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COMMUTATIVE NON-SINGULAR SEMIGROUPS

BY
C. S. JOHNSON, JR. AND F. R. McMORRIS

Introduction. It is well known (see [5]) that the maximal right quotient ring of a ring R is (von Neumann) regular if and only if R is (right) non-singular (every large right ideal is dense). In [8] it was shown that for a semigroup S, the regularity of $Q(S)$, the maximal right quotient semigroup [7], is independent of the non-singularity of S. Nevertheless, right non-singular semigroups form an important class of semigroups. Hinkle [3] showed for example that, as in the ring theory situation, if S is right non-singular then $Q(S)$ is the injective hull of S in the category of S-systems and is self-injective as a $Q(S)$-system. In addition, a right non-singular semigroup is weakly self-injective if and only if it is self-injective (Proposition 3.4 of [3]).

To date, non-singular semigroups have been determined among the following classes of semigroups: completely o-simple, primitive dependent [2], semilattices, and semilattices of groups [4]. In this paper we characterize commutative non-singular semigroups for which every ideal is finitely generated. As a corollary we improve a recent result of Lopez and Luedeman [6] and point out that if S is a commutative non-singular semigroup, then $Q(S)$ is regular.

Results. S will always denote a commutative semigroup with zero 0. In such a semigroup we recall the following definitions: An ideal I of S is (intersection) *large* if it has a non-zero intersection with every non-zero ideal of S, and is *dense* if $x \neq y$ $(x, y \in S)$ implies that there exists $d \in I$ such that $xd \neq yd$. S is *non-singular* if every large ideal is dense and *separative* if the relation $x \leq y$ if and only if $xy = x^2$ is antisymmetric. Notice that if S is separative then it has no non-zero nilpotent elements. We let z^* denote the annihilator of an element z and I^* the annihilator of an ideal I. S is *disjunctive* if for any $x, y \in S$, $x^* = y^*$ implies $x = y$.

Recall [5, p. 108] that a commutative ring is non-singular if and only if it contains no non-zero nilpotent elements. Thus it is easy to show that a commutative ring is non-singular if and only if its multiplicative semigroup is separative. This fact, together with our result [4] that a semilattice is non-singular if and only if it is disjunctive, led us to inquire whether disjunctive

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and/or separative properties characterize commutative non-singular semigroups in general.

DEFINITION. S is *quasi-disjunctive* if $x \neq y$ and $xz = yz$ imply the existence of $d \in z^*$ such that $xd \neq yd$.

PROPOSITION. *Assume S contains no non-zero nilpotent elements.*

(i) *If S is disjunctive, then S is quasi-disjunctive.*

(ii) *If S is quasi-disjunctive and satisfies the identity $x^2y = xy^2$, then S is disjunctive.*

Proof. (i) If $x \neq y$, then $x^* \neq y^*$ and we may assume without loss of generality that there exists an element a such that $ya = 0$ and $xa \neq 0$. Now $x^2a \neq 0$, and setting $d = ax$ we have $yd = 0$ and $xd \neq 0$. Suppose $xz = yz$. Then $zd = zxa = zya = 0$ so that $d \in z^*$.

(ii) Assume $x \neq y$. Then $x(xy) = y(xy)$ implies that there exists $d \in (xy)^*$ such that $xd \neq yd$. Suppose $yd \neq 0$. Then with $d' = yd$ we have $xd' = 0$ and $yd' \neq 0$.

We remark that disjunctive and quasi-disjunctive are equivalent when S is a semilattice and also when S is separative with $xy \leq x^2$ for all $x, y \in S$.

LEMMA. *If S is quasi-disjunctive, then S is separative.*

Proof. Suppose that $xy = x^2 = y^2$ and that $x \neq y$. Using $xy = y^2$ we get d_1 with $xd_1 \neq yd_1 = 0$. Using $(xd_1)x = (yd_1)x$ we get d_2 with $xd_1d_2 \neq yd_1d_2$ and $xd_2 = 0$, a contradiction.

THEOREM. *If S is non-singular, then S is quasi-disjunctive. The converse is true when all ideals of S are finitely generated.*

Proof. Assume S is non-singular, $xz = yz$, and $x \neq y$. Now $I = zS \cup z^*$ is easily seen to be a large ideal of S and thus it is dense. Hence we have $d \in I$ with $xd \neq yd$. If $d \notin z^*$, then $d \in zS$ gives $xd = xzw = yzw = yd$ for some $w \in S$, and this is a contradiction.

Suppose now that S is quasi-disjunctive with all ideals finitely generated and let I be a large ideal of S generated by $z_1, z_2, \ldots, z_n$. If $xd = yd$ for all $d \in I$ and $x \neq y$, then there exist $z'_1, z'_2, \ldots, z'_n \in S$ such that $xz'_1 \neq yz'_1$, $xz'_1z'_2 \neq yz'_1z'_2, \ldots, xz'_1z'_2 \cdots z'_n \neq yz'_1z'_2 \cdots z'_n$ and $z_iz'_i = 0$ for $i = 1, \ldots, n$. Now if $w = z'_1z'_2 \cdots z'_n$ we have $xw \neq yw$ so that $w \neq 0$. Since $w \in I^*$ we have $I^* \neq \{0\}$ and I large implies $I \cap I^* \neq \{0\}$. This is impossible since S is separative by the Lemma and thus contains no non-zero nilpotent elements. This contradiction gives $xd \neq yd$ for some $d \in I$ so that I is dense.

EXAMPLE. To see that the converse of the theorem fails in general let S be the semilattice Γ of groups G_γ $(\gamma \in \Gamma)$ [1, Theorem 4.11] where $\Gamma = \{(0, 1/n) : n$ positive integer$\} \cup \{(1/n, 0) : n$ positive integer$\} \cup \{0\}$ with $(0, 1/m) \leq (0, 1/n)$

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when $1/m \leq 1/n$, $(1/m, 0) \leq (0, 1/n)$ when $1/m \leq 1/n$, and 0 smaller than everything. Let G_γ be the cyclic group of order two when $\gamma = (0, 1/n)$ and the one element group otherwise, and define the linking homomorphism between two cyclic groups of order two to be the identity homomorphism. It is easy but slightly tedious to verify that Γ is a semilattice, that S is quasi-disjunctive, and that $\bigcup\{G_\gamma : \gamma \neq (0, 1/n)\}$ is a large ideal that fails to be dense.

COROLLARY. *If S is non-singular, then S is separative.*

Rompke in [9] showed that if S is separative and for all $a \in S$ the ideal $\Gamma(a) = \{a \in S$: there exists $b \in aS$ such that $bt = st$ for all $t \in aS\}$ is dense, then $Q(S)$ is regular. Lopez and Luedeman [6] have shown that if S is non-singular then $\Gamma(a)$ is dense for all $a \in S$. Thus we have the following.

COROLLARY. *If S is non-singular, then $Q(S)$ is regular.*

Example 1 of [8] suffices to show that the converse to this corollary is not true.

We can also drop the separative condition from Theorem 6 of [6], making it read

COROLLARY. *If S is non-singular, then $H = \mathrm{Hom}_S(I, I)$ is regular, where I is the injective hull of S in the category of S-systems.*

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DEPT. OF MATH.,
BOWLING GREEN STATE UNIVERSITY,
BOWLING GREEN, OHIO 43403
U.S.A.

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