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RATIONAL CLASSIFICATION OF SIMPLE FUNCTION SPACE COMPONENTS FOR FLAG MANIFOLDS

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ABSTRACT. Let $M(X, Y)$ denote the space of all continuous functions between X and Y and $M_f(X, Y)$ the path component corresponding to a given map $f\colon X \longrightarrow Y$. When X and Y are classical flag manifolds, we prove the components of $M(X, Y)$ corresponding to "simple" maps f are classified up to rational homotopy type by the dimension of the kernel of f in degree two cohomology. In fact, these components are themselves all products of flag manifolds and odd spheres.

1. **Introduction.** When X and Y are flag manifolds or, more generally, F_0-*spaces* (simply connected finite complexes with finite-dimensional rational homotopy and no rational cohomology in odd degrees), the rational classification problem for components of the function space $M(X, Y)$ intersects two basic areas of research. First, W. Meier [10] proved the identity component $M_1(X, X)$ for an F_0-space X is rationally a product of odd spheres if and only if the rational Serre spectral sequence collapses for any orientable fibration with fibre X. Thus identifying the rational homotopy type of this particular function space component is equivalent to resolving the Halperin conjecture for F_0-spaces. Second, the rational classification of components is directly related to the problem of describing the set $[X_{\mathbb{Q}}, Y_{\mathbb{Q}}]$ of maps between the rationalizations of X and Y. For convenience, we denote this set by $[X, Y]_{\mathbb{Q}}$. When $X = Y$ is a generalized complex flag manifold this latter problem has been studied extensively by several authors (see [4,9]) with particular emphasis on the group $\mathcal{E}(X_{\mathbb{Q}})$ of rational self-equivalences. By [1, Corollary 3.6] the set $[X, Y]_{\mathbb{Q}}$ for F_0-spaces is in bijection with $\mathrm{Hom}\big(H^*(Y, \mathbb{Q}), H^*(X, \mathbb{Q})\big)$ and so determining its structure is a purely algebraic problem. Nonetheless, there appears to be no general structure theorem in the literature for the rational maps between two different flag manifolds. In this paper, we focus on the large class of "simple" and "signed-simple" maps between flag manifolds and classify the components corresponding to these maps in the complex and symplectic cases.

Let $X = G_1/T$ and $Y = G_2/T$ be flag manifolds where G_1 and G_2 are compact, connected Lie groups and T denotes a maximal torus of appropriate rank. By [2], $H^*(G_i/T) = B_i/J_i$, $i = 1, 2$, where B_i is the polynomial algebra on $\mathrm{rank}(G_i)$ variables generated in degree two and J_i is the ideal consisting of polynomials invariant under the

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action of the Weyl group W_i of G_i on the subscripts of the variables. (Here and throughout, all (co)homology and homotopy groups are taken to have rational coefficients.) Write $B_1 = \Lambda_2(t_1,\ldots,t_k)$ and $B_2 = \Lambda_2(x_1,\ldots,x_n)$ where $k = \operatorname{rank}(G_1)$ and $n = \operatorname{rank}(G_2)$.

DEFINITION. *A map $f\colon X \longrightarrow Y$ between flag manifolds is* simple *(respectively,* signed-simple*) if there is $a \in \mathbb{Q}$ such that for each $x_i \in B_2$ if $\phi(x_i) \neq 0$ then $\phi(x_i) = a \cdot t_j$ (resp., $\phi(x_i) = \pm a \cdot t_j$) for some $t_j \in B_1$ where $\phi\colon B_2 \longrightarrow B_1$ is the map induced by f.*

Examples of simple maps arise naturally from the basic inclusions $U(k) \hookrightarrow U(n)$ and $\mathrm{Sp}(k) \hookrightarrow \mathrm{Sp}(n)$ for $k \leq n$ and of signed-simple maps via $\mathrm{Sp}(k) \hookrightarrow U(n)$ for $2k \leq n$. In Section 2, we show that the simple maps in $[U(k)/T, U(n)/T]$ and $[\mathrm{Sp}(k)/T, \mathrm{Sp}(n)/T]$ are classified by the *integer* $l = \Big(n - \dim\big(\ker\{H^2(f)\}\big)\Big)/k$ and the signed-simple maps in $[\mathrm{Sp}(k)/T, U(n)/T]$ by the integer $l = \Big(n - \dim\big(\ker\{H^2(f)\}\big)\Big)/2k$. Using an explicit construction of the Haefliger model for F_0-spaces (Section 3), we establish

THE CLASSIFICATION THEOREM. *Let f be a simple or signed-simple map between complex or symplectic flag manifolds and let l be as above. Then*

$$M_f\big(U(k)/T, U(n)/T\big) \simeq_{\mathbb{Q}} \big(U(l)/T\big)^k \times U(l-1)/T \times U(n-kl)/T \times \textit{odd spheres}$$

$$M_f\big(\mathrm{Sp}(k)/T, \mathrm{Sp}(n)/T\big) \simeq_{\mathbb{Q}} \big(U(l)/T\big)^k \times \mathrm{Sp}(n-kl)/T \times \textit{odd spheres}$$

$$M_f\big(\mathrm{Sp}(k)/T, U(n)/T\big) \simeq_{\mathbb{Q}} \big(U(l)/T\big)^{2k-1} \times U(l-1)/T \times U(n-2kl)/T \times \textit{odd spheres.}$$

∎

2. **Rational Maps Between Flag Manifolds.**

Let $X = G_1/T$ and $Y = G_2/T$ be flag manifolds, as above. Since rational self-equivalences of X and Y induce rational equivalences between components of $M(X,Y)$, for our purposes we need only determine the structure of the set $[X,Y]_{\mathbb{Q}}$ “modulo rational equivalences”. In other words, we identify rational maps $f\colon X \longrightarrow Y$ up to pre- and post-composition by rational self-equivalences in Y and X, respectively.

The Weyl group of a compact Lie group is a finite reflection group and so may be viewed as a subgroup of the orthogonal group. Thus the polynomials $P_{2,1}(t_1,\ldots,t_n) = t_1^2+\cdots+t_k^2$ and $P_{2,2}(x_1,\ldots,x_n) = x_1^2+\cdots+x_n^2$ are elements of the ideals J_1 and J_2 of grade four. If G_1 is simple then $P_{2,1}$ is (up to scalar multiple) the *unique* element of grade four in J_1 (see, *e.g.*, [8, p. 59]). If $G_1 = U(k)$ then the element $t_1+\cdots+t_k$ of grade two appears in J_1 and so $P_{2,1}$ is not unique. However, if we replace B_1 by $B_1' = \Lambda_2(t_1,\ldots,t_{k-1})$, J_1 by the appropriate subideal J_1' and set $t_k = -t_1 - \cdots - t_{k-1}$, then the polynomial $P_{2,1}(t_1,\ldots,t_k)$ appears and is the unique element of grade four in J_1'. We use this uniqueness to prove

THEOREM 2.1. *Let $X = G_1/T$ and $Y = G_2/T$ with G_1 simple or $U(k)$. Let $f\colon X \longrightarrow Y$ be any map. Then $f^*\colon H^*(Y) \longrightarrow H^*(X)$ is either trivial or surjective.*

PROOF. Suppose $f^*\colon H^2(Y) \longrightarrow H^2(X)$ is nontrivial. Let $a_j = \phi(x_j)$. Then, making the above replacement if necesssary, we have $\phi(P_{2,2}) = \alpha P_{2,1}$ for some $\alpha \in \mathbb{Q}$. If $\alpha = 0$

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then $a_1^2 + \cdots + a_n^2 = \phi\big(P_{2,2}(x_1, \ldots, x_n)\big) = 0$ which implies each $a_j = 0$, contrary to our assumption. Thus $\alpha \neq 0$ and we have a nontrivial identity of the form $P_{2,2}(a_1, \ldots, a_n) = \alpha P_{2,1}(t_1, \ldots, t_k)$. Viewing the a_j as linear endomorphisms $a_j(t_1, \ldots, t_k)$ of the vector space $\mathbb{Q}(t_1, \ldots, t_k)$, we take $\frac{\partial}{\partial t_i}$ of both sides and obtain

$$\alpha \cdot t_i = \sum_{j=1}^{n} a_j \cdot \frac{\partial a_j}{\partial t_i}.$$

Thus $\mathbb{Q}(a_1, \ldots, a_n) = H^2(X)$. ∎

COROLLARY 2.2. *Let $X = G/T$ with G simple or $G = U(k)$. Then $[X, X]_{\mathbb{Q}} = E(X_{\mathbb{Q}}) \cup \{0\}$.* ∎

While the set $[X, Y]_{\mathbb{Q}}$ modulo rational equivalences is fairly simple when $k = \operatorname{rank}(G_1) \geq q \operatorname{rank}(G_2) = n$, when $k < n$ the structure is apparently much more complicated. We focus on the simple maps.

THEOREM 2.3. *Let $X = G_1/T$ and $Y = G_2/T$ with G_1 simple or $U(k)$ such that the Weyl group of G_2 contains the symmetric group S_n. Then $[X, Y]_{\mathbb{Q}}$ contains at most $[n/k] + 1$ distinct simple maps modulo rational equivalences. If G_1 is simple and $G_2 = U(k)$ then $[X, Y]_{\mathbb{Q}}$ contains at most $[n/2k]+1$ distinct signed-simple maps modulo rational equivalences.*

PROOF. Let $f\colon X \longrightarrow Y$ be simple. Then, after permuting the subscripts of the x_i (rational equivalence in Y), we may assume $\phi(x_1) = \cdots = \phi(x_l) = a \cdot t_1$ for some l. Thus the coefficient of t_1^2 in $\phi(P_{2,2})$ is $l \cdot a^2$. By the uniqueness of $P_{2,1}$ in J_1, it follows that, after further permutation of subscripts, the list $\{x_1, \ldots, x_n\}$ splits into $k+1$ sublists, $\{x_1, \ldots, x_l\}, \ldots, \{x_{(k-1)l+1}, \ldots, x_{kl}\}$ and $\{x_{kl+1}, \ldots, x_n\}$ with the property that $\phi(x_j) = a \cdot t_i$ for x_j in the i-th sublist $i = 1, \ldots, k$ and $\phi(x_j) = 0$ for x_j in the $k+1$-st sublist. Since multiplication by a in degree two cohomology induces a rational equivalence of X, the first statement follows.

For the second statement, observe that $\phi(x_1 + \cdots + x_n) = 0$. Using the previous case, we see that, after permutation, $\{x_1, \ldots, x_n\}$ splits into $2k+1$ sublists, the first $2k$ of length l, such that $\phi(x_j) = a \cdot t_i$ for x_j in the i-th and $\phi(x_j) = -a \cdot t_i$ for x_j in the $2i+1$-st sublist, $i = 1, \ldots, k$, while $\phi(x_j) = 0$ for x_j in the $2k+1$-st sublist. ∎

For each $l = 0, \ldots, [n/k]$ the inclusion $\prod_{i=1}^{l} U(k) \hookrightarrow U(n)$ induces a map $i_l\colon \prod_{i=1}^{l} U(k)/T \longrightarrow U(n)/T$. Define $f_l\colon U(k)/T \longrightarrow U(n)/T$ by setting $f_l = \Delta \circ i_l$ where $\Delta\colon U(k)/T \longrightarrow \prod_{i=1}^{l} U(k)/T$ is the diagonal map. It is clear that the f_l are simple and rationally distinct. This construction can be applied, as well, to the other classical inclusions (Section 1) and so

COROLLARY 2.4. *Modulo rational equivalences, the sets $[U(k)/T, U(n)/T]_{\mathbb{Q}}$ and $[\mathrm{Sp}(k)/T, \mathrm{Sp}(n)/T]_{\mathbb{Q}}$ contain exactly $[n/k] + 1$ distinct simple maps while $[\mathrm{Sp}(k)/T, U(n)/T]_{\mathbb{Q}}$ contains exactly $[n/2k] + 1$ distinct signed-simple maps.* ∎

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3. **The Haefliger Model for F_0-Spaces.** By [6], the minimal model $(\mathcal{M}_Y, d_Y)$ for an F_0-space Y is a *two-stage* DGA; specifically, $\mathcal{M}_Y = \Lambda(V_0) \otimes_{d_Y} \Lambda(V_1)$ where V_0 is evenly graded, V_1 oddly graded and where the differential d_Y satisfies $d_Y|V_0 = 0$ and $d_Y(V_1) \subseteq \Lambda(V_0)$. This simple rational structure implies the Haefliger model for components of $M(X, Y)$ admits a direct and accessible construction when X and Y are F_0-spaces. Our argument follows the line of proof of [13, Theorem 3] which, in turn, was based on the methods of [11].

THEOREM 3.1. *Let $f: X \longrightarrow Y$ be a map between F_0-spaces with $\mathcal{M}_Y = \Lambda(V_0) \otimes_{d_Y} \Lambda(V_1)$. There is a two-stage model $\mathcal{A}_f = \Lambda(Z_0) \otimes_{d_f} \Lambda(Z_1)$ for the function space component $M_f(X, Y)$ with*

$$Z_0^m = \bigoplus_{i=0}^{\infty} H_{2i}(X) \otimes V_0^{2i+m} \text{ and } Z_1^m = \bigoplus_{i=0}^{\infty} H_{2i}(X) \otimes V_1^{2i+m}.$$

PROOF. By [7, Theorem B] we may assume Y is a rational space. We view Y as the total space of a principal fibration with base $K_0 = \Pi_i K\big(\mathrm{Hom}(V_0^i, \mathbb{Q}), i\big)$ and fibre $K_1 = \Pi_i K\big(\mathrm{Hom}(V_1^i, \mathbb{Q}), i\big)$. Observe that $M(X, K_1)$ is connected since $[X, K_1] = \oplus_i H^i(X) \otimes V_1^i$ and $H^*(X)$ is evenly graded while V_1 is oddly graded. Thus applying the mapping space functor to the classifying fibration for Y we obtain the diagram

$$\begin{array}{ccccc} M_f(X,Y) & \hookleftarrow & M(X,K_1) & \hookrightarrow & M(X,PK_1) \\ \downarrow{\scriptstyle \underline{p}} & & & & \downarrow{\scriptstyle \underline{p_\infty}} \\ M_{p\circ f}(X,K_0) & & \xrightarrow{\underline{k}} & & M_0(X,BK_1) \end{array}$$

Since the obstructions to lifting a homotopy between $p \circ f, p \circ g: X \longrightarrow K_0$ to a homotopy between $f, g: X \longrightarrow Y$ lie in the trivial groups $H^n\big(X, \pi_n(K_1)\big)$, this is a pull-back diagram.By the classical result of Thom [15] on the space of maps into an Eilenberg-Maclane space, $\underline{p_\infty}$ is a principal fibration. The Hirsch Lemma ([3 Lemma 4.1]) applied to $\underline{p}$ implies there is a two-stage model for $M_f(X, Y)$ of the form $\mathcal{A}_f = H^*\big(M_{p\circ f}(X, K_0)\big) \otimes_{d_f} H^*\big(M_0(X, K_1)\big)$. The result now follows from Thom's result. ∎

We pursue applications of this model for general F_0-spaces in [14] and consider here only the case when Y is cohomologically generated in degree two. In this case, $\mathcal{M}_Y = \Lambda_2(x_1, \ldots, x_n) \otimes_{d_Y} \Lambda(V_1)$ and so $\mathcal{A}_f = \Lambda_2(x_1, \ldots, x_n) \otimes_{d_f} \Lambda(Z_1)$ where $Z_1^m = \oplus_{i=0}^{\infty} H_{2i}(X) \otimes V_1^{2i+m}$. Given $b \in H_{2i}(X)$ we view b as an element of the dual space to $H^*(X)$ and write $b(a) \in \mathbb{Q}$ for the value of b on $a \in H^*(X)$. Let $a_i = f^*(x_i) \in H^2(X)$ and write $c = (c_1, \ldots, c_n)$ to denote an n-tuple of non-negative integers with $|c| = \sum_{i=1}^m c_i$. Regarding the differential d_f we have

THEOREM 3.2. *Let $f: X \longrightarrow Y$ be a map between F_0-spaces with Y cohomologically generated in degree two. Given $b \otimes y \in H_{2i}(X) \otimes V_1^{2i+m} \subset Z_1^m$ write $d_Y(y) = P(x_1, \ldots, x_n)$ for some homogeneous polynomial P. Then*

$$d_f(b \otimes y) = \sum_{|c|=i} \frac{1}{(c_1! \cdots c_n!)} \cdot b(a_1^{c_1} \cdots a_n^{c_n}) \cdot \frac{\partial^{|c|}}{\partial x_1^{c_1} \cdots \partial x_n^{c_n}} P(x_1, \ldots, x_n).$$

PROOF. By the Hirsch Lemma, $d_f(b \otimes y) = \underline{k}^*(b \otimes y) \in H^*\big(M_{p\circ f}(X, K_0)\big) \cong \Lambda_2(x_1, \ldots, x_n)$. Let $\varepsilon_{p\circ f}\colon X \times M_{p\circ f}(X, K_0) \longrightarrow K_0$ be the evaluation map. Then [13, Lemma 7.2] $\varepsilon^*_{p\circ f}(x_i) = 1\otimes x_i + (p \circ f)^*(x_i)\otimes 1 = 1\otimes x_i + a_i \otimes 1$, for $x_i \in H^2(K_0)$. Given $b \in H_*(X)$ and $a\otimes P \in H^*\big(X\times M_{p\circ f}(X, K_0)\big)$ following Haefliger write $b\cap(a\otimes P) = b(a)P \in H^*(K_0)$. By [13, Lemma 7.1]

$$\begin{aligned} d_f(b \otimes y) &= \underline{k}^*(b \otimes y) \\ &= b \cap \varepsilon^*_{p\circ f}\big(k^*(y)\big) \\ &= b \cap \varepsilon^*_{p\circ f}\big(P(x_1, \ldots, x_n)\big) \\ &= b \cap P\big(\varepsilon^*_{p\circ f}(x_1), \ldots, \varepsilon^*_{p\circ f}(x_n)\big) \\ &= b \cap P(a_1 \otimes 1 + 1 \otimes x_1, \ldots, a_n \otimes 1 + 1 \otimes x_n). \end{aligned}$$

The result now follows from the identity

$$P(a_1 \otimes 1 + 1 \otimes x_1, \ldots, a_n \otimes 1 + 1 \otimes x_n)$$

$$= \sum_{|c|\leq|P|} \frac{1}{(c_1! \cdots c_n!)} a_1^{c_1} \cdots a_n^{c_n} \otimes \frac{\partial^{|c|}}{\partial x_1^{c_1} \cdots \partial x_n^{c_n}} P(x_1, \ldots, x_n). \qquad \blacksquare$$

To determine the rational homotopy type of $M_f(X, Y)$ for F_0-spaces we must construct the minimal model for the DGA $(\mathcal{A}_f, d_f)$. In many cases, this can be done by simply computing the image of the differential d_f in $\Lambda(Z_0)$. The elements of Z_1 giving "superfluous relations" correspond to odd spheres whose degrees can be computed directly. We give some

EXAMPLES 3.3. (a) Let X and Y be F_0-spaces with Y cohomologically generated in degree two. Then $M_0(X, Y) \simeq_{\mathbb{Q}} Y\times$ *odd spheres*. See [14] for an extension of this result.

(b) When $X = G/T$ is a flag manifold with G simple or $U(k)$, Corollary 2.2 implies there are at most two rationally distinct components of $M(X, X)$. By (a) and the (known case of the) Halperin conjecture [12] we have

$$M_f(G/T, G/T) \simeq_{\mathbb{Q}} \begin{cases} G/T \times \textit{odd spheres} & \text{f rationally null} \\ \textit{odd spheres} & \text{otherwise.} \end{cases}$$

(c) Let $X = U(n+2)/U(1)^2 \times U(n)$ for $n \geq 1$. The minimal model $(\mathcal{M}_X, d_X)$ is $\mathcal{M}_X = \Lambda_2(x_1, x_2) \otimes_{d_X} \Lambda(y_{n+1}y_{n+2})$, with $d_X(y_m) = T_m(x_1, x_2)$ where $T_m(x_1, x_2) = \sum_{i=0}^{m} x_1^i x_2^{m-i}$ [4]. Thus $\mathcal{A}_f = \Lambda_2(x_1, x_2) \otimes_{d_f} \Lambda(Z_1)$. If n is odd [4, Theorems 1.3, 1.4] imply $[X_{\mathbb{Q}}, X_{\mathbb{Q}}] = \mathcal{E}(X_{\mathbb{Q}}) \cup \{0\}$. Let n be even and suppose f is rationally nontrivial. If f is not a rational equivalence then by [4, Theorem 1.4] f is a "projective map" and so (swapping subscripts if necessary) $f^*(x_1) = \alpha x_1$ and $f^*(x_2) = -\alpha x_1$ for some $\alpha \neq 0$. We may assume $\alpha = 1$. Let $b_k \in H_{2k}(X)$ be dual to $x_1^k \in H^{2k}(X)$ for $k = n, n+1$. By Theorem 3.2

$$\begin{aligned} d_f(b_n \otimes y_{n+1}) &= \sum_{|c|=n} \frac{1}{(c_1!\, c_2\,!)} b_n\big(x_1^{c_1}(-x_1)^{c_2}\big) \frac{\partial^n}{\partial x_1^{c_1} \partial x_2^{c_2}} T_{n+1}(x_1, x_2) \\ &= \sum_{c_2=0}^{n} (-1)^{c_2}[n+1-c_2]x_1 + [c_2+1]x_2. \end{aligned}$$

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Since n is even, we see $d_f(b_n \otimes y_{n+1}) = \frac{n+2}{2}(x_1 + x_2)$ and $d_f(b_{n+1} \otimes y_{n+2}) = \frac{n+2}{2}(x_1 - x_2)$. Thus $d_f\colon Z_1 \longrightarrow \Lambda_2(x_1, x_2)$ is surjective and we have shown

$$M_f(X,X) \simeq_{\mathbb{Q}} \begin{cases} X \times \textit{odd spheres} & \text{f rationally null} \\ \textit{odd spheres} & \text{otherwise.} \end{cases}$$

(d) Let $Y = \prod_{j=1}^{k} \mathbb{C}P^{n_j}$ and X any F_0-space. We show the components of the space $M(X,Y)$ are classified by the "heights" in $H^*(X)$ of the images of the generators of $H^*(Y)$ under f. Given $f\colon X \longrightarrow Y$ let h_j be zero if $f^*(x_j) = 0$ and otherwise $h_j = \max\{n \mid f^*(x_j^n) \neq 0\}$, where the $x_j \in H^2(Y)$ are the generators. Write $\mathcal{M}_Y = \Lambda_2(x_1, \ldots, x_k) \otimes_{d_Y} \Lambda(y_{n_1+1}, \ldots, y_{n_k+1})$ where $d_Y(y_{n_j+1}) = x_j^{n_j+1}$. Let $b_{j,m} \in H_{2m}(X)$ be dual to $f^*(x_j^m) \in H^{2m}(X)$. By Theorem 3.2, the image of d_f is generated by the elements

$$d_f(b_{j,m} \otimes y_{n_j+1}) = \frac{1}{m!}\frac{d^m}{dx_j^m} x_j^{n_j+1} = \binom{n_j+1}{m} \cdot x_j^{n_j-m+1},$$

for $m = 0, \ldots, h_j$; that is, by the monomial $x_j^{n_j-h_j+1}$. Thus

$$M_f\Big(X, \prod_{j=1}^{k} \mathbb{C}P^{n_j}\Big) = \prod_{j=1}^{k} \mathbb{C}P^{n_j-h_j} \times \textit{odd spheres.}$$

4. **Simple Components.** We classify the simple and signed-simple components of maps into a complex or symplectic flag manifold. Define

$$P_m(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^m, \qquad T_m(x_1, \ldots, x_n) = \sum_{|c|=m} x_1^{c_1} \cdots x_n^{c_n}$$

and

$$S_{2m}(x_1, \ldots, x_n) = \sigma_m(x_1^2, \ldots, x_n^2),$$

where σ_m is the m-th symmetric function in n variables. It is easy to prove the ideals $(T_1, \ldots, T_n)$, $(P_1, \ldots, P_n)$ and $(\sigma_1, \ldots, \sigma_n)$ coincide in the polynomial algebra $\Lambda_2(x_1, \ldots, x_n)$. The minimal model for $X = U(n)/T$ can thus be written $\mathcal{M}_X = \Lambda_2(x_1, \ldots, x_{n-1}) \otimes_{d_X} \Lambda(y_2, \ldots, y_n)$ where $|y_m| = 2m-1$ and $d_X(y_m) = T_m(x_1, \ldots, x_{n-1})$ [4]. The minimal model $Y = \mathrm{Sp}(n)/T$ is of the form $\mathcal{M}_Y = \Lambda_2(x_1, \ldots, x_n) \otimes_{d_Y} \Lambda(y_2, \ldots, y_{2n})$ where $|y_{2m}| = 4m-1$ and $d_Y(y_{2m}) = S_{2m}(x_1, \ldots, x_n)$.

We will partition variable lists like $\{x_1, \ldots, x_n\}$ into sublists like $\{x_1, \ldots, x_l\}$, $\{x_{l+1}, \ldots, x_{2l}\}, \ldots$. For convenience, we let $P_{m,i}$ denote P_m applied to the i-th variable sublist. Also, given a nonnegative integer c_i we define linear operators $D_i(c_i)$ on $\Lambda_2(x_1, \ldots, x_n)$ by

$$D_i(c_i) = \sum_{|d|=c_i} \frac{1}{(d_1! \cdots d_l!)} \frac{\partial^{c_i}}{\partial x_{l(i-1)+1}^{d_1} \cdots \partial x_{li}^{d_l}},$$

where the variables x_j are those in i-th sublist. The following formula regarding partial derivatives of $T_m = T_m(x_1, \ldots, x_n)$ and $S_{2m} = S_{2m}(x_1, \ldots, x_n)$ are proved consecutively by inductive arguments.

(1) $$\sum_{i=1}^{l} \frac{\partial}{\partial x_i} T_{m+1} \equiv P_{m,1} \mod (T_1, \ldots, T_m)$$

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(2) $$\sum_{i=1}^{l} \frac{\partial^2}{\partial x_i^2} T_{m+2} \equiv 2(m+1)P_{m,1} \mod (T_1,\dots,T_m)$$

(3) $$\sum_{i=1}^{l}\sum_{j=l+1}^{2l} \frac{\partial^2}{\partial x_i \partial x_j} T_{m+2} \equiv l(P_{m,1}+P_{m,2}) \mod (P_{1,1},\dots,P_{m-1,1},P_{1,2},\dots,P_{m-1,2},T_1,\dots T_m)$$

(4) $$\sum_{i=1}^{l-1}\sum_{j=i+1}^{l} \frac{\partial^2}{\partial x_i \partial x_j} T_{m+2} \equiv (l-1-[m/2])P_{m,1} \mod (P_{1,1},\dots,P_{m-1,1},T_1,\dots T_m)$$

(5) $$\sum_{j=1}^{l} \frac{\partial}{\partial x_j} S_{2m} \equiv (-1)^{m-1} 2P_{2m-1,i} \mod (S_2,\dots,S_{2m})$$

(6) $$\sum_{i=1}^{l-1}\sum_{j=i+1}^{l} \frac{\partial^2}{\partial x_i \partial x_j} S_{2m} \equiv (-1)^{m+1} 4mP_{2(m+1),1} \mod (P_{1,1},\dots,P_{2m-1,1},S_2,\dots,S_{2m})$$

THEOREM 4.1. *Let* $f\colon U(k)/T \longrightarrow U(n)/T$ *be a simple map where* $k > 2$. *Let* $l = \Big(n - \dim\big(\ker\{H^2(f)\}\big)\Big)/k$. *Then*

$$M_f\big(U(k)/T, U(n)/T\big) \simeq_{\mathbb{Q}} \big(U(l)/T\big)^k \times U(l-1)/T \times U(n-kl)/T \times \textit{odd spheres.}$$

PROOF. We split the list $\{x_1,\dots,x_{n-1}\}$ into $k+1$ sublists $\{x_1,\dots,x_l\},\dots,\{x_{l(k-2)+1},\dots,x_{l(k-1)}\}$, $\{x_{l(k-1)+1},\dots x_{lk-1}\}$ and $\{x_{lk},\dots,x_{n-1}\}$. By the Theorem 2.3 translated to cohomology, we may take $f^*(x_j) = t_i$ for x_j in the i-th sublist $i = 1,\dots,k-1$, $f^*(x_j) = -t_1 - \cdots - t_{k-1}$ for x_j in the k-th and $f^*(x_j) = 0$ for x_j in the $k+1$-st sublist. To determine the differential d_f in the model $A_f = \Lambda(x_1,\dots,x_{n-1}) \otimes_{d_f} \Lambda(Z_1)$ for $M_f\big(U(k)/T, U(n)/T\big)$ we must solve monomials in the $a_j = f^*(x_j)$ for monomials in the t_i. Let $c = (c_1,\dots,c_{k-1})$ be a $(k-1)$-tuple and let $b_c \in H_{2|c|}\big(U(K)/T\big)$ be dual to $t_1^{c_1}\cdots t_{k-1}^{c_{k-1}} \in H^{2|c|}\big(U(k)/T\big)$. The elements $b_c \otimes y_m$ span Z_1. Using Theorem 3.2 and the binomial formula for $(-t_1 - \cdots - t_{k-1})^d$ we find

$$d_f(b_c \otimes y_m) = \sum_{|e|\le|c|} (-1)^d \binom{d}{d_1} \cdots \binom{d}{d_{k-1}} D_1(e_1)\circ\cdots\circ D_{k-1}(e_{k-1})\circ D_k(d)(T_m)$$

where $d = |c| - |e|$ and $d_i = c_i - e_i$, $i = 1,\dots,k-1$.

The idea of the proof is to show that the image of d_f is precisely the ideal J of $\Lambda_2(x_1,\dots,x_{n-1})$ consisting of those polynomials which are symmetric in each of the $k+1$ variable lists separately. Now each operator $D_i(e_i)$ is clearly invariant under permutations of the variables in the i-th sublist and trivially invariant under permutations

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of the variables in the other k lists. Thus if $T \in J$ then $D_i(e_i)(T) \in J$ also. The inclusion $d_f\big(\Lambda(Z_1)\big) \subseteq J$ follows.

To show $d_f\big(\Lambda(Z_1)\big) \supseteq J$ we first write

$$J = (P_{1,1}, \ldots, P_{l,1}, \ldots, P_{1,k-1}, \ldots, P_{l,k-1}, P_{1,k}, \ldots, P_{l-1,k}, T_1, \ldots, T_n).$$

Regarding d_f in grade one, we must show $d_f(Z_1^1) = \mathbb{Q}(P_{1,1}, \ldots, P_{1,k}, T_1)$. Let $b_i \in H_2(X_k)$ be dual to $t_i \in H^2(X_k)$ so that $b_i \otimes y_2 \in Z_1^1$. Then

$$d_f(b_i \otimes y_2) = \sum_{j=l(i-1)+1}^{il} \frac{\partial}{\partial x_j} T_2 - \sum_{j=l(k-1)+1}^{lk-1} \frac{\partial}{\partial x_j} T_2 = P_{1,i} - P_{1,k} + T_1.$$

Next let $b_{(1,1,0,\ldots,0)}$ be dual to $t_1 t_2 \in H^4\big(U(k)/T\big)$. Then

$$\begin{aligned} d_f(b_{(1,1,0,\ldots,0)} \otimes y_3) &= \sum_{i=1}^{l} \sum_{j=l+1}^{2l} \frac{\partial^2}{\partial x_i \partial x_j} T_3 - \sum_{i=1}^{2l} \sum_{j=l(k-1)+1}^{lk-1} \frac{\partial^2}{\partial x_i \partial x_j} T_3 \\ &\quad + \sum_{i=l(k-1)+1}^{lk-1} \frac{\partial^2}{\partial x_i^2} T_3 + 2 \sum_{i=l(k-1)+1}^{lk-2} \sum_{j=i+1}^{l(k-1)} \frac{\partial^2}{\partial x_i \partial x_j} T_3 \\ &= P_{1,1} + P_{1,2} + (l+2) T_1. \end{aligned}$$

A similar calculation and equations (2)–(4) give

$$d_f(b_{(2,0,\ldots,0)} \otimes y_3) \equiv 2P_{1,1} \mod (T_1),$$

where $b_{(2,0,\ldots,0)}$ is dual to t_1^2. Thus $d_f\big(\Lambda(Z_1^1)\big) \supseteq J^{(2)}$.

Now since that $d_f(1 \otimes y_m) = T_m$ it remains only to show the elements $P_{m,i}$, $m > 1$, are in the image of d_f. Using equation (1) and computing as before we have

$$\begin{aligned} d_f(b_i \otimes y_{m+1}) &= \sum_{j=l(i-1)+1}^{il} \frac{\partial}{\partial x_j} T_{m+1} - \sum_{j=l(k-1)+1}^{lk-1} \frac{\partial}{\partial x_j} T_{m+1} \\ &\equiv P_{m,i} - P_{m,k} \mod (T_1, \ldots, T_m) \end{aligned}$$

for $i = 1, \ldots, k-1$. Similarly, using equations (2)–(4) we get

$$\begin{aligned} d_f(b_{(1,1,0,\ldots,0)} \otimes y_{m+2}) &\equiv P_{m,1} + P_{m,2} + 2(m - 2 - 2[m/2]) P_{m,k} \\ &\mod (P_{1,1}, \ldots, P_{m-1,1}, \ldots, P_{1,k}, \ldots, P_{m-1,k}, T_1, \ldots, T_m). \end{aligned}$$

The inclusion $d_f\big(\Lambda(Z_1)\big) \supseteq J$ follows. ∎

THEOREM 4.2. *Let $f\colon G/T \longrightarrow U(n)/T$ be a signed-simple map where G is simple of rank $k > 1$. Let $l = \Big(n - \dim\big(\ker\{H^2(f)\}\big)\Big)/2k$. Then*

$$M_f\big(G/T, U(n)/T\big) \simeq_{\mathbb{Q}} \big(U(l)/T\big)^{2k-1} \times U(l-1)/T \times U(n-2kl)/T \times \text{ odd spheres.}$$

PROOF. In this case, we split the list $\{x_1, \ldots, x_{n-1}\}$ into $2k+1$ sublists: the first $2k-1$ of length l, the $2k$-th of length $l-1$ and the last of length $n-2kl$. By Theorem 2.3, we

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may assume $f^*(x_j) = t_i$ for x_j in the i-th and $f^*(x_j) = -t_i$ for x_j in the $i+k$-th sublist $i = 1, \ldots, k$ while $f^*(x_j) = 0$ for x_j in the last sublist. In this case,

$$d_f(b_c \otimes y_m) = \sum_{|e| \leq |c|} (-1)^d D_1(e_1) \circ \cdots \circ D_k(e_k) \circ D_{k+1}(d_1) \circ \cdots \circ D_{2k}(d_k)(T_m)$$

where $d = |c| - |e|$ and $d_i = c_i - e_i, i = 1, \ldots, k$. The proof now proceeds in a similar manner to the above. ∎

THEOREM 4.3. *Let* $f\colon G/T \longrightarrow \mathrm{Sp}(n)/T$ *be a simple map where* G *is simple of* rank $k > 1$. *Let* $l = \Big(n - \dim\big(\ker\{H^2(f)\}\big)\Big)/k$. *Then*

$$M_f\big(G/T, \mathrm{Sp}(n)/T\big) \simeq_{\mathbb{Q}} \big(U(l)/T\big)^k \times \mathrm{Sp}(n-kl)/T \times \textit{odd spheres.}$$

PROOF. Here we split the variable list $\{x_1, \ldots, x_n\}$ into $k+1$ sublists; the first k of length l and the last of length $n - kl$ and take the map f^* to satisfy $f^*(x_j) = t_i$ for x_j in the i-th and $f^*(x_j) = 0$ for x_j in the $k+1$-st sublist. By Theorem 3.2., $d_f(b_c \otimes y_{2m}) = D_1(c_1) \circ \cdots \circ D_k(c_k)(S_{2m})$.

This time, we show that $d_f\big(\Lambda(Z_1)\big)$ equals the ideal J consisting of polynomials symmetric in our first k variable lists and in the squares of the elements of the $k+1$-st variable list separately. Since the operators $D_i(c_i)$ clearly preserve this ideal, $d_f\big(\Lambda(Z_1)\big) \supseteq J$. For the reverse inclusion, observe

$$J = (P_{1,1}, \ldots, P_{l,1}, \ldots, P_{1,k}, \ldots, P_{l,k}, P_{1,k+1}, \ldots, P_{l-1,k+1}, S_2, \ldots, S_{2n}).$$

Recalling that b_i is dual to $t_i \in H^2(G/T)$, we have

$$d_f(b_i \otimes y_2) = \sum_{j=l(i-1)+1}^{il} \frac{\partial}{\partial x_j} S_2 = P_{1,i}$$

and $d_f(1 \otimes y_m) = S_m$ so it remains to produce the $P_{m,i}$, $m > 1$. If m is odd, say $m = 2j-1$ for $j > 1$, then by equation (5)

$$d_f(b_i \otimes y_{2j}) = \sum_{h=l(i-1)+1}^{il} \frac{\partial}{\partial x_h} S_{2j} \equiv (-1)^{j-1} 2P_{m,i} \mod (S_2, \ldots, S_{2j}).$$

If m is even, say $m = 2j$ for $j > 1$, let $c_i = (0, \ldots, 2, \ldots, 0)$ be the k-tuple with a 2 in the i-th position so that $b_{c_i} \in H_4(G/T)$ denotes the element dual to $t_i^2 \in H^4(G/T)$. Using (5) and (6) we find

$$\begin{aligned} d_f(b_{c_i} \otimes y_{m+2}) &= \frac{1}{2} \sum_{h=(i-1)l}^{il} \frac{\partial^2}{\partial x_h^2} S_{m+2} + \sum_{h=(i-1)l}^{il-1} \sum_{k=h+1}^{il} \frac{\partial^2}{\partial x_h \partial x_k} S_{m+2} \\ &\equiv (2m-3)(-1)^j P_{m,i} \mod (P_{1,i}, \ldots, P_{m-1,i}, S_2, \ldots, S_{2n}). \end{aligned}$$ ∎

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