BULL. AUSTRAL. MATH. SOC. MOS 86A05, 34C15, 34A50
VOL. 3 (1970), 391-411.

# On a class of differential equations which model tide-well systems

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For three possible types of tide-well systems the non-dimensional head response, $Y(\tau)$, to a sinusoidal fluctuation of the sea-level is given by the differential equation

$$dY/d\tau + \beta_n^{-1}|Y|^{n/2}\operatorname{sgn}(Y) = \cos\tau\ , \quad n = 1, 2, 3\ .$$

Estimates of the non-dimensional well response

$$Z(\tau) = \sin\tau - Y(\tau)$$

are found by considering the steady state solutions of the above equation. With $n = 2$ the equation is linear and an exact solution can be found; for $n \neq 2$ the equation is non-linear and several methods which give approximate solutions are described. The methods used can be extended to cover other values of $n$; for example, with $n = 4$ the equation corresponds to one governing oscillations near resonance in open pipes.

## 1. Introduction

Recently attention has been focussed on the response of tide wells to sea level oscillations of different frequencies and amplitudes. The water inside the well is usually connected to that outside by means of a circular orifice near the bottom of the well; two other alternatives consist of a long horizontal pipe connection and a constant width vertical

Received 10 August 1970. The author thanks Dr E.O. Tuck and Professor J.R.M. Radok for their interest in the work presented in this paper.

slit. In the past it has been assumed that the water level recorded inside the well is the same as the sea level outside.

The governing differential equations for the three systems are

$$(1.1) \qquad dH/dt + C_n|H|^{n/2}\text{sgn}(H) = dh_0/dt \ , \quad n = 1, 2, 3 \ ,$$

where $h_0(t)$ is the fluctuating sea level, $h_w(t)$ is the level inside the well, $t$ is the time, and the excess external head is

$$(1.2) \qquad H(t) = h_0(t) - h_w(t) \ .$$

$n = 1$ applies for the well with orifice, $n = 2$ for a well with laminar flow in a uniform long horizontal pipe, and $n = 3$ for the well with a sharp edged uniform vertical slit. $C_n$ is a constant depending only on the well dimensions. It is shown in [2] that

$$C_1 = C_c(2g)^{\frac{1}{2}}A_p/A_w \ ,$$

$$C_2 = gA_p^2(8\pi\nu L_pA_w)^{-1} \ ,$$

where $C_c$ is the coefficient of contraction $(\simeq 0.6)$ , $g$ is the gravitational constant, $A_p$ is the area of cross-section of the connection, $A_w$ is the area of cross-section of the well, $L_p$ is the pipe length and $\nu$ the kinematic coefficient of viscosity. By extending the argument used for $n = 1$ to flow through a slit of constant width $W$ , one obtains (1.1) with $n = 3$ and

$$C_3 = (8g)^{\frac{1}{2}}WC_c/3A_w \ .$$

In each case we are interested in finding the well response

$$(1.3) \qquad h_w = h_0 - H$$

for a given input $h_0(t)$ . In particular, much can be learned about the response of the system to an arbitrary input from the steady state response to a sinusoidal input of amplitude $a$ and circular frequency $\omega$ , namely,

$$(1.4) \qquad h_0 = a\sin\omega t \ .$$

Substituting (1.4) in (1.1) and non-dimensionalising by the substitutions $Y = H/a$ and $\tau = \omega t$ gives

(1.5) $$dY/d\tau + \beta_n^{-1}|Y|^{n/2}\mathrm{sgn}(Y) = \cos\tau ,$$

where $\beta_n = \omega a^{1-n/2}C_n^{-1}$ .

The non-dimensional well response

$$Z = h_w/a$$

is then found from

(1.6) $$Z = \sin\tau - Y .$$

With $n = 2$ equation (1.1) is linear and it has an exact solution; for $n \neq 2$ the equation is non-linear and no exact solution has been found. A search of the literature revealed very little; attempts to solve (1.1) with $n = 1$ were reported in [1] and [3]. Several methods for finding approximate solutions for $n = 1, 3$ are developed in the following, each of these supplying additional information about the well response $Z(\tau)$ . These methods can be extended to other values of $n$ ; for example, to $n = 4$ , when (1.1) corresponds to a non-linear equation governing oscillations near resonance in an open pipe [5].

## 2. The linear equation

With $n = 2$ , (1.5) becomes

(2.1) $$dY/d\tau + \beta_2^{-1}Y = \cos\tau ,$$

with $\beta_2 = C_2^{-1}\omega$ .

The solution of this linear equation is

(2.2) $$Y = \alpha_{2h}\sin\left(\tau+\theta_{2h}\right) ,$$

where

(2.3) $$\alpha_{2h} = \left(1+\beta_2^{-2}\right)^{-\frac{1}{2}}$$

and

(2.4) $$\theta_{2h} = \arctan\left(\beta_2^{-1}\right) .$$

The non-dimensional well response is given by (1.6) and can be written

(2.5) $$Z = \alpha_{2w}\sin\left(\tau-\theta_{2w}\right)$$

where

(2.6) $$\alpha_{2w} = \left(1+\alpha_{2h}^2-2\alpha_{2h}\cos\theta_{2h}\right)^{\frac{1}{2}}$$

and

(2.7) $$\theta_{2w} = \arctan\left(\frac{\alpha_{2h}\sin\theta_{2h}}{1-\alpha_{2h}\cos\theta_{2h}}\right) .$$

Substitution of $\alpha_{2h}$ and $\theta_{2h}$ in (2.6) and (2.7) gives the amplitude response of the water in the tide-well

(2.8) $$\alpha_{2w} = \left(1+\beta_2^2\right)^{-\frac{1}{2}} ,$$

and the phase lag of the response behind the sinusoidal input,

(2.9) $$\theta_{2w} = \arctan\beta_2 .$$

Because the system is linear both $\alpha_{2w}$ and $\theta_{2w}$ are independent of the amplitude of the incident oscillations, and the principle of superposition of solutions holds. Therefore the amplitudes $a$ and phases $\phi$ of (say) tidal components obtained from a Fourier analysis of the record of the water level in the well can be corrected to give their true values, namely, $a/\alpha_{2w}$ and $\phi - \theta_{2w}$ .

## 3. The non-linear equations

For $n = 1$ and $3$ (1.1) is non-linear. The concept of a response function defined for a linear system no longer applies; a sine-wave input gives an output with a fundamental oscillation of the same frequency but distorted by the presence of higher harmonics and the principle of superposition of solutions no longer holds.

Four ways of finding approximate solutions to (1.1), with arbitrary

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$n$, are described in the following sections. Each method concerns different ranges of frequencies and amplitudes. For each method the approximate solution with $n = 2$ is compared with the exact solution obtained in Section 2 as a check. The first two methods, one being an asymptotic solution for small $\beta_n$ and the other an exact solution for an 'almost-sinusoidal' input, give an indication of the way in which the fundamental and the harmonics in the output are related to the input. The two remaining methods, one using collocation and the other being numerical, indicate the way in which the output as a whole is related to the input.

## 4. An asymptotic method for small $\beta_n$

Rewriting (1.5) gives

$$\varepsilon_n^n dY/d\tau + |Y|^{n/2}\operatorname{sgn}(Y) = \varepsilon_n^n \cos\tau \tag{4.1}$$

where $\varepsilon_n^n = \beta_n$. Successive approximations to the solution of this equation when $\varepsilon_n$ is small can be found by the method of matched asymptotic expansions described in [4]. In the following work only the outer asymptotic expansion, which applies for $\tau$ larger than $O\left(\varepsilon_n^2\right)$, is derived since this gives the steady state response.

Let $Y = \lim_{k\to\infty} Y^{(k)}$, where $Y^{(k)} = \sum_{r=1}^{k} Y_r$ and $Y_{r+1} = o\left(Y_r\right)$ with respect to $\varepsilon_n$. Then (4.1) becomes

$$\varepsilon_n^n \frac{d}{d\tau}\left(Y_1+Y_2+\ldots\right) + \left|Y_1+Y_2+\ldots\right|^{n/2}\operatorname{sgn}\left(Y_1+Y_2+\ldots\right) = \varepsilon_n^n\cos\tau . \tag{4.2}$$

Letting $\varepsilon_n \to 0$ we obtain to leading order on $\varepsilon_n$

$$\left|Y_1\right|^{n/2}\operatorname{sgn}\left(Y_1\right) = \varepsilon_n^n\cos\tau ,$$

from which follows

$$Y_1 = \varepsilon_n^2|\cos\tau|^{2/n}\operatorname{sgn}(\cos\tau) .$$

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To find $Y_2$ this value of $Y_1$ is substituted into a re-arranged form of (4.2), namely,

$$|Y_1 + Y_2 + \ldots| = \varepsilon_n^2 |\cos\tau - \frac{d}{d\tau}(Y_1 + Y_2 + \ldots)|^{2/n}$$

with

$$\mathrm{sgn}(Y_1 + Y_2 + \ldots) = \mathrm{sgn}\left\{\cos\tau - \frac{d}{d\tau}(Y_1 + Y_2 + \ldots)\right\} .$$

Again taking the limit $\varepsilon_n \to 0$, we see that $Y_2 = O\left(\varepsilon_n^4\right)$ and obtain

$$(4.3) \quad Y = \varepsilon_n^2 |\cos\tau|^{2/n} \mathrm{sgn}(\cos\tau)\left\{1+2n^{-1}\varepsilon_n^2 |\cos\tau|^{(2-2n)/n} \sin\tau\,\mathrm{sgn}(\cos\tau)\right\}^{2/n} + O\left(\varepsilon_n^6\right) .$$

This process can be continued giving $Y$ correct to successively higher orders of approximation.

Solutions for the non-dimensionalised head, $Y$, to $O\left(\varepsilon_n^4\right)$ are shown in Figure 1 for $\varepsilon_n = 0.2$, $n = 1, 2, 3$.

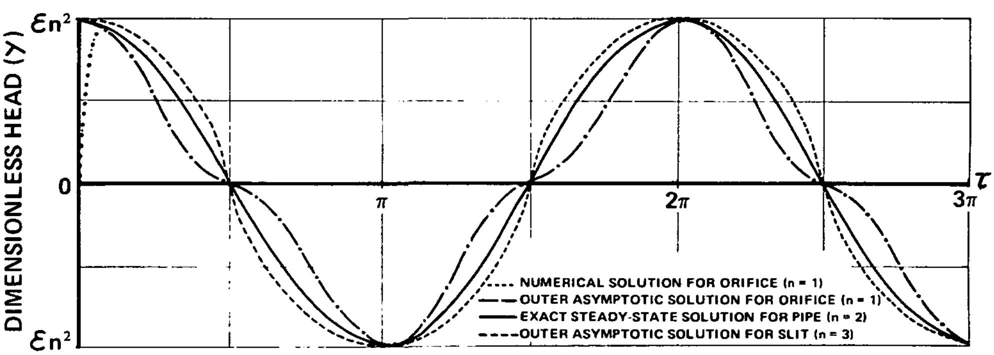


Figure 1. The dimensionless head $(Y)$ for the three systems with $\beta_n = 0.2$ .

For $n = 2$ the solution obtained by modifying (2.2) to (2.4) for $\beta_2 \ll 1$, gives to $O\left(\beta_2^2\right)$,

$$(4.4) \qquad Y = \beta_2 \cos(\tau - \beta_2) + O\left(\beta_2^3\right) .$$

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On expansion, (4.4) yields the same result as (4.3) with $n = 2$, namely,

$$Y = \varepsilon_2^2\cos\tau + \varepsilon_2^4\sin\tau + O\left(\varepsilon_2^6\right) .$$

The three head responses differ noticeably in the neighbourhood of the turning points and of the zero heads. For the tide-well with an orifice $(n = 1)$ the head response tends to dwell around the zero value, and changes most rapidly at the turning points. For the tide-well with a vertical slit $(n = 3)$ the head response changes most rapidly when the head is zero, and tends to dwell at the turning points. For the tide-well with a pipe connection $(n = 2)$ the sinusoidal head response lies between these two extremes.

The tide-well output given by (1.6) is

$$Z = \sin\tau - \varepsilon_n^2|\cos\tau|^{2/n}\mathrm{sgn}(\cos\tau) - 4n^{-2}\varepsilon_n^4|\cos\tau|^{(4/n-2)}\sin\tau + O\left\{\varepsilon_n^6\right\}$$

which applies for all $\tau$, except in the neighbourhood of $(2k+1)\pi/2$ when $n > 2$. Expanding $|\cos\tau|^{2/n}\mathrm{sgn}(\cos\tau)$ and $4n^{-2}|\cos\tau|^{(4/n-2)}\sin\tau$ in their Fourier series gives

$$Z = \sin\tau - \varepsilon_n^2 \sum_{m=1}^{\infty} C_{nm}\cos(2m-1)\tau - \varepsilon_n^4 \sum_{m=1}^{\infty} K_{nm}\sin(2m-1)\tau + O\left(\varepsilon_n^6\right)$$

or

$$(4.5) \quad Z = \alpha_{n\omega}^{(1)}\sin\left(\tau-\theta_{n\omega}^{(1)}\right) - \varepsilon_n^2 \sum_{m=2}^{\infty} \left(C_{nm}^2+\varepsilon_n^4K_{nm}^2\right)^{\frac{1}{2}}\sin\{(2m-1)\tau-\phi_{nm}\} + O\left(\varepsilon_n^6\right)$$

where

$$\alpha_{n\omega}^{(1)} = 1 - \left(K_{n1} - \tfrac{1}{2}C_{n1}^2\right)\varepsilon_n^4 + O\left(\varepsilon_n^6\right)$$

and

$$\theta_{n\omega}^{(1)} = C_{n1}\varepsilon_n^2 + O\left(\varepsilon_n^6\right) .$$

Since all other harmonics apart from the fundamental oscillation are of $O\left(\varepsilon_n^2\right)$ then $\alpha_{n\omega}^{(1)}$ gives an estimate of the amplitude response, and $\theta_{n\omega}^{(1)}$ an estimate of the phase lag, for small $\varepsilon_n$ .

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In particular, for the tide-well with an orifice, $n = 1$ , and

$$C_{1m} = \frac{8(-1)^m}{\pi(2m-3)(2m-1)(2m+1)}\ , \quad m = 1, 2, 3 \ldots ,$$

$$K_{1m} = 1 \quad \text{for} \quad m = 1, 2 ,$$

$$= 0 \quad \text{otherwise.}$$

Then the estimate of the amplitude response is, to $O\left(\varepsilon_1^6\right)$ ,

$$\alpha_{1\omega}^{(1)} = 1 - 0.64\varepsilon_1^4$$

with corresponding phase lag

$$\theta_{1\omega}^{(1)} = 0.85\varepsilon_1^2 .$$

The values for $\alpha_{1\omega}^{(1)}$ for small $\varepsilon_1$ may be compared with the values of Keulegan's approximation [1] to the solution of (1.5) with $n = 1$ . Basically, his method consisted of finding a solution of the form

$$Y = \sum_{n=1}^{\infty} A_{2n-1}\sin(2n-1)\phi + \sum_{n=1}^{\infty} B_{2n-1}\left[\cos(2n-1)\phi-\cos(2n+1)\phi\right]$$

where $\phi = \tau + \psi$ , and $\psi$ is a zero of $Y(\tau) = 0$ . The coefficients $A_1$, $A_3$ and $B_1$ , were evaluated by truncation of the series for $Y$ and using a process of iteration, substituting approximate values of these coefficients in the given differential equation to obtain more accurate values. Besides the complete omission of all harmonics with frequency greater than 3 , there is partial omission of the contribution of the harmonic of frequency 3 , since this involves the coefficient $B_3$ . This explains why, for $\beta_1 < 0.6$ in Figure 2, (see page 399), the amplitude response of the fundamental oscillation in the output is greater than unity using his estimate.

The ratio of the amplitude of the harmonic with a frequency three times the fundamental of the output to the amplitude of the fundamental is a measure of the distortion of the output. The amplitude of this harmonic is found from (4.5) to be

$$\alpha^* = \varepsilon_1^2\left(C_{12}^2+\varepsilon_1^4K_{12}^2\right)^{\frac{1}{2}} + O\left(\varepsilon_1^6\right)$$

so the required ratio is

$$\alpha^*/\alpha_{1w}^{(1)} = 0.17\varepsilon_1^2 + O\left(\varepsilon_1^6\right) .$$

The same ratio calculated for $\varepsilon_1 < 0.4$ from Keulegan's data is approximately 25 percent larger over most of the range.

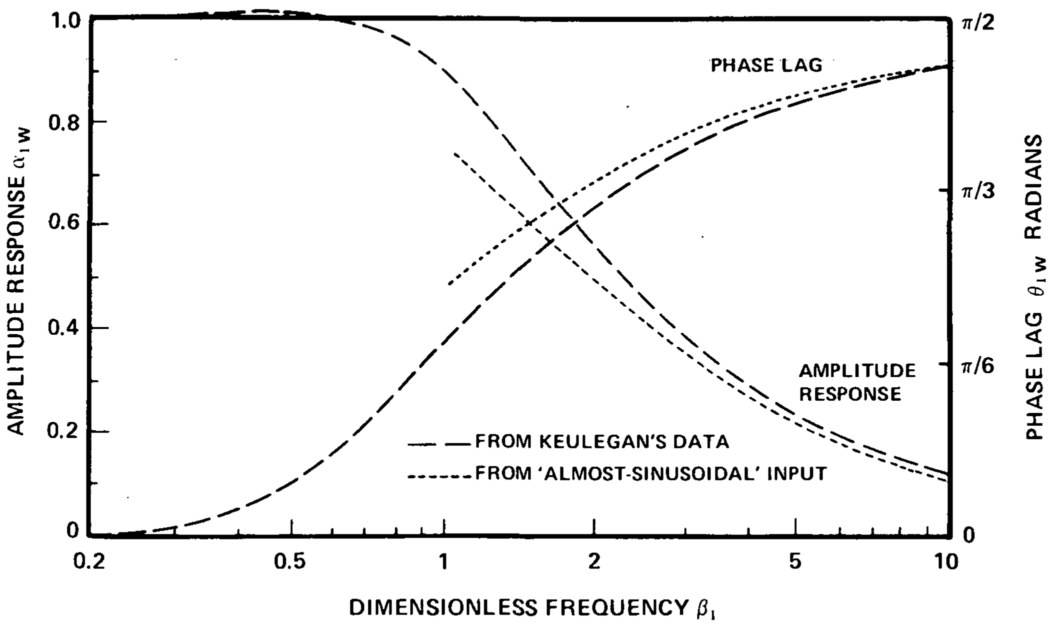


Figure 2. The amplitude response and phase lag at the input frequency, for $n = 1$ .

The results of this section indicate the presence of higher harmonics in the response of a tide-well with an orifice or a vertical slit to a sinusoidal input, when $\varepsilon_n$ is small. Since most tide gauges consist of a well with an orifice, we must therefore be careful when interpreting the results of an analysis of tide-records. Some of the apparent tidal components obtained from a tide-well record may not exist in the fluctuations of the sea-level outside the well; these harmonics may appear in the oscillations of the water inside the well simply because the tide-well with an orifice is a non-linear device.

## 5. An exact solution for an "almost-sinusoidal" input

Non dimensionalising (1.1) with arbitrary $h_0$ yields

(5.1) $$dY/d\tau + \beta_n^{-1}|Y|^{n/2}\operatorname{sgn}(Y) = dX/d\tau$$

where $X = h_0/a$ is given by

(5.2) $$X = Y + Z .$$

If it is assumed that the head difference $Y$ is a sinusoid

(5.3) $$Y = \sin\tau$$

we can solve for $X$. Substituting in (5.1) we obtain

$$dX/d\tau = \cos\tau + \beta_n^{-1}|\sin\tau|^{n/2}\operatorname{sgn}(\sin\tau) ,$$

the steady state solution for which is

$$X = \sin\tau + \beta_n^{-1}\int_0^{\pi} |\sin x|^{n/2}\operatorname{sgn}(\sin x)dx .$$

If the Fourier series representation of the integrand is

$$|\sin x|^{n/2}\operatorname{sgn}(\sin x) = \sum_{r=1}^{\infty} a_{nr}\sin(2r-1)x$$

then

(5.4) $$X = \left\{\sin\tau - b_{n1}\beta_n^{-1}\cos\tau\right\} - \beta_n^{-1}\sum_{r=2}^{\infty} b_{nr}\cos(2r-1)\tau ,$$

where

$$b_{nr} = a_{nr}/r \quad \text{for} \quad r = 1, 2, 3, \ldots .$$

This may be rewritten

$$X = \left\{1+b_{n1}^2\beta_n^{-2}\right\}^{\frac{1}{2}}\sin\left(\tau-\gamma_n\right) - \beta_n^{-1}\sum_{r=2}^{\infty} b_{nr}\cos(2r-1)\tau ,$$

where $\gamma_n = \arctan\left(b_{n1}\beta_n^{-1}\right)$. The input $X$ becomes more truly sinusoidal, approaching unit frequency, unit amplitude, and zero phase, as $\beta_n$

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becomes larger.

The well response, $Z$, to this "almost-sinusoidal" input is found by substituting $X$ and $Y$ from (5.4) and (5.3) in (5.2), giving

$$Z = b_{n1}\beta_n^{-1}\sin(\tau-\pi/2) - \beta_n^{-1}\sum_{r=2}^{\infty} b_{nr}\cos(2r-1)\tau \ .$$

The amplitude response of the well at the fundamental frequency is therefore

$$\alpha_{nw}^{(2)} = \left\{1+\beta_n^2 b_{n1}^{-2}\right\}^{-\frac{1}{2}} ,$$

with the output lagging the input at this frequency, by

$$\theta_{nw}^{(2)} = \arctan\left(\beta_n b_{n1}^{-1}\right) \ .$$

For $n = 1$ we obtain

$$b_{1r} = 1.113,\ 0.053,\ 0.014,\ 0.006,\ \ldots\ ,\ \text{for}\ \ r = 1,\ 2,\ 3,\ 4,\ \ldots\ .$$

Therefore, for a tide-well with an orifice, an estimate of the amplitude response to a sinusoidal input is, for large $\beta_1$,

$$\alpha_{1w}^{(2)} = \left(1+0.81\beta_1^2\right)^{-\frac{1}{2}}$$

with corresponding phase lag

$$\theta_{1w}^{(2)} = \arctan\left(0.90\beta_1\right) \ .$$

Graphs of these estimates are displayed in Figure 2 (see page 399), for $\beta_1 > 1$, where they are compared with estimates obtained using Keulegan's data.

For $n = 2$ we obtain $b_{21} = 1$ with $b_{2r} = 0$ for $r = 2, 3, \ldots$ . This gives the same result for the amplitude response and phase lag of a tide-well with a long pipe connection as (2.8) and (2.9).

For $n = 3$ we find

$$b_{3r} = 0.915,\ -0.034,\ -.005,\ -.001,\ \ldots\ ,\ \text{for}\ \ r = 1,\ 2,\ 3,\ 4,\ \ldots\ .$$

This gives the following estimates for a tide-well with a vertical slit, for large $\beta_3$ ,

$$\alpha_{3\omega}^{(2)} = \left(1+1.19\beta_3^2\right)^{-\frac{1}{2}} ,$$

$$\theta_{3\omega}^{(2)} = \arctan\left(1.09\beta_3\right) .$$

The ratio of the amplitude of the third harmonic to the amplitude of the fundamental oscillation in the response of the conventional tide-well system to a sinusoidal input is, for large $\beta_1$ ,

$$\frac{\alpha^*}{\alpha_{1\omega}^{(2)}} \sim \frac{0.053\beta_1^{-1}}{\left[1+0.81\beta_1^2\right]^{-\frac{1}{2}}} \sim 0.048 .$$

The same ratio calculated from Keulegan's approximate data is 0.042 .

## 6. A collocation method

To obtain an estimate of the head response as a whole we may seek a solution to

$$(6.1) \qquad dY/d\tau + \beta_n^{-1}|Y|^{n/2}\operatorname{sgn}(Y) - \cos\tau = 0$$

which has the form

$$Y^* = \alpha_{nh}\sin\left(\tau+\theta_{nh}\right) .$$

The preceding sections have shown that such a function cannot satisfy (6.1) for all $\tau$ , even though the solution of (6.1) is nearly sinusoidal and has the same period as the input, $X = \sin\tau$ .

Estimates of the amplitude response $\alpha_{nh}$ and phase lead $\theta_{nh}$ of the head response are obtained by collocating $Y^*$ and $Y$ in the neighbourhood of the zero-crossings and turning points of $Y^*$ . This requires that

$$(6.2) \qquad dY^*/d\tau + \beta_n^{-1}|Y^*|^{n/2}\operatorname{sgn}(Y^*) - \cos\tau = 0 .$$

for values of $\tau$ that make $Y^* = 0$ and $dY^*/d\tau = 0$ . When $Y^* = 0$ , substitution of the appropriate values of $\tau$ in (6.2) yields

$$\alpha_{nh} = \cos\theta_{nh}$$

and when $dY^*/d\tau = 0$ , we obtain

$$\beta_n^{-1}\alpha_{nh}^{n/2} = \sin\theta_{nh} \ .$$

Squaring and adding gives

$$\alpha_{nh}^{n} + \beta_n^2\alpha_{nh}^2 - \beta_n^2 = 0 \ , \tag{6.3}$$

the appropriate solution being in the range $0 < \alpha_{nh} \le 1$ . Furthermore, by division we obtain

$$\theta_{nh} = \arctan\left(\alpha_{nh}^{n/2-1}\beta_n^{-1}\right) \ . \tag{6.4}$$

The response of the water level inside the well is now found using the same reasoning as in the linear case. Application of (2.6) gives the amplitude response of the water in the tide-well,

$$\alpha_{nw}^{(3)} = \beta_n^{-1}\alpha_{nh}^{n/2} \ , \tag{6.5}$$

and (2.7) gives the phase lag

$$\theta_{nw}^{(3)} = \arctan\left(\beta_n\alpha_{nh}^{1-n/2}\right) \ . \tag{6.6}$$

In particular, for the tide-well with an orifice, (6.3) becomes

$$\beta_1^2\alpha_{1h}^2 + \alpha_{1h} - \beta_1^2 = 0 \ ,$$

from which is obtained

$$\alpha_{1h} = \left\{\left(1+4\beta_1^4\right)^{\frac{1}{2}} - 1\right\}/2\beta_1^2 \ .$$

Substitution of this result in (6.5) and (6.6) yields

$$\alpha_{1w}^{(3)} = \left\{\left(1+4\beta_1^4\right)^{\frac{1}{2}} - 1\right\}^{\frac{1}{2}}/\left(\sqrt{2}\beta_1^2\right)$$

and

$$\theta_{1w}^{(3)} = \arctan\left(\alpha_{1h}^{\frac{1}{2}}\beta_1\right) \ .$$

For the tide-well with a vertical slit (6.3) becomes

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$$\alpha_{3h}^3 + \beta_3^2\alpha_{3h}^2 - \beta_3^2 = 0 ,$$

the only positive real root being

$$\alpha_{3h} = \sqrt{3}/(2\cosh\delta) , \text{ where } \delta = \tfrac{1}{3}\operatorname{arcosh}\left\{3\sqrt{3}/\left(2\beta_3^2\right)\right\} \text{ for } \beta_3^2 \le 3\sqrt{3}/2 ,$$

$$= \sqrt{3}/(2\cos\delta) , \text{ where } \delta = \tfrac{1}{3}\arccos\left\{3\sqrt{3}/\left(2\beta_3^2\right)\right\} \text{ for } \beta_3^2 > 3\sqrt{3}/2 .$$

Substitution of these values in (6.5) and (6.6) yields

$$\alpha_{3w}^{(3)} = \alpha_{3h}^{3/2}\beta_3^{-1} ,$$

with

$$\theta_{3w}^{(3)} = \arctan\left(\alpha_{3h}^{-\frac{1}{2}}\beta_3\right) .$$

Figure 3 compares the graphs of the estimated overall amplitude response, $\alpha_{1w}^{(3)}$ , and phase lag, $\theta_{1w}^{(3)}$ , for the conventional tide-well system, with the results obtained by numerical means described in the next section.

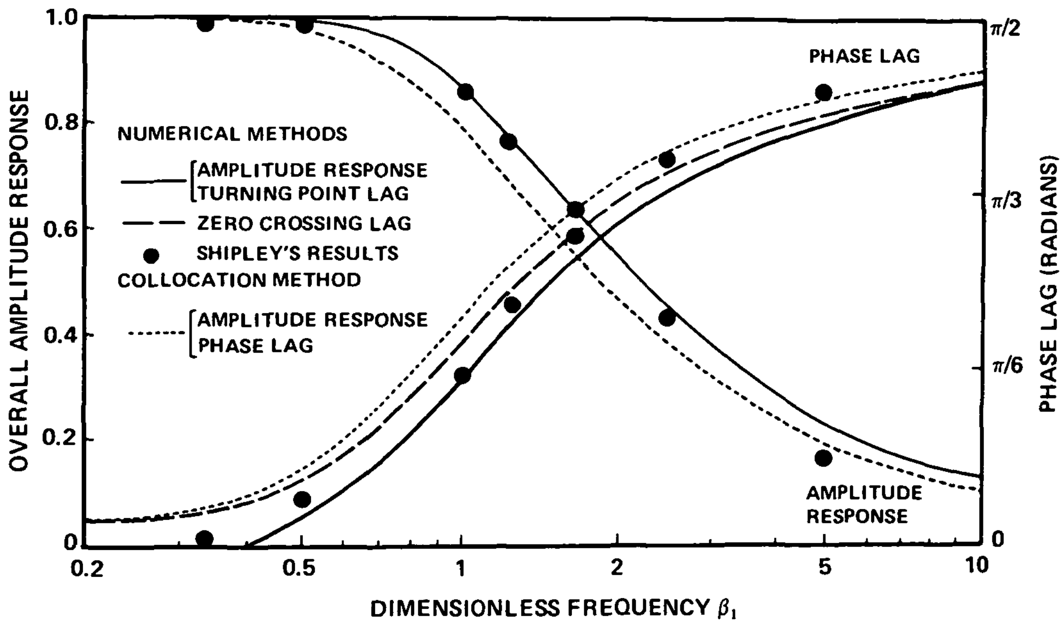


Figure 3. The overall amplitude response and phase lag to a sinusoidal input, with $n = 1$ .

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Substitution of $n = 2$ in (6.3) and (6.4) yields the exact solution for $\alpha_{2h}$ and $\theta_{2h}$ in the linear case $\left(cf.\ (2.3) \text{ and } (2.4)\right)$.

## 7. A numerical method

If, for a given system, the output due to a step input is known, an approximate output for an arbitrary input can be found by replacing the input by a series of steps. This provides a numerical method of solving (1.5) with (1.6) to obtain the well response $Z$ to the given sinusoidal input $\sin\tau$. Over the interval $r\Delta\tau \le \tau < (r+1)\Delta\tau$, $r = 0, 1, 2, \ldots$, the input is taken to be

$$X_r = \sin(r+1)\Delta\tau \ . \tag{7.1}$$

If $Z_r$ is the well response at $\tau = r\Delta\tau$, the step input at the start of the above interval is

$$Y_r = X_r - Z_r \ , \tag{7.2}$$

and from the known step response the increase in water level in the well over the interval $\Delta\tau$ can be computed. Let this increase be $W_r$. Then the well response at $\tau = (r+1)\Delta\tau$ is

$$Z_{r+1} = Z_r + W_r \ . \tag{7.3}$$

Commencing with initial conditions $Z_0 = 0$, successive applications of (7.1) to (7.3) with $r = 0, 1, 2, \ldots$, yields the well response $Z_r$ at $\tau = r\Delta\tau$ to the stepped input (7.1).

In the case of the tide-well with an orifice $n = 1$ in the relevant equations, and the response of the water in the well to the step $Y_r$ is

$$\begin{aligned} W_r &= \frac{\Delta\tau}{4\beta_1^2}\left\{4\beta_1|Y_r|^{\frac{1}{2}} - \Delta\tau\right\}\operatorname{sgn}(Y_r) \ , \quad |Y_r| \ge \left(\frac{\Delta\tau}{2\beta_1}\right)^2 \ , \\ &= Y_r \ , \quad |Y_r| < \left(\frac{\Delta\tau}{2\beta_1}\right)^2 \ . \end{aligned} \tag{7.4}$$

For the well with a vertical slit, $n = 3$ in the relevant equations, and

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the response $W_r$ to the step $Y_r$ is given by

(7.5) $$W_r = Y_r\left\{1 - \left(1+\tfrac{1}{2}|Y_r|\beta_3^{-1}\Delta\tau\right)^{-2}\right\} .$$

In both these cases the system is non-linear, so the principle of superposition of solutions does not hold. The addition of step-responses to give the required solution therefore requires justification; we must show that as $\Delta\tau \to 0$ the output converges to that due to $X = \sin\tau$ . For $n = 1$ , this is seen by taking $\Delta\tau < 2\beta_1|Y_r|^{\frac{1}{2}}$ , and substituting in (7.4) the expression for $W_r$ from (7.3), which gives

(7.6) $$Z_{r+1} - Z_r = \Delta\tau\beta_1^{-1}|Y_r|^{\frac{1}{2}}\mathrm{sgn}(Y_r) + O(\Delta\tau^2) .$$

Use of (7.2) then yields

$$\left(\frac{Y_{r+1}-Y_r}{\Delta\tau}\right) + \beta_1^{-1}|Y_r|^{\frac{1}{2}}\mathrm{sgn}(Y_r) = \left(\frac{X_{r+1}-X_r}{\Delta\tau}\right) + O(\Delta\tau) .$$

This is the discrete analogue of (5.1) with $n = 1$ , which is obtained in the limit as $\Delta\tau \to 0$ .

The stability of the finite difference scheme for $n = 1$ is evident from the following. Let there be a small error $\delta Z_r$ in the dimensionless output at the $r$-th step so that $Z_r + \delta Z_r$ replaces $Z_r$ and $Y_r - \delta Z_r$ replaces $Y_r$ in all computations. To $O(\Delta\tau)$ , equation (7.6) may be written

(7.7) $$Z_{r+1} = Z_r + \Delta\tau\beta_1^{-1}|Y_r|^{\frac{1}{2}}\mathrm{sgn}(Y_r) ,$$

so that the error $\delta Z_{r+1}$ in the output at the $(r+1)$-th step is given by

(7.8) $$Z_{r+1} + \delta Z_{r+1} = Z_r + \delta Z_r + \Delta\tau\beta_1^{-1}|Y_r-\delta Z_r|^{\frac{1}{2}}\mathrm{sgn}(Y_r-\delta Z_r) .$$

Using the relation

$$|Y_r-\delta Z_r|^{\frac{1}{2}}\mathrm{sgn}(Y_r-\delta Z_r) = |Y_r|^{\frac{1}{2}}\mathrm{sgn}(Y_r)\left\{1 - \tfrac{1}{2}\delta Z_r Y_r^{-1}\right\} + O\left[\delta Z_r^2\right] ,$$

and substracting (7.7) from (7.8) gives

$$\delta Z_{r+1} = \delta Z_r\left\{1 - \tfrac{1}{2}\Delta\tau\beta_1^{-1}|Y_r|^{-\frac{1}{2}}\right\} + O\left[\delta Z_r^2\right] .$$

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With $\Delta\tau < 2\beta_1 |Y_r|^{\frac{1}{2}}$ it follows that

$$|\delta Z_{r+1}/\delta Z_r| < 1$$

and the error decreases as the process continues.

Similar results for convergence and stability are obtained for $n = 3$. Figure 4 shows the well responses computed for $\beta_1 = 2.2$ and $\beta_3 = 1.4$, with the sinusoidal input for comparison.

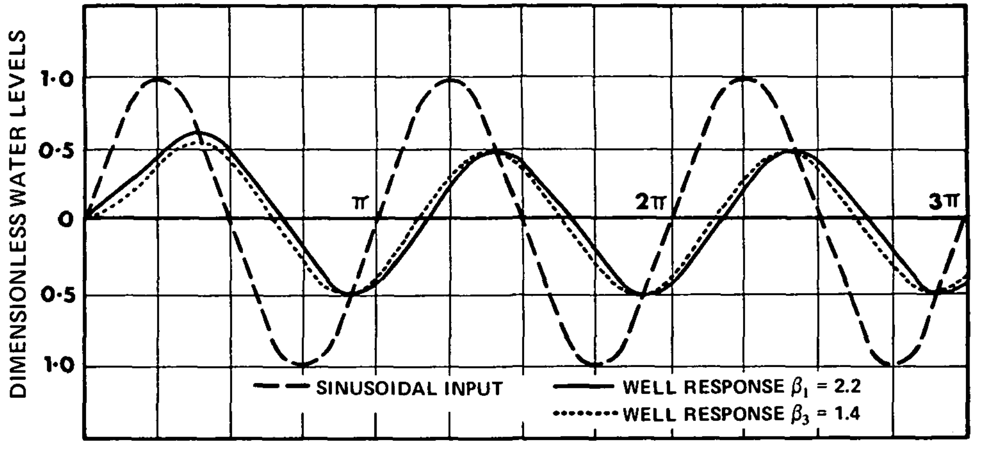


Figure 4. The computed well response to a sinusoidal input, for $n = 1$ ($\beta_1 = 2.2$) and $n = 3$ ($\beta_3 = 1.4$).

In the output from the well with an orifice the wave-form has been distorted, the amplitude attenuated, the output lags the input by a greater amount at the zero-crossings than at the turning points, and there is a transient response for about one cycle after which the output settles down to its steady state value. The output from a well with a slit is similarly effected, with the difference that it lags the input by a greater amount at the turning points than at the zero-crossings.

In both cases the overall amplitude response and phase lag were found in the following way. The well response $Z$ has turning points when $Y_r . Y_{r-1} < 0$. If the corresponding values of $r$, in ascending order, are $r(k)$, $k = 1, 2, \ldots$, then the amplitude response is given by

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$$\alpha_{r\omega}^{(4)} = \lim_{k\to\infty} |Z_{r(k)}| ,$$

and the apparent phase lag at the turning points by

$$\theta_{r\omega}^{(4)} = \lim_{k\to\infty} \{r(k)\Delta\tau - (2k-1)\pi/2\} .$$

Zero crossings in the well response occur when $Z_r . Z_{r-1} < 0$ . If this occurs for values of $r$ , in order, $r'(k)$ , $k = 1, 2, \ldots$ , then the apparent phase lag at the zero crossings is given by

$$\theta_{r\omega}^{(4)'} = \lim_{k\to\infty} \{r'(k)\Delta\tau - k\pi\} .$$

The graphs of the overall amplitude response, $\alpha_{1\omega}^{(4)}$ , and phase lags $\theta_{r\omega}^{(4)}$ and $\theta_{r\omega}^{(4)'}$ for a tide-well with an orifice are shown in Figure 3, (see p. 404), where they are seen to compare favourably with the estimates $\alpha_{1\omega}^{(3)}$ and $\theta_{1\omega}^{(3)}$ . The values for $\alpha_{1\omega}^{(4)}$ agree with five of the seven values computed by Shipley [3] using the Runge-Kutta method, with a modified first step, to solve the differential equation for $Z$ . His amplitude response values at $\beta_1 = 2.5$ and 5 are low and his results did not indicate the difference in phase lags at the turning points and zero-crossings.

Graphs of $\alpha_{1\omega}^{(4)}$ , $\theta_{1\omega}^{(4)}$ and $\alpha_{3\omega}^{(4)}$ , $\theta_{3\omega}^{(4)}$ , the overall amplitude response and lag at turning point of the non-linear systems, are compared with $\alpha_{2\omega}$ , $\theta_{2\omega}$ , the amplitude response and phase lag of the linear system, in Figure 5, (see p. 409). The non-dimensional frequencies, $\beta_n$ , $n = 1, 2, 3$ , have been normalised by dividing by $f_n$ , the value of $\beta_n$ at which the energy of the output is half the energy of the input. This permits comparison of the nature of the response, and shows that the tide-well with an orifice retains a unit amplitude response much longer than the other types of well, has a sharper "cut-off", and that it retains a zero lag at the turning point much longer than the other types. In all these regards the tide-well with a slit is worst.

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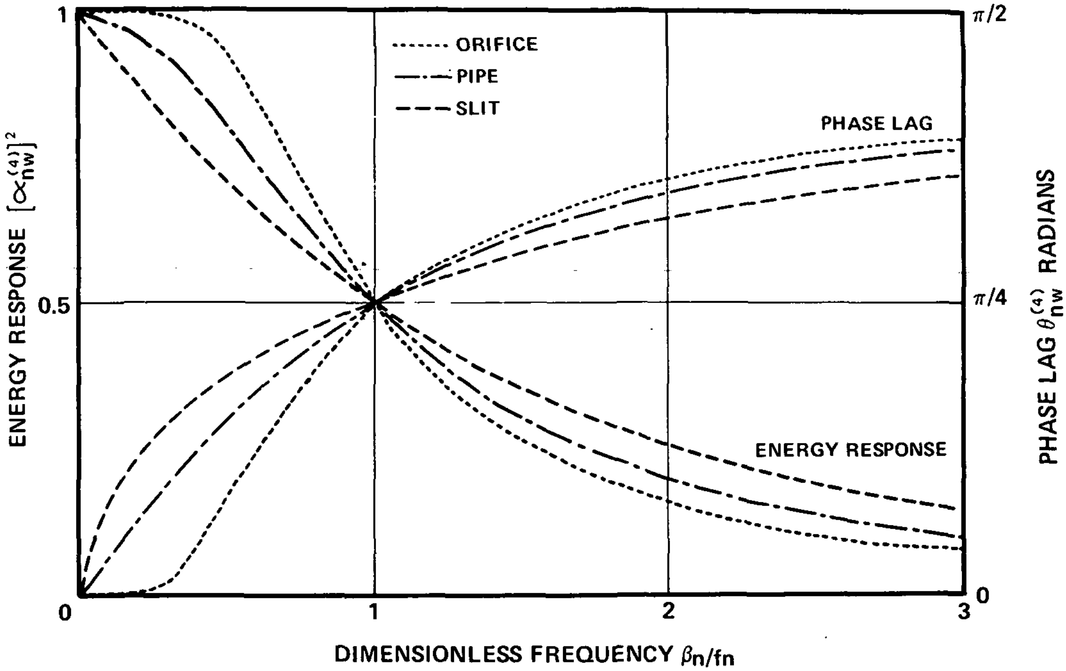


Figure 5. Pseudo energy response curves and lag at turning points for the three tide-well systems.

## 8. Discussion

When comparing the solutions for the equation governing the response of the three types of wells, the most important difference concerns the linearity of the systems. For the conventional type of tide-well (with an orifice) and also for the tide-well with a vertical slit, there is a non-linear response. The output from a sinusoidal input of fixed amplitude and frequency consists of a fundamental oscillation at the same frequency plus odd harmonics of decreasing amplitude. The principle of superposition of solutions does not hold, so no unique response function can be defined which relates the output to an input which consists of the superposition of a number of different waves.

In the previous sections estimates have been found for two types of pseudo-response functions. These give a rough idea of the attenuation which occurs with increasing frequency, and can be used to correct approximately records of harbour oscillations and tsunamis read directly from tide-records obtained from conventional tide-wells. They can also be used to determine the necessary sampling frequency for digital recording.

A comparison of the response estimates for the three systems show that the well with an orifice retains a unit amplitude response and a zero phase lag at the turning points to a higher frequency than the other types. The amplitude response of this system also has the sharpest cut-off to almost zero response. In both these regards the well with a slit is worst, and this combined with its non-linear properties makes it unacceptable for tidal work.

The linearity of an appropriately dimensioned well with a long pipe connection is an advantage not easily out-weighed. Since the principle of superposition of solutions holds, a direct application of the response function to a spectral analysis of the tidal record will establish the true amplitudes and phases of the sea level oscillations.

Great care must be taken when interpreting the results of a spectral analysis of a record from a conventional type tide-well. Firstly, the response of the well must be such that there is negligible energy in the recorded oscillations above the Nyquist frequency, which is the reciprocal of twice the sampling period of digitisation. If this is not so aliasing can occur and some of the Fourier components produced by the analysis may be fictitious. Secondly, the non-linearity of the system implies that harmonics of any incident oscillation may appear in the record even though they do not exist in the sea-level outside the well. One cannot be sure whether small peaks which appear in the Fourier spectrum of the record are contributions of the sea level oscillations or are due to the non-linear effects of the orifice.

Being a linear system, the well with a long horizontal pipe connection is preferable to the other alternatives considered in this paper.

## References

[1] G.H. Keulegan, *Tidal flow in entrances; water level fluctuations of basins in communication with seas* (Technical Bulletin no. 14, U.S. Army Corps of Engineers, Committee on Tidal Hydraulics, 1967).

[2] B.J. Noye, "The frequency response of a tide-well", *Proc. 3rd Aust. Conf. on Hydraulics and Fluid Mechanics, Sydney*, 1968, 65-71.

[3] A.M. Shipley, "On measuring long waves with a tide gauge", *Deutsche Hydrographische Zeitschrift* 16 (1963), 136-140.

[4] M. van Dyke, "Perturbation methods in fluid mechanics", *Applied Mathematics and Mechanics*, 8 (Academic Press, New York, London, 1964).

[5] L. van Wijngaarden, "On the oscillations near and at resonance in open pipes", *J. Engrg. Math.* 2 (1968), 225-240.

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