines of each null polarity of a pencil form a linear complex. The ∞^1 linear complexes which arise from a pencil are referred to as a *pencil of linear complexes* (**2**, pp. 92-93; **5**, pp. 332-333). Theorem 2 implies that each point X may be associated with a line x, called *the axis of* X, which is self-polar for all polarities in the pencil. (The axis of X is precisely the line OX mentioned in the proof of Theorem 2.) Further, every point of x is associated with the same axis. Thus, there are ∞^2 axes, and we may state

THEOREM 5. *The set of lines self-polar for all polarities of a pencil form a linear congruence.*

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There are exactly three types of linear congruences in complex projective space (**3**, pp. 140-141). Our main result is that all three can be achieved as the axes of the three types of pencils of null polarities. More precisely, we state

THEOREM 6. (i) *The axes of a general pencil form a general linear congruence.*

(ii) *The axes of a parabolic pencil form a parabolic linear congruence.*

(iii) *The axes of a degenerate pencil form a degenerate linear congruence.*

Proof. (i) Each linear complex of the pencil has among its self-polar lines a flat pencil in η with vertex Q and a flat pencil in ρ with vertex R. Thus, the lines $R(t\cdot\eta)$ and $Q(t\cdot\rho)$ are the directrices of the linear congruence of axes. Further, $t\cdot\eta \neq t\cdot\rho$, from which it follows that the directrices are skew. Hence, the axes form a general linear congruence.

It is interesting to note that the line t is a member of the congruence, so that we could deduce—independently of the definition—that the plane τ must be fixed in a general pencil.

(ii) The two directrices of (i) become coincident (with QR), and all the axes meet QR. Since the polar plane of $QR\cdot t$ is QRT in every polarity of the pencil, we know that t is also an axis. Hence, we have the parabolic linear congruence established by the four linearly independent axes PQ, QR, RS, and t.

(iii) Let the degenerate system be defined by having t meet both PQ and QR. In §2, we saw that all the linear complexes of this pencil are special with directrix RT. Hence, all the lines of η belong to the linear congruence of lines which are common to all the linear complexes of the pencil.

Now consider a point X in general position. Its polar plane is XRT. Therefore, XR is self-polar in every polarity under consideration. The desired degenerate congruence is then established as consisting of the bundle of lines with vertex R plus the set of all lines in η.

Before concluding we turn attention to pencils of linear complexes in *real* projective space. In this case the general linear congruence is classified as hyperbolic or elliptic according as its directrices are proper (real) or improper (**2**, p. 93; **5**, pp. 315-318). We therefore consider whether both the hyperbolic and the elliptic congruences can be attained as the axes of different types of general pencils.

An examination of the proof of Theorem 6 (i) shows that nothing is altered by the condition that $PQRST$ be a real pentagon. The congruence of axes has as its directrices the lines $R(t\cdot\eta)$ and $Q(t\cdot\rho)$, both of which are real. Hence, *a hyperbolic congruence results*. However, the elliptic congruence is unattainable for the very reason that lines $R(t\cdot\eta)$ and $Q(t\cdot\rho)$ are real lines (and are always the directrices). A further argument which also easily establishes the unattainability of the elliptic congruence (when P, Q, R, S, and T are real) is the following: An elliptic congruence is generated by four linearly independent skew lines, such that no one of them meets the regulus containing the other three in a proper point (**5**, p. 315). Yet our general system —indeed, all of our cases—always include three non-skew linearly independent lines PQ, QR, and RS, as generators. Hence *it is impossible that the congruence of axes be elliptic.*

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