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APPROXIMATE CONTROLLABILITY OF POPULATION DYNAMICS WITH SIZE DEPENDENCE AND SPATIAL DISTRIBUTION

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Abstract

We investigate the approximate controllability of a size- and space-structured population model, for which the control function acts on a subdomain and corresponds to the migration of individuals. We establish the main result via the unique continuation property of the adjoint system. The desired controller is the minimizer of an infinite-dimensional optimization problem.

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1. Introduction

Age-structured population models have been a popular area of research over the past decade [3]. For many species (for example, fish and plants), however, size-structured models are likely to be more appropriate tools for analysis than age-structured ones [6]. So far, size-structured population modelling has been an active and fruitful field since the 1960s [12]. Barbu et al. [4] worked on the exact controllability by migration and birth control. However, they took no diffusion into account, and the method there cannot be applied to the case where the control acts on a space area. Some controllability results for a linear age- and space-structured population dynamics were obtained by Ainseba and Langlais [1, 2]. They showed that a certain set of profiles is approximately reachable at any given time. Traore [13] studied an application of the approximate controllability to data assimilation problems. The main result was derived from a new Carleman's inequality [7, 14]. As far as we know, there is only one article, by Boulite et al. [5], treating the controllability of population systems with size structure, where the exact controllability by boundary control was proved by the

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semigroup theory, and no space distribution was considered. On the other hand, there is a lot of work involving the controllability of age-structured systems (see [8] and the references therein).

This paper explores the approximate controllability for a kind of population model with size structure and spatial location. A description of the problem and some basic hypotheses are given in the next section, while Section 3 consists of the controllability analysis and Section 4 concludes the paper.

2. Description of the problem

We consider the following model, governing the dynamics of a single species:

$$\begin{cases} \dfrac{\partial p(s,t,x)}{\partial t} + \dfrac{\partial (g(s)p(s,t,x))}{\partial s} - \Delta p + \mu(s)p(s,t,x) = v(s,t,x)\chi_\omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (s,t,x) \in Q, \\ g(0)p(0,t,x) = \displaystyle\int_0^m \beta(s)p(s,t,x)\,ds, \qquad (t,x) \in (0,T)\times\Omega, \\ p(s,0,x) = p_0(s,x), \qquad (s,x) \in (0,m)\times\Omega, \\ \dfrac{\partial p}{\partial \eta}(s,t,x) = 0, \qquad (s,t,x) \in \Sigma, \end{cases} \tag{2.1}$$

where $Q = (0,m)\times(0,T)\times\Omega$; $p(s,t,x)$ is the distribution of individuals of size s at time t and location x in Ω, which is a bounded open set in $\mathbb{R}^d$; $d \in \{1,2,3\}$ with a suitably smooth boundary $\partial\Omega$. The (finite) maximum size of individuals is $m > 0$ and the horizon of control is $T > 0$. The function $g(s) = ds/dt$ models the size-specific growth rate and the rates $\mu(s)$ and $\beta(s)$ denote the mortality and fertility, respectively. The Laplacian operator of p, Δp, describes spatial diffusion of individuals, and $v(s,t,x)$ stands for emigration. By χ_ω, we denote the characteristic function of an open subdomain, ω, of population habitat Ω. The initial distribution of the population is $p_0(s,x)$. For the sake of simplicity, we suppose that the size of newborns is zero. In the domain $\Sigma = (0,m)\times(0,T)\times\partial\Omega$, the no-flux boundary condition, $\partial p(s,t,x)/\partial\eta = 0$, implies that the population inhabits an isolated environment, where $\partial/\partial\eta$ is the outward normal derivative.

Throughout this paper the following assumptions hold:

- (H1) $p_0(s,x) \geq 0$ almost everywhere (a.e.), $(s,x) \in (0,m)\times\Omega$ and $p_0 \in L^\infty((0,m)\times\Omega)$;
- (H2) $\beta \in L^\infty(0,m), \beta(s) \geq 0$ a.e., $s \in (0,m)$; moreover, there exist $m_0 < m_1$ such that $\mathrm{supp}(\beta) \subset [m_0,m_1] \subset (0,m)$, where $\mathrm{supp}(\beta) = \{x \in \mathbf{R}, \beta(\mathbf{x}) \neq \mathbf{0}\}$;
- (H3) $\mu \in L^\infty_{\mathrm{loc}}([0,m)), \mu(s) \geq 0, \int_0^m \mu(s)\,ds = +\infty$;
- (H4) $g \in C^1[0,m], 0 < g(s) \leq g^*, g'(s) + \mu(s) \geq 0$ a.e., $s \in [0,m]$, g^* is a constant;
- (H5) $\omega \subset \bar{\omega} \subset \Omega$, ω has positive Lebesgue measure with same dimension as Ω. $\mathrm{supp}(v) \subset (0,m)\times(0,T)\times\omega$;
- (H6) μ, β, g, p_0 are extended by zero outside their domains. Hereafter $\Gamma(s) = \int_0^s (dv/g(v)), \Gamma(m) < \infty$.

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DEFINITION 2.1. For any given T, distribution $p_d(s,x)$ and $\varepsilon > 0$, if there exists a control $v \in L^2((0,m)\times(0,T)\times\omega)$ such that the corresponding solution to (2.1) satisfies

$$\|p(s,T,x) - p_d(s,x)\| \le \varepsilon, \tag{2.2}$$

then system (2.1) is called approximately controllable.

REMARK 2.2. In a similar manner as in Aniţa's work [3, pp. 148–154], one can show that, under (H1)–(H6), there exists a unique solution to (2.1) for given parameters.

3. Controllability

For convenience, we assume that the initial distribution $p_0(s,x)$ is zero; otherwise, one can make a translation of data and obtain the corresponding results.

The controllability result is mainly based on the Hilbert uniqueness method [10, 11], so we need some auxiliary results as follows.

LEMMA 3.1 [9]. *Let $\{c_j\}_{j\ge1}$ be a sequence of complex numbers such that $\sum_{j\ge1} e^{2\lambda_j\tau}|c_j|^2 < \infty$ for some $\tau > 0$. Then the function $z = \sum_{j\ge1} c_j\varphi_j \in L^2(\Omega)$ is well defined and, if $z \equiv 0$ on a nonempty open subset $\omega \subset \Omega$, then $z \equiv 0$ on Ω and $c_j = 0$ for all $j \ge 1$. Here λ_j and φ_j are the eigenvalue and corresponding eigenvector, respectively, of $-\Delta$, where Δ is the Laplacian operator.*

REMARK 3.2. For $0 < s < m$, let

$$\pi(s) = \exp\Big\{-\int_0^s \frac{\mu(\tau)+g'(\tau)}{g(\tau)}\,d\tau\Big\};$$

here we assume that $\pi^{-1}p_0 \in L^2((0,m)\times\Omega)$.

REMARK 3.3. Set $\hat{p}(s,t,x) = \pi(s)^{-1}p(s,t,x), (s,t,x) \in Q$. Then $\hat{p}$ is the solution of the following system:

$$\begin{cases} \dfrac{\partial \hat{p}(s,t,x)}{\partial t} + g(s)\dfrac{\partial \hat{p}(s,t,x)}{\partial s} - \Delta\hat{p} = \hat{v}(s,t,x)\chi_\omega, (s,t,x) \in Q, \\ g(0)\hat{p}(0,t,x) = \displaystyle\int_0^m \hat{\beta}(s)\hat{p}(s,t,x)\,ds, & (t,x) \in (0,T)\times\Omega, \\ \hat{p}(s,0,x) = \hat{p}_0(s,x), & (s,x) \in (0,m)\times\Omega, \\ \dfrac{\partial \hat{p}}{\partial \eta}(s,t,x) = 0, & (s,t,x) \in \Sigma, \end{cases} \tag{3.1}$$

where $\hat{p}_0(s,x) = \pi(s)^{-1}p_0(s,x), \hat{\beta}(s) = \pi(s)\beta(s)$ and $\hat{v}(s,t,x) = \pi(s)^{-1}v(s,t,x)$.

REMARK 3.4. According to the Hilbert uniqueness method, we formulate an adjoint system of (3.1) as follows:

$$\begin{cases} -\dfrac{\partial \rho(s,t,x)}{\partial t} - \dfrac{\partial (g(s)\rho(s,t,x))}{\partial s} - \Delta\rho = \rho(0,t,x)\hat{\beta}(s), & (s,t,x) \in Q, \\ \rho(s,T,x) = h(s,x), & (s,x) \in (0,m)\times\Omega, \\ \rho(m,t,x) = 0, & (t,x) \in (0,T)\times\Omega, \\ \dfrac{\partial \rho}{\partial \eta}(s,t,x) = 0, & (s,t,x) \in \Sigma, \end{cases} \tag{3.2}$$

where $h(s,x) \in L^2((0,m)\times\Omega)$ is any fixed function.

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Proposition 3.5. *Let the hypotheses* (H1)–(H6) *be satisfied. If the solution* ρ *of* (3.2) *satisfies* $\rho(0,t,x)=0$ *in* $(0,T)\times\omega$, *then* $\rho(s,t,x)\equiv 0$ *in* $(0,m)\times(0,T)\times\Omega$.

Proof. It is more convenient to prove the result for a forward system; hence, by setting $z(s,t,x)=\rho(m-s,T-t,x)$, we just prove that $z\equiv 0$ when $z(m,t,x)=0$ in $(0,T)\times\omega$. Note that z satisfies the following system:

$$\begin{cases} \dfrac{\partial z(s,t,x)}{\partial t}+g(s)\dfrac{\partial z(s,t,x)}{\partial s}-\Delta z=z(m,T-t,x)\hat{\beta}(m-s), \\ \qquad\qquad\qquad\qquad\qquad\qquad (s,t,x)\in Q, \\ z(s,0,x)=h(m-s,x), \quad (s,x)\in(0,m)\times\Omega, \\ z(0,t,x)=0, \quad (t,x)\in(0,T)\times\Omega, \\ \dfrac{\partial z}{\partial\eta}(s,t,x)=0, \quad (s,t,x)\in\Sigma. \end{cases} \tag{3.3}$$

It is well-known that $(\varphi_j)_{j\geq 1}$ is a Hilbert basis of $L^2(\Omega)$, so we may write

$$z(s,t,x)=\sum_{j=1}^{\infty} z_j(s,t)\varphi_j(x).$$

The first equation in the system (3.3) can be changed into the form

$$\frac{\partial\Big\{\sum_{j=1}^{\infty} z_j(s,t)\varphi_j(x)\Big\}}{\partial t}+g(s)\frac{\partial\sum_{j=1}^{\infty} z_j(s,t)\varphi_j(x)}{\partial s}-\Delta\Big\{\sum_{j=1}^{\infty} z_j(s,t)\varphi_j(x)\Big\}$$
$$=z(m,T-t,x)\hat{\beta}(m-s),$$

which simplifies to

$$\sum_{j=1}^{\infty}\frac{\partial z_j(s,t)}{\partial t}\varphi_j(x)+g(s)\sum_{j=1}^{\infty}\frac{\partial z_j(s,t)}{\partial s}\varphi_j(x)-\sum_{j=1}^{\infty} z_j(s,t)\Delta\varphi_j(x)=z(m,T-t,x)\hat{\beta}(m-s)$$

and then

$$\Big\langle\varphi_j(x),\sum_{j=1}^{\infty}\frac{\partial z_j(s,t)}{\partial t}\varphi_j(x)+g(s)\sum_{j=1}^{\infty}\frac{\partial z_j(s,t)}{\partial s}\varphi_j(x)-\sum_{j=1}^{\infty} z_j(s,t)\Delta\varphi_j(x)\Big\rangle$$
$$=\langle\varphi_j(x),z(m,T-t,x)\hat{\beta}(m-s)\rangle.$$

The system (3.3) is equivalent to the following linear hyperbolic system:

$$\begin{cases} \dfrac{\partial z_j(s,t)}{\partial t}+g(s)\dfrac{\partial z_j(s,t)}{\partial s}+\lambda_j z_j(s,t)=\gamma_j(s,t), & (s,t)\in(0,m)\times(0,T), \\ z_j(s,0)=k_j(s), & s\in(0,m), \\ z_j(0,t)=0, & t\in(0,T), \end{cases}$$

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where

$$\gamma_j(s,t) = \int_\Omega \varphi_j(x)\hat{\beta}(m-s)z(m,T-t,x)\,dx,$$
$$k_j(s) = \int_\Omega \varphi_j(x)h(m-s,x)\,dx.$$

By means of characteristic curves,

$$z_j(m,t) = \begin{cases} k_j(\Gamma^{-1}(\Gamma(m)-t))e^{-\lambda_j t} + \displaystyle\int_0^t \gamma_j(\Gamma^{-1}(\Gamma(m)-\tau), t-\tau)e^{-\lambda_j\tau}\,d\tau, & t \le \Gamma(m), \\ \displaystyle\int_0^{\Gamma(m)} \gamma_j(\Gamma^{-1}(\Gamma(m)-\tau), t-\tau)e^{-\lambda_j\tau}d\tau, & t > \Gamma(m). \end{cases} \tag{3.4}$$

From the hypothesis (H2), it follows that

$$\gamma_j(\Gamma^{-1}(\Gamma(m)-\tau), t-\tau) = 0, \quad 0 \le t \le \min\{m_0, m - \Gamma^{-1}(\Gamma(m)-m_0)\}$$

and

$$z_j(m,t) = k_j(\Gamma^{-1}(\Gamma(m)-t))e^{-\lambda_j t},$$
$$0 \le t \le \min\{m_0, m-\Gamma^{-1}(\Gamma(m)-m_0)\}.$$

Since $k_j \in L^2(\Omega)$, based on Lemma 3.1, we conclude that

$$z_j(m,t,x) = 0, \quad (t,x) \in (0, \min\{m_0, m-\Gamma^{-1}(\Gamma(m)-m_0)\}) \times \Omega. \tag{3.5}$$

Next we consider the case $\min\{m_0, m-\Gamma^{-1}(\Gamma(m)-m_0)\} \le t < m$ and set the coefficients

$$\alpha_j(t) = \int_0^t \gamma_j(\Gamma^{-1}(\Gamma(m)-\tau), t-\tau)e^{-\lambda_j\tau}\,d\tau.$$

From (3.5), it follows that

$$\alpha_j(t) = \int_{\min\{m_0, m-\Gamma^{-1}(\Gamma(m)-m_0)\}}^t \gamma_j(\Gamma^{-1}(\Gamma(m)-\tau), t-\tau)e^{-\lambda_j\tau}\,d\tau.$$

Obviously,

$$|\alpha_j(t)| \le e^{-\lambda_j \min\{m_0, m-\Gamma^{-1}(\Gamma(m)-m_0)\}} \max(m-m_0, \Gamma^{-1}(\Gamma(m)-m_0))\|z(m,t,x)\|.$$

Combining Lemma 3.1 with (3.4) yields

$$z_j(m,t,x) = 0, \quad (t,x) \in \{\min\{m_0, m-\Gamma^{-1}(\Gamma(m)-m_0)\}, m\} \times \Omega.$$

Thus, we have shown that

$$z(m,t,x) = 0, \quad (t,x) \in (0,m) \times \Omega. \tag{3.6}$$

As for the case $\Gamma(m) \le t \le T$, the same method can be used to obtain equation (3.6). This completes the proof. □

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Now let J be the functional defined on $L^2((0,m)\times\Omega)$ by

$$J(h)=\frac{1}{2}\int_{(0,m)\times(0,T)\times\omega}\rho^2(s,t,x)\,dx\,dt\,ds+\varepsilon\|h\|-\int_{(0,m)\times\Omega}\hat{p}_d(s,x)h(s,x)\,ds\,dx, \tag{3.7}$$

where $\hat{p}_d(s,x)=\pi^{-1}(s)p_d(s,x)$ and ρ is the solution of (3.2) corresponding to h. Then we have the following result.

Proposition 3.6. *The functional J is continuous, strictly convex and coercive. More precisely,*

$$\liminf_{\|h\|\to\infty}\frac{J(h)}{\|h\|}\geq\varepsilon, \tag{3.8}$$

where $\|h\|=\|h\|_{L^2((0,m)\times\Omega)}$. The functional J achieves its minimum at a unique point $\hat{h}\in L^2((0,m)\times\Omega)$.

Proof. Consider a sequence $\{h_n\}_{n\geq1}\subset L^2((0,m)\times\Omega)$ such that $\|h_n\|\to\infty$ as $n\to\infty$. Let

$$\hat{h}_n=\frac{h_n}{\|h_n\|}$$

and denote by $\hat{\rho}_n$ the associated solution of (3.2) with $h=\hat{h}_n$. Then

$$\frac{J(h_n)}{\|h_n\|}=\frac{\|h_n\|}{2}\int_{(0,m)\times(0,T)\times\omega}\hat{\rho}_n^2(s,t,x)\,dx\,dt\,ds+\varepsilon-\int_{(0,m)\times\Omega}\hat{p}_d(s,x)\hat{h}_n(s,x)\,ds\,dx.$$

If

$$\int_{(0,m)\times(0,T)\times\omega}\hat{\rho}_n^2(s,t,x)\,dx\,dt\,ds>0,$$

then (3.8) holds. If

$$\int_{(0,m)\times(0,T)\times\omega}\hat{\rho}_n^2(s,t,x)\,dx\,dt\,ds=0,$$

then there must be a subsequence, still denoted by $\{\hat{h}_n\}_{n\geq1}$, such that

$$S-\lim_{n\to\infty}\int_{\omega\times(0,T)}\hat{\rho}_n^2(0,t,x)\,dx\,dt=0\ (\textit{strong convergence}),$$

$$W-\lim_{n\to\infty}\hat{h}_n(s,x)=\hat{h}(s,x)\quad\text{in } L^2((0,m)\times\Omega)\ (\textit{weak convergence}),$$

$$W-\lim_{n\to\infty}\hat{\rho}_n(s,t,x)=\hat{\rho}_n(s,t,x)\quad\text{in } L^2((0,m)\times(0,T),H_0^1(\Omega)),$$

$$W-\lim_{n\to\infty}\left(\frac{\partial\hat{\rho}_n}{\partial t}+\frac{\partial(g\hat{\rho}_n)}{\partial s}\right)=\frac{\partial\hat{\rho}}{\partial t}+\frac{\partial(g\hat{\rho})}{\partial s}\quad\text{in } L^2((0,m)\times(0,T),H^{-1}(\Omega)).$$

Using the unique continuation result in Proposition 3.5, we conclude that $\hat{\rho}=0$ and $\hat{h}=0$. Therefore, $\hat{h}_n$ is weakly convergent to 0, which implies that

$$\lim_{n\to\infty}\int_{\Omega\times(0,m)}\hat{p}_d(s,x)\hat{h}_n(s,x)\,ds\,dx=0.$$

So, the relation (3.8) is also true and J achieves its minimum at a unique point. □

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Proposition 3.7. *Let $\hat{h}$ be the unique minimizer of J on $L^2((0,m)\times\Omega)$. There exists a control $\hat{v} \in L^2(Q)$ such that the corresponding solution $\hat{p}$ of (3.1) satisfies*

$$\|\hat{p}(s,T,x) - \hat{p}_d(s,x)\| \le \varepsilon. \tag{3.9}$$

More precisely, if $\|\hat{p}_d(s,x)\| \le \varepsilon$, we take $\hat{v} = 0$; if $\|\hat{p}_d(s,x)\| > \varepsilon$, we choose $\hat{v} = \hat{\rho}$, where $\hat{\rho}$ is the solution of system (3.2) with $h = \hat{h}$.

Proof. If $\|\hat{p}_d(s,x)\| \le \varepsilon$, taking $\hat{v} = 0$, then $\hat{p} = 0$ and (3.9) holds. If $\|\hat{p}_d(s,x)\| > \varepsilon$, since $\hat{h}$ minimizes J, it follows that $\partial_h J(\hat{h}) = 0$ when $\hat{h} \ne 0$ and so

$$\begin{aligned}\langle(\partial_h J)(\hat{h}), h\rangle = \int_{\omega\times(0,T)\times(0,m)} \hat{\rho}\rho(s,t,x)\,dx\,dt\,ds + \frac{\varepsilon}{\|\hat{h}\|}\int_{\Omega\times(0,m)} \hat{h}h\,ds\,dx \\ - \int_{\Omega\times(0,m)} \hat{p}_d(s,x)h(s,x)\,ds\,dx = 0.\end{aligned} \tag{3.10}$$

Combining the equations (3.1) and (3.2), we rewrite (3.10) as

$$\int_{\Omega\times(0,m)} \hat{p}(s,T,x)h(s,x)\,dx\,dt + \int_{\Omega\times(0,m)}\Big(\frac{\varepsilon\hat{h}}{\|\hat{h}\|} - \hat{p}_d(s,x)\Big)h(s,x)\,ds\,dx = 0.$$

This means that

$$\hat{p}(s,T,x) - \hat{p}_d(s,x) = -\frac{\varepsilon\hat{h}}{\|\hat{h}\|}$$

holds and so (3.9) follows. □

Now the main result follows.

Theorem 3.8. *Let the assumptions* (H1)–(H6) *hold and $T > m$. Then, for $p_d \in L^2((0,m)\times\Omega)$ and $\varepsilon > 0$, there exists v such that the corresponding solution of the system* (2.1) *satisfies condition* (2.2).

Proof. Since $p = \pi(s)\hat{p}$, $p_d = \pi(s)\hat{p}_d$ and $0 < \pi(s) \le 1$,

$$\begin{aligned}\|p(s,T,x) - p_d(s,x)\| &= \|\pi(s)(\hat{p}(s,T,x) - \hat{p}_d(s,x))\| \\ &\le \|\hat{p}(s,T,x) - \hat{p}_d(s,x)\| \le \varepsilon,\end{aligned}$$

which completes the proof. □

4. Conclusions

For the population model presented here, we have proved that the system is approximately controllable. The conditions supporting the main result are reasonable and biologically meaningful. The controllability property is helpful for biological diversity and population balance, since populations can be driven into a neighbourhood of some desired states in a given period of time, by means of migration. On the other hand, it supplies us with a solid theoretical ground for optimal control and stability problems. Furthermore, we have suggested an appropriate controller, in Proposition 3.7, via the use of an adjoint variable. To find the desired controller, one needs to deal with the infinite-dimensional minimizing problem (3.7). Constructing an effective algorithm will be a future task for us. It should be emphasized that, although the choice of ω does not affect the theoretical result, some other factors must be considered in its practical use, such as feasibility and implementing costs. Generally speaking, ω cannot be too small, otherwise the migration rate ν approximates a delta function, which is impractical in the context. For example, we cannot prevent an infestation by eradicating a pest in a small area.

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