



Isomorphic Structure of Cesàro and Tandori Spaces

Sergey V. Astashkin, Karol Lesnik, and Lech Maligranda

Abstract. We investigate the isomorphic structure of the Cesàro spaces and their duals, the Tandori spaces. The main result states that the Cesàro function space Ces_∞ and its sequence counterpart ces_∞ are isomorphic. This is rather surprising since Ces_∞ (like Talagrand's example) has no natural lattice predual. We prove that ces_∞ is not isomorphic to ℓ_∞ nor is Ces_∞ isomorphic to the Tandori space \tilde{L}_1 with the norm $\|f\|_{\tilde{L}_1} = \|\tilde{f}\|_{L_1}$, where $\tilde{f}(t) := \text{ess sup}_{s \geq t} |f(s)|$. Our investigation also involves an examination of the Schur and Dunford–Pettis properties of Cesàro and Tandori spaces. In particular, using results of Bourgain we show that a wide class of Cesàro–Marcinkiewicz and Cesàro–Lorentz spaces have the latter property.

1 Introduction and Contents

The classical Cesàro spaces most commonly appear as optimal domains of the Cesàro (Hardy) operator or some of its versions (see [DS07, NP10, LM15b]; see also [ORSP08, CR16, CR17], where the vector measures' point of view is presented). Moreover, they can coincide with the so-called down spaces, which were introduced and investigated by Sinnamon [KMS07, MS06, Si94, Si01, Si07], but have their roots in the papers of Halperin and Lorentz. In comparison to the function case, there is a much richer literature devoted to Cesàro sequence spaces and their duals (see the classical paper of Bennett [Be96] and also [CH01, CMP00, Ja74, KK12, MPS07]). Development of this topic related to the weighted case, including the so-called blocking technique, can be found in [GE98]. In this paper we investigate the isomorphic structure of abstract Cesàro spaces CX and their duals, Tandori spaces \tilde{X} , on three separable measure spaces \mathbb{N} , $[0, \infty)$ and $[0, 1]$. For a Banach ideal space X of measurable functions on $I = [0, \infty)$ or $I = [0, 1]$, CX is defined as the space of all measurable functions f on I such that $C|f| \in X$, equipped with the norm $\|f\|_{CX} := \|C|f|\|_X$, where C denotes the Cesàro operator, i.e., $(Cf)(x) := \frac{1}{x} \int_0^x f(t) dt$ for $0 < x \in I$. For a Banach ideal sequence space X , one uses the corresponding discrete Cesàro operator $(C_d x)_n := \frac{1}{n} \sum_{k=1}^n x_k$, $n \in \mathbb{N}$ in the definition of the Cesàro space.

The study of abstract Cesàro function spaces, under this name, started in [LM15a], where a description of their Köthe duals by the so-called Tandori spaces can be found. It is worth noting here that the results obtained differ substantially in the

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cases $I = [0, \infty)$ and $I = [0, 1]$. Continuing the same direction of research, the authors examined interpolation properties of these spaces [LM16]. Investigation of the isomorphic structure of classical Cesàro function spaces $\text{Ces}_p := CL_p$ was initiated in [AM09] (see also [AM14]); however, studying classical Cesàro sequence spaces $\text{ces}_p := C\ell_p$ started much earlier (see [MPS07] and the references therein). Among other things, the existence of an isomorphism between the spaces $\text{Ces}_p[0, \infty)$ and $\text{Ces}_p[0, 1]$ for all $1 < p \leq \infty$ has been proved [AM09]. On the other hand, $\text{Ces}_p(I)$ and ces_p for any $1 < p < \infty$ are clearly not isomorphic because, in contrast to ces_p , the space $\text{Ces}_p(I)$ is not reflexive.

Therefore, the only remaining question (already formulated in [AM09, AM14]) was whether or not Ces_∞ is isomorphic to ces_∞ . Theorem 5.1, one of the main results of the present paper, answers this problem in the affirmative. It is instructive to compare this result with the well-known Pełczyński theorem on the existence of an isomorphism between the spaces L_∞ and ℓ_∞ [Pe58] and also with Leung's result that showed, using Pełczyński decomposition method, that the weak L_p -spaces: $L_{p,\infty}[0, 1]$, $L_{p,\infty}[0, \infty)$, and $\ell_{p,\infty}$ for every $1 < p < \infty$ are all isomorphic [Le93].

On the other hand, we prove that ces_∞ and $\text{Ces}_\infty(I)$ are not isomorphic to ℓ_∞ or to the Tandori space $\tilde{L}_1(I)$ with the norm $\|f\|_{\tilde{L}_1} = \|\tilde{f}\|_{L_1}$, where

$$\tilde{f}(t) := \text{ess sup}_{s \in I, s \geq t} |f(s)|.$$

Moreover, if X is a reflexive symmetric space on $[0, 1]$ and the Cesàro operator C is bounded on X , then CX is not isomorphic to any symmetric space on $[0, 1]$. The main tool in proving these results comes from the fact that an arbitrary Cesàro space CX (and, in particular, ces_∞ and $\text{Ces}_\infty(I)$) contains a complemented copy of $L_1[0, 1]$, provided the Cesàro operator C is bounded in X (see Proposition 2.2), but the other ones do not. We also make use of a characterization due to Hagler–Stegall [HS73] of dual Banach spaces containing complemented copies of $L_1[0, 1]$; the Cembranos–Mendoza result [CM08], stated that the mixed-norm space $\ell_\infty(\ell_1)$ contains a complemented copy of $L_1[0, 1]$, while the space $\ell_1(\ell_\infty)$ does not.

In this paper, along with the isomorphic structure of abstract Cesàro and Tandori spaces we also study their Schur and Dunford–Pettis properties (being isomorphic invariants). In particular, we are able to find a new rather natural Banach space non-isomorphic to ℓ_1 with the Schur property, namely, the sequence Tandori space $\tilde{\ell}_1$ with the norm $\|(a_k)\|_{\tilde{\ell}_1} = \|(\tilde{a}_k)\|_{\ell_1}$, where $\tilde{a}_k := \sup_{i \geq k} |a_i|$. Regarding the Dunford–Pettis property, we note that, generally, it is not easy to determine whether a given space has this property. We apply here a deep result of Bourgain [Bo81] that shows that every ℓ_∞ -sum of L_1 -spaces has the Dunford–Pettis property. Using this, Bourgain also deduced that the spaces of vector-valued functions $L_1(\mu, C(K))$ and $C(K, L_1(\mu))$ and their duals, where μ and K are a σ -finite measure and any compact Hausdorff set, respectively, have the Dunford–Pettis property. Using these facts and a suitable description of Cesàro and Tandori spaces obtained in this paper, we will prove that $\text{Ces}_\infty(I)$, $\tilde{L}_1(I)$ and, under some conditions on dilation indices of a function φ , Cesàro–Marcinkiewicz spaces $CM_\varphi[0, \infty)$, their separable parts $C(M_\varphi^0)(I)$, Cesàro–Lorentz spaces $CA_\varphi(I)$ as well as Tandori–Lorentz spaces $\tilde{\Lambda}_\varphi(I)$ (in both cases $I = [0, \infty)$ and $I = [0, 1]$) all enjoy the Dunford–Pettis property. Recall that Kamińska and

Mastyło [KM00] proved that ℓ_1 , c_0 and ℓ_∞ are the only symmetric sequence spaces with the Dunford–Pettis property and there exist exactly six non-isomorphic symmetric spaces on $[0, \infty)$ enjoying the latter property: L_1 , L_∞ , $L_1 \cap L_\infty$, $L_1 + L_\infty$, and the closures of $L_1 \cap L_\infty$ in both L_∞ and $L_1 + L_\infty$. The paper is organized as follows. Following this Introduction, Section 2 collects some necessary preliminaries, first, on Banach ideal and symmetric spaces and then, on Cesàro and Tandori spaces. Here, we recall also Theorem 2.1 related to the duality from [LM15a] and prove Proposition 2.2 on the existence of a complemented copy of $L_1[0, 1]$ in an arbitrary Cesàro space CX provided the Cesàro operator C is bounded in X . These results are frequently used throughout the paper.

Section 3 contains results related to studying the Schur and Dunford–Pettis properties of Tandori and Cesàro sequence spaces. We prove that $\tilde{\ell}_1$ has the Schur property (Theorem 3.1) and ces_∞ contains a complemented copy of $L_1[0, 1]$ (Theorem 3.7). Moreover, we investigate the conditions under which both Cesàro–Marcinkiewicz and Cesàro–Lorentz sequence spaces as well as their duals have the Dunford–Pettis property (see Theorems 3.8 and 3.9). Finally, we show that the spaces CX and \tilde{X} fail to have the Dunford–Pettis property whenever a symmetric sequence space X is reflexive and the discrete Cesàro operator is bounded in X or in X' , respectively (Theorem 3.11).

Section 4 deals with the Dunford–Pettis property of Cesàro and Tandori function spaces. It is proved that, under the assumption $q_\varphi < 1$, both Tandori–Lorentz space $\tilde{\Lambda}_\varphi[0, \infty)$ and Cesàro–Marcinkiewicz space $CM_\varphi[0, \infty)$ have the Dunford–Pettis property (Theorem 4.1). In particular, two non-isomorphic spaces $\text{Ces}_\infty(I)$ and $\tilde{L}_1(I)$ have the latter property (see Theorem 4.3). A similar result also holds for the separable parts of the Cesàro–Marcinkiewicz spaces $CM_\varphi[0, \infty)$ and $CM_\varphi[0, 1]$ provided $\lim_{t \rightarrow 0^+} \varphi(t) = 0$ and $q_\varphi < 1$ or $q_\varphi^0 < 1$, respectively (Theorems 4.6 and 4.8). Moreover, if X is a reflexive symmetric function space satisfying some conditions, then CX and \tilde{X} fail to have the Dunford–Pettis property (Theorem 4.9).

Section 5, contains one of the main results of the paper, showing that the spaces Ces_∞ and ces_∞ are isomorphic (Theorem 5.1). This gives a positive answer to the question posed in [AM09, Problem 1] and repeated in [AM14, Problem 4]. An interesting consequence of this result is the fact that the space Ces_∞ is isomorphic to a dual space, although $[(\text{Ces}_\infty)']^0 = (\tilde{L}_1)^0 = \{0\}$ (Corollary 5.2) and so there is no natural candidate for its predual. For ces_∞ , however, the predual is $\tilde{\ell}_1$, because $(\tilde{\ell}_1)^* = (\tilde{\ell}_1)' = \text{ces}_\infty$. Observing that this phenomenon has its counterpart in the general theory of Banach lattices, we discuss its relation to Lotz’s result [Lo75] and to Talagrand’s example of a separable Banach lattice being a dual space (and hence having the Radon–Nikodym property) such that for each $x^* \in E^*$, the interval $[0, |x^*|]$ is not weakly compact [Ta81] (see Proposition 5.4). Finally, we prove that $\text{Ces}_\infty(I)$ is isomorphic to the space $(\bigoplus_{k=1}^\infty \mathcal{M}[0, 1])_{\ell_\infty}$, where $\mathcal{M}[0, 1]$ is the space of regular Borel measures on $[0, 1]$ of finite variation (Theorem 5.5).

2 Definitions and Basic Facts

2.1 Banach Ideal Spaces and Symmetric Spaces

By $L^0 = L^0(I)$ we denote the set of all equivalence classes of real-valued Lebesgue measurable functions defined on $I = [0, 1]$ or $I = [0, \infty)$. A *Banach ideal space* $X = (X, \|\cdot\|)$ on I is understood as a Banach space contained in $L^0(I)$ that satisfies the so-called ideal property: if $f, g \in L^0(I)$, $|f| \leq |g|$ almost everywhere with respect to the Lebesgue measure on I and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$. Sometimes we write $\|\cdot\|_X$ to be sure which norm in the space is considered. If not stated otherwise, we suppose that a Banach ideal space X contains a function $f \in X$ with $f(x) > 0$ almost everywhere on I (such a function is called the *weak unit* in X), which implies that $\text{supp} X = I$. Similarly we define a Banach ideal sequence space, *i.e.*, on $I = \mathbb{N}$ with the counting measure.

Since an inclusion of two Banach ideal spaces is continuous, we prefer to write in this case $X \hookrightarrow Y$ rather than $X \subset Y$. Moreover, the symbol $X \overset{A}{\hookrightarrow} Y$ indicates that $X \hookrightarrow Y$ with the norm of the inclusion operator not bigger than A , *i.e.*, $\|f\|_Y \leq A\|f\|_X$ for all $f \in X$. Also, $X = Y$ (resp. $X \equiv Y$) means that the spaces X and Y have the same elements with equivalent (resp. equal) norms. By $X \simeq Y$ we denote the fact that the Banach spaces X and Y are isomorphic.

For a Banach ideal space $X = (X, \|\cdot\|)$ on I , the *Köthe dual space* (or *associated space*) X' is the space of all $f \in L^0(I)$ such that the *associated norm*

$$\|f\|' = \sup_{\substack{g \in X \\ \|g\|_X \leq 1}} \int_I |f(x)g(x)| dx$$

is finite. The Köthe dual $X' = (X', \|\cdot\|')$ is a Banach ideal space. Moreover, $X \overset{1}{\hookrightarrow} X''$ and we have equality $X = X''$ with $\|f\| = \|f\|''$ if and only if the norm in X has the *Fatou property*, meaning that the conditions $0 \leq f_n \nearrow f$ almost everywhere on I and $\sup_{n \in \mathbb{N}} \|f_n\| < \infty$ imply that $f \in X$ and $\|f_n\| \nearrow \|f\|$.

For a Banach ideal space $X = (X, \|\cdot\|)$ on I with the Köthe dual X' , the following *generalized Hölder–Rogers inequality* holds: if $f \in X$ and $g \in X'$, then fg is integrable and

$$(2.1) \quad \int_I |f(x)g(x)| dx \leq \|f\|_X \|g\|_{X'}.$$

A function f in a Banach ideal space X on I is said to have an *order continuous norm* in X if for any decreasing sequence of Lebesgue measurable sets $A_n \subset I$ with $m(\bigcap_{n=1}^{\infty} A_n) = 0$, where m is the Lebesgue measure, we have $\|f\chi_{A_n}\| \rightarrow 0$ as $n \rightarrow \infty$. The set of all functions in X with order continuous norm is denoted by X^0 . If $X^0 = X$, then the space X is said to be *order continuous*. For an order continuous Banach ideal space X , the Köthe dual X' , and the dual space X^* coincide. Moreover, a Banach ideal space X with the Fatou property is reflexive if and only if both X and its Köthe dual X' are order continuous.

For a weight $w(x)$, *i.e.*, a measurable function on I with $0 < w(x) < \infty$ almost everywhere and for a Banach ideal space X on I , the *weighted Banach ideal space* $X(w)$

is defined as the set $X(w) = \{f \in L^0 : fw \in X\}$ with the norm $\|f\|_{X(w)} = \|fw\|_X$. Of course, $X(w)$ is also a Banach ideal space and $[X(w)]' \equiv X'(1/w)$.

A Banach ideal space $X = (X, \|\cdot\|_X)$ is said to be a *symmetric* (or *rearrangement invariant*) space on I if, from the conditions $f \in X, g \in L^0(I)$, and the equality $d_f(\lambda) = d_g(\lambda)$ for all $\lambda > 0$, where $d_f(\lambda) := m(\{x \in I : |f(x)| > \lambda\})$, $\lambda \geq 0$, it follows that $g \in X$ and $\|f\|_X = \|g\|_X$. In particular, $\|f\|_X = \|f^*\|_X$, where

$$f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) < t\}, \quad t \geq 0.$$

For a symmetric function space X on I , its fundamental function φ_X is defined as

$$\varphi_X(t) = \|\chi_{[0,t]}\|_X, \quad t > 0,$$

where χ_E , throughout, will denote the characteristic function of a set E .

Let us recall some classical examples of symmetric spaces. Each increasing concave function φ on I , $\varphi(0) = 0$, generates the *Lorentz space* Λ_φ (resp. *Marcinkiewicz space* M_φ) on I endowed with the norm $\|f\|_{\Lambda_\varphi} = \int_I f^*(s) d\varphi(s)$, respectively,

$$(2.2) \quad \|f\|_{M_\varphi} = \sup_{t \in I} \frac{\varphi(t)}{t} \int_0^t f^*(s) ds.$$

In the case when $\varphi(t) = t^{1/p}, 1 < p < \infty$, the Marcinkiewicz space is also called the weak- L_p space (briefly denoted by $L_{p,\infty}$) and the norm (2.2) is equivalent to the quasi-norm $\|f\|_{L_{p,\infty}} = \sup_{t \in I} t^{1/p} f^*(t)$. In general, a space M_φ is not separable (for example, when $\lim_{t \rightarrow 0^+} \frac{t}{\varphi(t)} = \lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} = 0$), but the spaces

$$\left\{ f \in M_\varphi : \lim_{t \rightarrow 0^+, \infty} \frac{\varphi(t)}{t} \int_0^t f^*(s) ds = 0 \right\}$$

in the case when $I = [0, \infty)$

and

$$\left\{ f \in M_\varphi : \lim_{t \rightarrow 0^+} \frac{\varphi(t)}{t} \int_0^t f^*(s) ds = 0 \right\}$$

in the case when $I = [0, 1]$

with the norm (2.2) are the separable symmetric spaces which, in fact, coincide with the space M_φ^0 on $I = [0, \infty)$ or $I = [0, 1]$, respectively, provided $\lim_{t \rightarrow 0^+} \varphi(t) = 0$ [KPS82, pp. 115–116].

Let Φ be an increasing convex function on $[0, \infty)$ such that $\Phi(0) = 0$. Denote by L_Φ the *Orlicz space* on I [KR61, Ma89] endowed with the Luxemburg–Nakano norm $\|f\|_{L_\Phi} = \inf\{\lambda > 0 : \int_I \Phi(|f(x)|/\lambda) dx \leq 1\}$.

For a given symmetric space X with the fundamental function φ (recall that every such a function is equivalent to a concave function), we have $\Lambda_\varphi \xrightarrow{2} X \xrightarrow{1} M_\varphi$ and $(M_\varphi)' = \Lambda_\psi$ with $\psi(t) = \frac{t}{\varphi(t)}, t > 0$.

Similarly one can define Banach ideal and symmetric sequence spaces and all the above notions. In particular, the fundamental function of a symmetric sequence space X is the function $\varphi_X(n) = \|\sum_{k=1}^n e_k\|_X, n \in \mathbb{N}$, where $\{e_n\}_{n=1}^\infty$ is the canonical basic sequence of X . Moreover, the *Lorentz sequence space* λ_φ (resp. *Marcinkiewicz sequence space* m_φ) is defined as the space of all sequences $x = (x_n)_{n=1}^\infty$, for which the following norm is finite $\|x\|_{\lambda_\varphi} = \sum_{k=1}^\infty x_k^*(\varphi(k+1) - \varphi(k))$, respectively,

$$(2.3) \quad \|x\|_{m_\varphi} = \sup_{n \in \mathbb{N}} \frac{\varphi(n)}{n} \sum_{k=1}^n x_k^*,$$

where φ is an increasing concave function on $[0, \infty)$ and (x_k^*) is the decreasing re-arrangement of the sequence $(|x_k|)_{k=1}^\infty$. In the case when $\varphi(n) = n^{1/p}$ for $1 < p < \infty$, the Marcinkiewicz space m_φ is also called the weak- ℓ_p space (denoted by $\ell_{p,\infty}$ for short) and the norm (2.3) is equivalent to the quasi-norm $\|x\|_{\ell_{p,\infty}} = \sup_{k \in \mathbb{N}} k^{1/p} x_k^*$.

The dilation operators σ_s ($s > 0$) defined on $L^0(I)$ by $\sigma_s f(x) = f(x/s)$ if $I = [0, \infty)$ and $\sigma_s f(x) = f(x/s)\chi_{[0, \min(1,s)]}(x)$ if $I = [0, 1]$ are bounded in any symmetric space X on I and $\|\sigma_s\|_{X \rightarrow X} \leq \max(1, s)$ [BS88, p. 148] and [KPS82, pp. 96–98]. These operators are also bounded in some Banach ideal spaces that are not symmetric. For example, if $X = L_p(x^\alpha)$, then $\|\sigma_s\|_{X \rightarrow X} = s^{1/p+\alpha}$ (see [Ru80] for more examples). The Boyd indices of a symmetric space X are defined by

$$\alpha_X = \lim_{s \rightarrow 0^+} \frac{\ln \|\sigma_s\|_{X \rightarrow X}}{\ln s}, \quad \beta_X = \lim_{s \rightarrow \infty} \frac{\ln \|\sigma_s\|_{X \rightarrow X}}{\ln s},$$

and we have $0 \leq \alpha_X \leq \beta_X \leq 1$ [KPS82, pp. 96–98], [LT79, p. 139].

For every $m \in \mathbb{N}$, let σ_m and $\sigma_{1/m}$ be the dilation operators defined in spaces of sequences $a = (a_n)$ by

$$\begin{aligned} \sigma_m a &= ((\sigma_m a)_n)_{n=1}^\infty = (a_{\lceil \frac{n-1+n}{m} \rceil})_{n=1}^\infty = (\overbrace{a_1, a_1, \dots, a_1}^m, \overbrace{a_2, a_2, \dots, a_2}^m, \dots), \\ \sigma_{1/m} a &= ((\sigma_{1/m} a)_n)_{n=1}^\infty = \left(\frac{1}{m} \sum_{k=(n-1)m+1}^{nm} a_k\right)_{n=1}^\infty \\ &= \left(\frac{1}{m} \sum_{k=1}^m a_k, \frac{1}{m} \sum_{k=m+1}^{2m} a_k, \dots, \frac{1}{m} \sum_{k=(n-1)m+1}^{nm} a_k, \dots\right), \end{aligned}$$

[LT79, p. 131], [KPS82, p. 165]. They are bounded in any symmetric sequence space and also in some non-symmetric Banach ideal sequence spaces. For example,

$$\|\sigma_m\|_{\ell_p(n^\alpha) \rightarrow \ell_p(n^\alpha)} \leq m^{1/p} \max(1, m^\alpha), \quad \|\sigma_{1/m}\|_{\ell_p(n^\alpha) \rightarrow \ell_p(n^\alpha)} \leq m^{-1/p} \max(1, m^{-\alpha}).$$

The lower index p_φ and upper index q_φ of an arbitrary positive function φ on $[0, \infty)$ are defined as

$$(2.4) \quad p_\varphi = \lim_{s \rightarrow 0^+} \frac{\ln \bar{\varphi}(s)}{\ln s}, \quad q_\varphi = \lim_{s \rightarrow \infty} \frac{\ln \bar{\varphi}(s)}{\ln s}, \quad \text{where } \bar{\varphi}(s) = \sup_{t>0} \frac{\varphi(st)}{\varphi(t)}.$$

It is known [KPS82, Ma85, Ma89] that for a concave function φ on $[0, \infty)$, we have $0 \leq p_\varphi \leq q_\varphi \leq 1$. Moreover, the estimate

$$(2.5) \quad \int_0^t \frac{1}{\varphi(s)} ds \leq C \frac{t}{\varphi(t)} \quad \text{for all } t > 0$$

is equivalent to the condition $q_\varphi < 1$ [KPS82, Lemma 1.4], [Ma85, Theorem 11.8], [Ma89, Theorem 6.4(a)]. If an increasing concave function φ is defined on $[0, 1]$ (resp. on $[1, \infty)$), then the corresponding indices p_φ^0, q_φ^0 (resp. $p_\varphi^\infty, q_\varphi^\infty$) are the numbers defined as the limits in (2.4), where instead of $\bar{\varphi}$ we take the function $\bar{\varphi}^0(s) = \sup_{0 < t \leq \min(1, 1/s)} \varphi(st)/\varphi(t)$ (resp. $\bar{\varphi}^\infty(s) = \sup_{t \geq \max(1, 1/s)} \varphi(st)/\varphi(t)$). Of course, $0 \leq p_\varphi^0 \leq q_\varphi^0 \leq 1$ (resp. $0 \leq p_\varphi^\infty \leq q_\varphi^\infty \leq 1$), and the estimate (2.5) for all $0 < t \leq 1$ are equivalent to the condition $q_\varphi^0 < 1$.

If $X_n, n = 1, 2, \dots$, are Banach spaces and $1 \leq p \leq \infty$, then $(\bigoplus_{n=1}^{\infty} X_n)_{\ell_p}$ is the Banach space of all sequences $\{x_n\}, x_n \in X_n, n = 1, 2, \dots$, such that

$$\|\{x_n\}\| := \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p} < \infty,$$

with natural modification in the case when p is infinite.

For general properties of Banach ideal and symmetric spaces we refer to [BS88, KA77, KPS82, LT79, Ma89].

2.2 Cesàro and Tandori Spaces

The Cesàro and Copson operators C and C^* are defined, respectively, as

$$Cf(x) = \frac{1}{x} \int_0^x f(t) dt, \quad 0 < x \in I,$$

$$C^*f(x) = \int_{I \cap [x, \infty)} \frac{f(t)}{t} dt, \quad x \in I.$$

By \tilde{f} we will understand the decreasing majorant of a given function f , i.e., $\tilde{f}(x) = \text{ess sup}_{t \in I, t \geq x} |f(t)|$. For a Banach ideal space X on I we define the *abstract Cesàro space* $CX = CX(I)$ as

$$(2.6) \quad CX = \{f \in L^0(I) : C|f| \in X\} \text{ with the norm } \|f\|_{CX} = \|C|f|\|_X,$$

and the *abstract Tandori space* $\tilde{X} = \tilde{X}(I)$ as

$$(2.7) \quad \tilde{X} = \{f \in L^0(I) : \tilde{f} \in X\} \text{ with the norm } \|f\|_{\tilde{X}} = \|\tilde{f}\|_X.$$

In particular, if $X = L_p, 1 < p \leq \infty$, we come to classical Cesàro spaces $\text{Ces}_p = CL_p$, which were investigated in [AM08, AM09, AM13, AM14]. Note that the case when $p = 1$ is not interesting here because it is easy to see that $\text{Ces}_1[0, \infty) = \{0\}$ and $\text{Ces}_1[0, 1] = L_1(\ln \frac{1}{t})$. The space $\text{Ces}_{\infty}[0, 1]$ appeared already in 1948 and is known as the Korenblyum–Krein–Levin space [KKL48, LZ66, Wn99].

It is clear that $\tilde{X} \xrightarrow{1} X$ and $X \xrightarrow{A} CX$, provided C is bounded on X with $A = \|C\|_{X \rightarrow X}$. Moreover, if X is a symmetric space on I , then for every $0 < a < b, b \in I$ we have

$$(2.8) \quad \|\chi_{[a,b]}\|_{\tilde{X}} = \|\widetilde{\chi_{[a,b]}}\|_X = \|\chi_{[0,b]}\|_X = \varphi_X(b).$$

In the sequence case, the discrete Cesàro and Copson operators C_d and C_d^* are defined by $(C_d a)_n = \frac{1}{n} \sum_{k=1}^n a_k$ and $(C_d^* a)_n = \sum_{k=n}^{\infty} \frac{a_k}{k}, n \in \mathbb{N}$, and also the decreasing majorant $\tilde{a} = (\tilde{a}_n)$ of a given sequence $a = (a_n)$ is defined by $\tilde{a}_n = \sup_{k \in \mathbb{N}, k \geq n} |a_k|$. Then the corresponding *abstract Cesàro sequence space* CX and *abstract Tandori sequence space* \tilde{X} are defined as in (2.6) and (2.7). Again a lot of information is known about classical Cesàro sequence spaces $\text{ces}_p := C\ell_p, 1 < p \leq \infty$ ([AM08, AM13, MPS07] and references given therein).

Abstract Cesàro and Copson spaces were investigated in [LM15a, LM15b], where the following results on their Köthe duality were proved [LM15a, Theorems 3, 5, 6].

- Theorem 2.1** (i) If X is a Banach ideal space on $I = [0, \infty)$ such that the Cesàro operator C and the dilation operator σ_τ for some $\tau \in (0, 1)$ are bounded on X , then $(CX)' = \widetilde{X}'$.
- (ii) If X is a symmetric space on $[0, 1]$ with the Fatou property such that the operators $C, C^*: X \rightarrow X$ are bounded, then $(CX)' = \widetilde{X}'(w_1)$, where $w_1(x) = \frac{1}{1-x}$, $x \in [0, 1)$.
- (iii) If X is a Banach ideal sequence space such that the dilation operator σ_3 is bounded on X' , then $(CX)' = \widetilde{X}'$. Moreover, in the extreme case when $X = L_\infty(I)$, the above duality results hold with the equality of norms.

The corresponding results on the Köthe duality of the classical spaces $\text{Ces}_p(I)$ for $1 < p < \infty$ were proved in [AM09] (see also [KK12]) showing a surprising difference among them in the cases $I = [0, \infty)$ and $I = [0, 1]$. Much earlier, the identifications $(\text{Ces}_\infty[0, 1])' \equiv \widetilde{L}_1[0, 1]$ and $(\text{Ces}_\infty^0[0, 1])^* \equiv \widetilde{L}_1[0, 1]$ were obtained by Luxemburg and Zaanan [LZ66] and by Tandori [Ta55], respectively. A simple proof of the latter results both for $I = [0, 1]$ and $I = [0, \infty)$ was given in [LM15a]. Moreover, according to Theorem 7 from [LM15a], if w is a weight on I such that $W(x) = \int_0^x w(t) dt < \infty$ for any $x \in I$, then setting $v(x) = x/W(x)$ we obtain

$$(\text{Ces}_\infty(v))' := [C(L_\infty(v))]' \equiv \widetilde{L}_1(\overline{w}).$$

A close identification for weighted Cesàro sequence spaces follows from a result by Alexiewicz [Al57], who showed that for a weight $w = (w_n)$ with $w_n \geq 0, w_1 > 0$, we have

$$(2.9) \quad (\widetilde{\ell}_1(\overline{w}))' \equiv \text{ces}_\infty(v) := C(\ell_\infty(v)), \quad \text{where } v(n) = \frac{n}{\sum_{k=1}^n w_k}.$$

In particular, using the Fatou property of the space $\widetilde{\ell}_1(\overline{w})$, from (2.9) we infer

$$(2.10) \quad (\text{ces}_\infty(v))' \equiv (\widetilde{\ell}_1(\overline{w}))'' \equiv \widetilde{\ell}_1(\overline{w}).$$

In [LM15a, Theorem 1(d)], it was shown that if a Banach ideal space X has the Fatou property, then the Cesàro and Tandori function spaces CX and \widetilde{X} also have it. Moreover, if a space X is order continuous, then the Cesàro function space CX is order continuous as well [LM16, Lemma 1]. However, the Tandori function space \widetilde{X} is never order continuous [LM15a, Theorem 1(e)], which implies immediately that this space contains an isomorphic copy of ℓ_∞ .

Next, we repeatedly make use of the following fact.

Proposition 2.2 If X is a Banach ideal function space on $I = [0, 1]$ or $I = [0, \infty)$ such that the operator C is bounded on X , then CX contains a complemented copy of $L_1[0, 1]$. Moreover, if $\chi_{[0, a]} \in X$ for $0 < a < 1$, then $\widetilde{X} \neq \{0\}$ and \widetilde{X} contains a complemented copy of $L_\infty[0, 1]$.

Proof Suppose that $I = [0, 1]$. Since $\text{supp } X = I$ and the operator C is bounded on X , then $\chi_{[a, 1]} \in X$ for any $0 < a < 1$. In fact, let $f_0 \in X$ with $f_0(x) > 0$ almost everywhere

on I . Then $f_0\chi_{[0,a]} \in X$ and $\int_0^a f_0(t) dt = c > 0$. Therefore, from the estimate

$$\begin{aligned} C(f_0\chi_{[0,a]})(x) &\geq \frac{1}{x} \int_0^x f_0(t)\chi_{[0,a]}(t) dt \geq \frac{1}{x} \int_0^x f_0(t)\chi_{[0,a]}(t) dt \cdot \chi_{[a,1]}(x), \\ &= \frac{1}{x} \int_0^a f_0(t) dt \cdot \chi_{[a,1]}(x) \geq c\chi_{[a,1]}(x), \quad 0 < x \leq 1, \end{aligned}$$

and from the boundedness of C on X it follows that $\chi_{[a,1]} \in X$. Now for $0 < a < b < 1$, one has

$$\begin{aligned} C(f\chi_{[a,b]})(x) &= \frac{1}{x} \int_a^x |f(t)| dt \chi_{[a,b]}(x) + \frac{1}{x} \int_a^b |f(t)| dt \chi_{[b,1]}(x) \\ &\leq \frac{1}{a} \int_a^b |f(t)| dt [\chi_{[a,b]}(x) + \chi_{[b,1]}(x)] = \frac{1}{a} \|f\|_{L_1[a,b]} \cdot \chi_{[a,1]}(x), \\ C(f\chi_{[a,b]})(x) &= \frac{1}{x} \int_a^x |f(t)|\chi_{[a,b]}(t) dt \geq \frac{1}{x} \int_a^x |f(t)|\chi_{[a,b]}(t) dt \cdot \chi_{[b,1]}(x) \\ &\geq \frac{1}{x} \int_a^b |f(t)| dt \cdot \chi_{[b,1]}(x) = \|f\|_{L_1[a,b]} \cdot \chi_{[b,1]}(x). \end{aligned}$$

Thus,

$$d \|f\|_{L_1[a,b]} \leq \|f\chi_{[a,b]}\|_{CX} \leq \frac{D}{a} \|f\|_{L_1[a,b]},$$

where $d = \|\chi_{[b,1]}\|_X$ and $D = \|\chi_{[a,1]}\|_X$ are finite. Therefore, $CX|_{[a,b]} = L_1[a, b]$ with equivalence of norms and, since $L_1[a, b] \simeq L_1[0, 1]$ and the projection $P: f \mapsto f\chi_{[a,b]}$ is bounded in CX , the first claim of the proposition is proved if $I = [0, 1]$. The case when $I = [0, \infty)$ can be treated in the same way, only the norm $\|\chi_{[b,1]}\|_X$ should be replaced with $\|\frac{1}{x}\chi_{[b,\infty)}(x)\|_X$.

Regarding the space \widetilde{X} , we note that under the conditions imposed on X by (2.8), we have $\widetilde{f\chi_{[a,b]}} \leq \|f\|_{L_\infty[a,b]} \cdot \widetilde{\chi_{[a,b]}} = \|f\|_{L_\infty[a,b]} \cdot \chi_{[0,b]}$ and conversely

$$\widetilde{f\chi_{[a,b]}} \geq \widetilde{f\chi_{[a,b]}} \cdot \chi_{[0,a]} = \|f\|_{L_\infty[a,b]} \cdot \chi_{[0,a]},$$

whence $\|\chi_{[0,a]}\|_X \|f\|_{L_\infty[a,b]} \leq \|f\chi_{[a,b]}\|_{\widetilde{X}} \leq \|\chi_{[0,b]}\|_X \|f\|_{L_\infty[a,b]}$. Thus, the image of the same projection $Pf = f\chi_{[a,b]}$ is isomorphic to $L_\infty[0, 1]$. Since P is bounded, the proof is complete. ■

3 On the Schur and Dunford–Pettis Properties of Cesàro and Tandori Sequence Spaces

A Banach space X is said to have the *Dunford–Pettis property* if, for all sequences $x_n \xrightarrow{w} 0$ in X and $x_n^* \xrightarrow{w} 0$ in X^* , we have $\langle x_n^*, x_n \rangle \rightarrow 0$ as $n \rightarrow \infty$ or, equivalently, if any weakly compact operator $T: X \rightarrow Y$, where Y is an arbitrary Banach space, is completely continuous, *i.e.*, from $x_n \xrightarrow{w} 0$, it follows that $T(x_n)$ converges to 0 in the norm of Y . Examples of spaces satisfying the Dunford–Pettis property are $\ell_1, c_0, \ell_\infty, L_1(\mu), L_\infty(\mu)$ for every σ -finite measure μ and $C(K), C(K)^* = \mathcal{M}(K)$ for an arbitrary compact Hausdorff space K [AK06, pp. 116–117], [Li04, pp. 57–67]). It is well known that infinite-dimensional reflexive spaces fail to have the Dunford–Pettis property. Moreover, if a dual space X^* has the Dunford–Pettis property, then

so does X (the reverse implication is not true) and complemented subspaces of spaces with the Dunford–Pettis property also have it.

Recall that a Banach space X has the *Schur property* if for any sequence $x_n \xrightarrow{w} 0$ in X we have $\|x_n\| \rightarrow 0$ as $n \rightarrow \infty$ or, equivalently, if every weakly compact operator from X to arbitrary Banach space Y is compact. Of course, spaces with the Schur property have also the Dunford–Pettis property. Even though it has been known [Ba32, pp. 137–139] that the space ℓ_1 has the Schur property, only a few natural infinite-dimensional spaces enjoying it were found. See [Di80] and [Wn93] for a survey of results related to the Dunford–Pettis property and the Schur property, respectively, (see also [CG94]).

We start with proving the Schur property of the space $\tilde{\ell}_1$. Note that $\tilde{\ell}_1$ is not isomorphic to ℓ_1 . In fact, $\{\frac{1}{n}e_n\}$ is a normalized, unconditional basis in $\tilde{\ell}_1$. On the other hand, ℓ_1 has a unique unconditional structure, *i.e.*, each normalized unconditional basis in ℓ_1 is equivalent to $\{e_n\}$ [LT77, Theorem 2.b.9]. Therefore, if we assume that $\tilde{\ell}_1$ is isomorphic to ℓ_1 , then $\{\frac{1}{n}e_n\}$ would be equivalent to $\{e_n\}$. But it is not the case, since we have both $\|\sum_{n=1}^k \frac{1}{n}e_n\|_{\tilde{\ell}_1} \approx \ln k$ and $\|\sum_{n=1}^k e_n\|_{\ell_1} = k, k \in \mathbb{N}$.

Theorem 3.1 *The space $\tilde{\ell}_1$ has the Schur property.*

Proof First, using (2.9) and the fact that $\tilde{\ell}_1$ has an order continuous norm [LM16], we obtain $(\tilde{\ell}_1)^* = (\tilde{\ell}_1)' = \text{ces}_\infty$. Now let $\|x^{(n)}\|_{\tilde{\ell}_1} \leq 1$ with $x^{(n)} \rightarrow 0$ weakly in $\tilde{\ell}_1$ as $n \rightarrow \infty$. By the Banach–Alaoglu theorem the closed unit ball B in ces_∞ is w^* -compact and metrizable, so, in particular, it is a w^* -complete metric space. For any $\varepsilon > 0$ we put $B_m = \bigcap_{n \geq m} \{f \in B : |\langle f, x^{(n)} \rangle| \leq \varepsilon\}$. Then the sets B_m are w^* -closed, $B_1 \subset B_2 \subset \dots$, and $B = \bigcup_{m=1}^\infty B_m$. Thus, by the Baire theorem, there are $N, m_1 \in \mathbb{N}, g = (g_k) \in B_{m_1}$, and $\delta > 0$ such that $U := \{f = (f_k) \in B : |f_k - g_k| < \delta, 1 \leq k \leq N\} \subset B_{m_1}$. Consequently, $U \subset B_m$ for each $m \geq m_1$. Fix $N_1 > N$ such that

$$(3.1) \quad \frac{\sum_{k=1}^N |g_k|}{N_1} < \varepsilon.$$

Clearly, the weak convergence of $\{x^{(n)}\}$ implies the coordinate convergence, so that there is $m_2 \in \mathbb{N}$ such that for $n \geq m_2$

$$(3.2) \quad \|x^{(n)} \chi_{[1, N_1]}\|_{\tilde{\ell}_1} \leq \varepsilon.$$

For every $n \in \mathbb{N}$, there is $f^{(n)} \in B$ such that $\|x^{(n)} \chi_{[N_1, \infty)}\|_{\tilde{\ell}_1} = \langle f^{(n)}, x^{(n)} \chi_{[N_1, \infty)} \rangle$. Without loss of generality, we may assume that $\text{supp } f^{(n)} \subset [N_1, \infty)$. Setting

$$g^{(n)} = g \chi_{[1, N]} + (1 - \varepsilon)f^{(n)},$$

we will show that $g^{(n)} \in U$ for all $n \in \mathbb{N}$. Since $g_k^{(n)} = g_k$ for each $1 \leq k \leq N$, it is enough to check only that $\|g^{(n)}\|_{\text{ces}_\infty} \leq 1$. We have

$$C|g^{(n)}|(j) = \begin{cases} C|g|(j) & \text{if } j < N, \\ \frac{1}{j} \sum_{k=1}^N |g_k| & \text{if } N \leq j < N_1, \\ \frac{1}{j} (\sum_{k=1}^N |g_k| + (1 - \varepsilon) \sum_{k=N_1}^j |f_k^{(n)}|) & \text{if } j \geq N_1. \end{cases}$$

Consequently, from (3.1) it follows that $C|g^{(n)}|(j) \leq 1$ for each $j \in \mathbb{N}$, i.e., $g^{(n)} \in U$ and therefore $g^{(n)} \in B_m$ if $m > m_1$. Finally, applying (3.2), for $n \geq m_0 := \max\{m_1, m_2\}$, we get

$$\begin{aligned} \|x^{(n)}\|_{\tilde{\ell}_1} &\leq \|x^{(n)}\chi_{[1, N_1]}\|_{\tilde{\ell}_1} + \|x^{(n)}\chi_{[N_1, \infty)}\|_{\tilde{\ell}_1} \leq \varepsilon + \left| \sum_{k=N_1}^{\infty} x_k^{(n)} f_k^{(n)} \right| \\ &= \varepsilon + \left| \frac{1}{1-\varepsilon} \left((1-\varepsilon) \sum_{k=N_1}^{\infty} x_k^{(n)} f_k^{(n)} + \sum_{k=1}^N x_k^{(n)} g_k \right) - \frac{1}{1-\varepsilon} \sum_{k=1}^N x_k^{(n)} g_k \right| \\ &\leq \varepsilon + \frac{1}{1-\varepsilon} \left| \sum_{k=1}^{\infty} x_k^{(n)} g_k^{(n)} \right| + \frac{1}{1-\varepsilon} \|x^{(n)}\chi_{[1, N]}\|_{\tilde{\ell}_1} \|g\|_{\text{ces}_{\infty}} \\ &= \varepsilon + \frac{1}{1-\varepsilon} |\langle g^n, x^n \rangle| + \frac{1}{1-\varepsilon} \|x^{(n)}\chi_{[1, N]}\|_{\tilde{\ell}_1} \|g\|_{\text{ces}_{\infty}} \leq \varepsilon + \frac{2\varepsilon}{1-\varepsilon}, \end{aligned}$$

which shows that $\lim_{n \rightarrow \infty} \|x^{(n)}\|_{\tilde{\ell}_1} = 0$, as desired. ■

Corollary 3.2 *The space ces_{∞}^0 has the Dunford–Pettis property.*

Proof From (2.10) we have $(\text{ces}_{\infty}^0)^* = (\text{ces}_{\infty}^0)' = \tilde{\ell}_1$ and by the fact that a Banach space has the Dunford–Pettis property whenever its dual space has it, the result follows from Theorem 3.1. ■

Although the spaces $\tilde{\ell}_1$ and ℓ_1 are not isomorphic, ℓ_1 is isomorphic to a subspace of $\tilde{\ell}_1$ and so $\tilde{\ell}_1$ can be treated as an extension of ℓ_1 preserving the Schur property.

Theorem 3.3 *The basic sequence $\{2^{-i} e_{2^i}\}_{i=0}^{\infty}$ is equivalent in the space $\tilde{\ell}_1$ to the canonical ℓ_1 -basis.*

Proof We prove that for all $n \in \mathbb{N}$ and $c_i \geq 0, i = 0, 1, \dots, n$,

$$\frac{1}{2} \sum_{i=0}^n c_i \leq \left\| \sum_{i=0}^n c_i 2^{-i} e_{2^i} \right\|_{\tilde{\ell}_1} \leq \sum_{i=0}^n c_i.$$

Since $\|e_m\|_{\tilde{\ell}_1} = m$, for every $m \in \mathbb{N}$, the right-hand side inequality is obvious. Thus, it is enough to check only the opposite inequality. If $c_i \geq 0$, then

$$\begin{aligned} \left(\sum_{i=0}^n c_i 2^{-i} e_{2^i} \right)^{\sim} &= \left(c_0, \frac{c_1}{2}, 0, \frac{c_2}{4}, 0, 0, 0, \frac{c_3}{8}, \dots, \frac{c_{n-1}}{2^{n-1}}, 0, \dots, \frac{c_n}{2^n}, 0 \dots \right)^{\sim} \\ &= \left(c_0, \frac{c_1}{2}, \frac{c_2}{4}, \frac{c_2}{4}, \frac{c_3}{8}, \frac{c_3}{8}, \frac{c_3}{8}, \frac{c_3}{8}, \dots, \frac{c_{n-1}}{2^{n-1}}, \frac{c_n}{2^n}, \dots, \frac{c_n}{2^n}, \frac{c_n}{2^n}, 0 \dots \right)^{\sim} \\ &\geq \left(c_0, \frac{c_1}{2}, \frac{c_2}{4}, \frac{c_2}{4}, \frac{c_3}{8}, \frac{c_3}{8}, \frac{c_3}{8}, \frac{c_3}{8}, \dots, \frac{c_{n-1}}{2^{n-1}}, \frac{c_n}{2^n}, \dots, \frac{c_n}{2^n}, \frac{c_n}{2^n}, 0 \dots \right) \end{aligned}$$

so

$$\left\| \sum_{i=0}^n c_i 2^{-i} e_{2^i} \right\|_{\tilde{\ell}_1} = \left\| \left(\sum_{i=0}^n c_i 2^{-i} e_{2^i} \right)^{\sim} \right\|_{\ell_1} \geq c_0 + \frac{1}{2} \sum_{i=1}^n c_i \geq \frac{1}{2} \sum_{i=0}^n c_i. \quad \blacksquare$$

Since the space $\tilde{\ell}_1$ is order continuous, then from (2.9) it follows that $(\tilde{\ell}_1)^* = (\tilde{\ell}_1)' = \text{ces}_{\infty}$. Therefore, taking into account that the space ces_{∞} has the Fatou property, we obtain the following.

Corollary 3.4 *The basic sequence $\{2^i e_{2^i}\}_{i=0}^\infty$ is equivalent in the space ces_∞ to the canonical c_0 -basis and ℓ_∞ is embedded into ces_∞ .*

Corollary 3.5 *The space $\tilde{\ell}_1$ is isomorphic to the space $(\bigoplus_{n=0}^\infty \ell_\infty^{2^n})_{\ell_1}$. Therefore, the space ces_∞ is isomorphic to the space $(\bigoplus_{n=0}^\infty \ell_1^{2^n})_{\ell_\infty}$ and contains a complemented subspace isomorphic to $L_1[0, 1]$.*

Proof Let us define the linear operator T from $\tilde{\ell}_1$ to $(\bigoplus_{n=0}^\infty \ell_\infty^{2^n})_{\ell_1}$ as follows: if $c = (c_k)_{k=1}^\infty \in \tilde{\ell}_1$, then $Tc = (d^{(n)})_{n=0}^\infty$, where

$$d^{(n)} = (d_j^{(n)})_{j=1}^{2^n}, d_j^{(n)} := (j - 1 + 2^n) \cdot c_{j-1+2^n}, j = 1, 2, \dots, 2^n.$$

Assuming that $c_k \geq 0, k = 1, 2, \dots$, by Theorem 3.3, we obtain

$$\begin{aligned} \|c\|_{\tilde{\ell}_1} &= \left\| \sum_{k=0}^\infty c_k e_k \right\|_{\tilde{\ell}_1} = \left\| \sum_{n=0}^\infty \sum_{j=1}^{2^n} \frac{d_j^{(n)}}{j - 1 + 2^n} e_{2^n+j-1} \right\|_{\tilde{\ell}_1} \\ &\leq \left\| \sum_{n=0}^\infty \max_{j=1, \dots, 2^n} d_j^{(n)} \cdot 2^{-n} e_{2^{n+1}} \right\|_{\tilde{\ell}_1} \leq 2 \sum_{n=0}^\infty \max_{j=1, \dots, 2^n} d_j^{(n)} = 2 \|Tc\|_{(\bigoplus_{n=0}^\infty \ell_\infty^{2^n})_{\ell_1}}. \end{aligned}$$

On the other hand, by the definition of the norm in $\tilde{\ell}_1$ and Theorem 3.3, we have

$$\begin{aligned} \|c\|_{\tilde{\ell}_1} &= \left\| \sum_{n=0}^\infty \sum_{j=1}^{2^n} \frac{d_j^{(n)}}{j - 1 + 2^n} e_{2^n+j-1} \right\|_{\tilde{\ell}_1} \geq \frac{1}{2} \left\| \sum_{n=0}^\infty \max_{j=1, \dots, 2^n} d_j^{(n)} \cdot 2^{-n} e_{2^n} \right\|_{\tilde{\ell}_1} \\ &\geq \frac{1}{72} \sum_{n=0}^\infty \max_{j=1, \dots, 2^n} d_j^{(n)} = \frac{1}{72} \|Tc\|_{(\bigoplus_{n=0}^\infty \ell_\infty^{2^n})_{\ell_1}}, \end{aligned}$$

and therefore T is an isomorphism from $\tilde{\ell}_1$ onto $(\bigoplus_{n=0}^\infty \ell_\infty^{2^n})_{\ell_1}$. Since $(\tilde{\ell}_1)' = \text{ces}_\infty$, by duality, we deduce that ces_∞ is isomorphic to the space $(\bigoplus_{n=0}^\infty \ell_1^{2^n})_{\ell_\infty}$.

To get the last result of the corollary, we note that $(\bigoplus_{n=0}^\infty \ell_\infty^{2^n})_{\ell_1}$ is a complemented subspace of $(\bigoplus_{n=0}^\infty \ell_\infty^{2^n})_{\ell_1}$ and hence of $\tilde{\ell}_1$. Thus, applying the Hagler–Stegall theorem [HS73, Theorem 1], we conclude that the dual space, *i.e.*, ces_∞ , contains a complemented subspace isomorphic to $L_1[0, 1]$. ■

Remark 1 Bourgain [B81, p. 19] proved that an arbitrary ℓ_1 -sum of finite-dimensional Banach spaces has the Schur property (see also [CI90, pp. 60–61] for a simpler proof). Hence, from Corollary 3.5 we can infer that $\tilde{\ell}_1$ has the Schur property and thereby we get another proof of Theorem 3.1.

From Corollary 3.5 and the fact that ℓ_∞ is a prime space [AK06, Theorem 5.6.5] and [LT77, Theorem 2.a.7]) we have the following.

Corollary 3.6 *The spaces ces_∞ and ℓ_∞ are not isomorphic.*

The fact that the space ces_∞ contains a complemented subspace isomorphic to $L_1[0, 1]$ will be a crucial tool in proving the existence of an isomorphism between

Ces_∞ spaces of functions and sequences. So, it is worth giving its direct proof without referring to the general theorem of Hagler-Stegall [HS73, Theorem 1], especially, because the following proof is interesting on its own.

Theorem 3.7 *The space ces_∞ contains a complemented subspace isomorphic to*

$$L_1[0, 1].$$

Proof Thanks to Corollary 3.5, it is sufficient to prove that in the space $(\bigoplus_{n=0}^{\infty} \ell_1^{2^n})_{\ell_{\infty}}$ there is a complemented subspace isomorphic to $L_1[0, 1]$. Denote the collection of dyadic intervals of $[0, 1]$ by $B_k^n = \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right)$, where $k = 1, \dots, 2^n$ and $n = 0, 1, 2, \dots$ and define a sequence of operators $H_n: L^1 \rightarrow \ell_1^{2^n}$ by

$$H_n: f \mapsto \left\{ \int_{B_k^n} f(t) dt \right\}_{k=1}^{2^n}.$$

Then $\|H_n\| = 1$ for each n . Moreover, put $H: f \mapsto \bigoplus_{n=0}^{\infty} H_n f$. Then H maps $L_1[0, 1]$ into the space $(\bigoplus_{n=0}^{\infty} \ell_1^{2^n})_{\ell_{\infty}}$ with $\|H\| = 1$. Moreover, let us show that H is an isometry between the spaces $L_1[0, 1]$ and $H(L_1[0, 1])$. In fact, denoting by $\{h_k\}$, the Haar basis, for a given $f \in L_1[0, 1]$ and any $\varepsilon > 0$, one can find a function $g = \sum_{k=1}^N a_k h_k$ such that $\|g - f\| \leq \varepsilon$. Then, for n large enough, there holds $\|H_n g\| = \|g\|$, which, in consequence, gives $\|H_n f\| \geq \|H_n g\| - \varepsilon \geq \|f\| - 2\varepsilon$ and proves our claim.

To see that $H(L_1[0, 1])$ is complemented in $(\bigoplus_{n=0}^{\infty} \ell_1^{2^n})_{\ell_{\infty}}$ a little more work is required. For a given n we set $T_n: (\bigoplus_{n=0}^{\infty} \ell_1^{2^n})_{\ell_{\infty}} \rightarrow L_1[0, 1]$ by

$$T_n: x \mapsto \sum_{k=1}^{2^n} 2^n x_n^k \chi_{B_k^n},$$

where $x_n = (x_n^k)_{k=1}^{2^n}$ and $x = \bigoplus_{n=0}^{\infty} x_n$. Of course, $\|T_n\| = 1$ for each n . Let η be a free ultrafilter. Then for a given $x \in (\bigoplus_{n=0}^{\infty} \ell_1^{2^n})_{\ell_{\infty}}$ we define the functional R_x on $C[0, 1]$ by the formula $R_x(g) = \lim_{\eta} \langle T_n x, g \rangle$ for $g \in C[0, 1]$. Since $\|T_n\| = 1, n \in \mathbb{N}$, we get $\|R_x\| = 1$.

Recalling that the space $C[0, 1]^*$ consists of all regular Borel measures on $[0, 1]$ with finite variation, denote by Q the Lebesgue projection that maps any such measure into its absolutely continuous part. Now one can verify that $P: x \mapsto H(Q(R_x))$ is the required projection from $(\bigoplus_{n=0}^{\infty} \ell_1^{2^n})_{\ell_{\infty}}$ onto $H(L_1[0, 1])$. In fact, since $\{T_n(Hf)\}_{n=0}^{\infty}$ is a uniformly integrable martingale associated with a function $f \in L_1[0, 1]$, there holds $T_n(Hf) \rightarrow f$ in $L_1[0, 1]$ -norm. In consequence,

$$R_{Hf}(g) = \lim_{\eta} \langle T_n Hf, g \rangle = \lim_{n \rightarrow \infty} \langle T_n Hf, g \rangle = \langle f, g \rangle \quad \text{for } g \in C[0, 1].$$

Therefore, $Q(R_{Hf}) = f$, which proves our claim. ■

The second named author learned yet another original proof of the above theorem from Professor Yves Raynaud. Namely, it is possible to construct a projection from $(\bigoplus_{n=0}^{\infty} \ell_1^{2^n})_{\ell_{\infty}}$ onto the space, say M , of all bounded dyadic martingales in L_1 and then, in the second step, from M onto L_1 .

Now we investigate the conditions under which Cesàro and Tandori sequence spaces have or do not have the Dunford–Pettis property.

Observe that under the assumption of nontriviality of indices of a function φ , that is, when $0 < p_\varphi \leq q_\varphi < 1$, in the case when $I = [0, \infty)$, we have $C\Lambda_\varphi = L_1(\varphi(t)/t)$ with equivalent norms [DS07, Theorem 4.4] and [LM15a, Theorem 8]). And hence, by Theorem 2.1 (i), the corresponding Tandori space $\widetilde{M}_\varphi = L_\infty(\varphi)$, so they both have the Dunford–Pettis property. An inspection of the proof of [LM15a, Theorem 8] combined with duality (see Theorem 2.1 (iii)) shows that for the respective sequence spaces the following result holds.

Theorem 3.8 *Let φ be an increasing concave function on $[0, \infty)$.*

- (i) *If $p_\varphi^\infty > 0$, then $C\lambda_\varphi = \ell_1(\varphi(n)/n)$ and therefore has the Dunford–Pettis property.*
- (ii) *If $q_\varphi^\infty < 1$, then $\widetilde{m}_\varphi = \ell_\infty(\varphi(n))$ and therefore has the Dunford–Pettis property.*

In the proof of a similar result related to the spaces $\widetilde{\lambda}_\varphi$ and Cm_φ we will make use of a suitable isomorphic description of these spaces and the well-known results of Bourgain mentioned in the introduction [Bo81].

Theorem 3.9 *For an arbitrary increasing concave function φ on $[0, \infty)$ the spaces $\widetilde{\lambda}_\varphi$ and Cm_φ have the Dunford–Pettis property.*

Proof At first, in the case when $\lim_{t \rightarrow \infty} \varphi(t) < \infty$ we have $\lambda_\varphi = \ell_\infty$, whence $\widetilde{\lambda}_\varphi = \ell_\infty$, and the result follows. So, let $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. Moreover, the function $\varphi(t)$ is strictly increasing and, without loss of generality, we can assume that $\varphi(1) = 1$. Let us define the increasing sequence $\{n_k\}_{k=1}^\infty$, where $n_1 = 1$, as follows

$$n_{k+1} := \sup\{i > n_k : \varphi(i) - \varphi(n_k) \leq 2^k\}, \quad k = 1, 2, \dots$$

Then, since $\varphi(n_{k+1} + 1) - \varphi(n_k) > 2^k$ and, by subadditivity of φ ,

$$\varphi(n_{k+1} + 1) - \varphi(n_{k+1}) \leq \varphi(1) = 1,$$

we have

$$(3.3) \quad 2^{k-1} \leq \varphi(n_{k+1}) - \varphi(n_k) \leq 2^k, \quad k = 1, 2, \dots$$

Therefore, by (3.3) and [GHS96, Proposition 2.1] (see also [GP03, Lemma 3.2]), for every $x = (x_n)_{n=1}^\infty \in \widetilde{\lambda}_\varphi$ we have

$$\begin{aligned} \|x\|_{\widetilde{\lambda}_\varphi} &= \sum_{n=1}^\infty \widetilde{x}_n (\varphi(n+1) - \varphi(n)) = \sum_{k=1}^\infty \sum_{n=n_k}^{n_{k+1}-1} \widetilde{x}_n (\varphi(n+1) - \varphi(n)) \\ &\leq \sum_{k=1}^\infty \widetilde{x}_{n_k} (\varphi(n_{k+1}) - \varphi(n_k)) \leq \sum_{k=1}^\infty 2^k \sup_{j \geq k} \max_{n_j \leq i \leq n_{j+1}} |x_i| \\ &\leq \sum_{k=1}^\infty 2^k \sum_{j=k}^\infty \max_{n_j \leq i \leq n_{j+1}} |x_i| = \sum_{j=1}^\infty \max_{n_j \leq i \leq n_{j+1}} |x_i| \sum_{k=1}^j 2^k \\ &\leq 2 \sum_{j=1}^\infty 2^j \max_{n_j \leq i \leq n_{j+1}} |x_i| = 2 \|x\|_{\oplus}, \end{aligned}$$

where $\|x\|_{\oplus} := \sum_{j=1}^\infty 2^j \max_{n_j \leq i \leq n_{j+1}} |x_i|$.

Conversely, again, by (3.3),

$$\begin{aligned} \|x\|_{\oplus} &= 2 \max_{1 \leq i \leq n_2} |x_i| + \sum_{j=2}^{\infty} 2^j \max_{n_j \leq i \leq n_{j+1}} |x_i| \leq 2\|x\|_{\ell_{\infty}} + 4 \sum_{j=2}^{\infty} \widetilde{x}_{n_j} [\varphi(n_j) - \varphi(n_{j-1})] \\ &\leq 2\|x\|_{\ell_{\infty}} + 4 \sum_{j=2}^{\infty} \sum_{n=n_{j-1}}^{n_j-1} \widetilde{x}_n [\varphi(n+1) - \varphi(n)] \leq \left(\frac{2}{\varphi(2)-1} + 4 \right) \|x\|_{\widetilde{\lambda}_{\varphi}}. \end{aligned}$$

These inequalities show that $\widetilde{\lambda}_{\varphi}$ is isomorphic to the space $(\oplus_{k \in \mathbb{N}} \ell_{\infty}^{m_k})_{\ell_1}$, where $m_k = n_{k+1} - n_k$, $k \in \mathbb{N}$. Hence, applying [Bo81, Corollary 7], we obtain that the space $\widetilde{\lambda}_{\varphi}$ has the Dunford–Pettis property.

Regarding Cm_{φ} we first note that in the case when $\lim_{t \rightarrow \infty} \varphi(t)/t > 0$, the latter space coincides with ℓ_1 and hence has the Dunford–Pettis property. If

$$\lim_{t \rightarrow \infty} \varphi(t)/t = 0,$$

then from Theorem 2.1 (iii) and the first part of the proof it follows that

$$(Cm_{\varphi})' = \widetilde{\lambda}_{\psi} \simeq (\oplus_{k \in \mathbb{N}} \ell_{\infty}^{m_k})_{\ell_1},$$

where $\psi(t) = t/\varphi(t)$. Combining this with the fact that Cm_{φ} has the Fatou property, we infer $Cm_{\varphi} \simeq [(\oplus_{k \in \mathbb{N}} \ell_{\infty}^{m_k})_{\ell_1}]' = (\oplus_{k \in \mathbb{N}} \ell_1^{m_k})_{\ell_{\infty}}$. Hence, from Bourgain’s result [Bo81, Theorem 1] it follows that Cm_{φ} has the Dunford–Pettis property. ■

Corollary 3.10 For any increasing concave function φ on $[0, \infty)$ the space $C(m_{\varphi}^0)$ has the Dunford–Pettis property.

Proof Since the space $C(m_{\varphi}^0)$ is order continuous, by Theorem 2.1 (iii), we have

$$[C(m_{\varphi}^0)]^* = [C(m_{\varphi}^0)]' = \widetilde{\lambda}_{\psi},$$

where $\psi(t) = t/\varphi(t)$. Now the desired result follows from the preceding theorem. ■

Now we show that in the case of reflexive spaces the situation is completely different.

Theorem 3.11 Let X be a reflexive symmetric sequence space.

- (i) If the operator C_d is bounded on X , then CX does not have the Dunford–Pettis property.
- (ii) If the operator C_d is bounded on X' , then \widetilde{X} does not have the Dunford–Pettis property.

Proof (i) Since X is reflexive, it follows that $C_d: CX \rightarrow X$ is a weakly compact operator. Therefore, it is sufficient to show that C_d is not a Dunford–Pettis operator.

Consider the sequence $x_n = \varphi_{X'}(n)e_n$, where $\{e_n\}$ is the canonical basis in X and $\varphi_{X'}$ is the fundamental function of X' . Let us show that $x_n \rightarrow 0$ weakly in CX .

Since the space X has an order continuous norm, CX also has order continuous norm and, by Theorem 2.1 (iii), we obtain $(CX)^* = (CX)' = \widetilde{X}'$. Therefore, by the definition of \widetilde{X}' , it is sufficient to prove that

$$(3.4) \quad \langle y, x_n \rangle = \varphi_{X'}(n) y_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

for each non-increasing positive sequence $y = (y_n) \in X'$. Observe that $X' \subset m_{\varphi_{X'}}$, where $m_{\varphi_{X'}}$ is the Marcinkiewicz space with the fundamental function $\varphi_{X'}$. Moreover, by reflexivity of X , $X' = (X')^0$, and thus

$$X' \subset m_{\varphi_{X'}}^0 = \{(z_n) : \lim_{n \rightarrow \infty} \varphi_{X'}(n) z_n^* = 0\}.$$

Clearly, this embedding implies (3.4) and hence $x_n \rightarrow 0$ weakly in CX .

On the other hand, $C_d e_n = \sum_{k=n}^{\infty} e_k/k$ for every $n \in \mathbb{N}$. Since X is a symmetric space, we have

$$\begin{aligned} \|C_d e_n\|_X &= \left\| \sum_{k=1}^{\infty} \frac{e_k}{n+k-1} \right\|_X \geq \left\| \sum_{k=1}^n \frac{e_k}{n+k-1} \right\|_X \\ &\geq \frac{1}{2n} \left\| \sum_{k=1}^n e_k \right\|_X = \frac{\varphi_X(n)}{2n} = \frac{1}{2\varphi_{X'}(n)}. \end{aligned}$$

Hence, $\|C_d x_n\|_X \geq 1/2$ for all n , and the proof is completed

(ii) Since X' is reflexive and C_d is bounded on X' , from Theorem 3.11 (i) it follows that $C(X')$ does not have the Dunford–Pettis property. By duality, Theorem 2.1 (iii) and the fact that $C(X')$ is order continuous, we obtain $\tilde{X} = (CX')' = (CX')^*$. Thus, \tilde{X} is the dual space to a space without the Dunford–Pettis property. So, \tilde{X} also fails to have it. ■

4 On the Dunford–Pettis Property of Cesàro and Tandori Function Spaces

As mentioned above, under the assumption of nontriviality of indices of a function φ , the spaces $C\Lambda_{\varphi}$ and \widetilde{M}_{φ} are some weighted L_1 -spaces and L_{∞} -spaces, respectively, and so they both have the Dunford–Pettis property (see [DS07, Theorem 4.4], [LM15a, Theorem 8]). Similarly, as in Theorem 3.8, we are able to prove the latter property also for their counterparts, $\widetilde{\Lambda}_{\varphi}$ and CM_{φ} .

Theorem 4.1 *Let φ be an increasing concave function on $[0, \infty)$.*

- (i) *The space $\widetilde{\Lambda}_{\varphi}[0, \infty)$ is isomorphic to the space $(\bigoplus_{n \in \mathbb{N}} L_{\infty}[0, 1])_{\ell_1}$ and has the Dunford–Pettis property.*
- (ii) *If $q_{\varphi} < 1$, then the space $CM_{\varphi}[0, \infty)$ is isomorphic to the space $(\bigoplus_{n \in \mathbb{N}} L_1[0, 1])_{\ell_{\infty}}$ and has the Dunford–Pettis property.*

First, we prove the following auxiliary result.

Proposition 4.2 *Let w be a locally integrable function on $[0, \infty)$, $w(t) > 0$ almost everywhere, such that $\int_0^{\infty} w(t) dt = \infty$. For $1 \leq p < \infty$, we consider the weighted L_p -space with the norm $\|f\|_{L_p(w)} := (\int_0^{\infty} |f(t)|^p w(t) dt)^{1/p}$. Then the space $\widetilde{L}_p(w)$ is isomorphic to the space $(\bigoplus_{n \in \mathbb{Z}} L_{\infty}[0, 1])_{\ell_p}$, and the constant of isomorphism depends only on w .*

Proof Without loss of generality, we can assume that $\int_0^1 w(t) dt = \frac{1}{a-1}$, where $a > 1$. Thanks to the assumptions, there is an increasing sequence $\{t_k\}_{k \in \mathbb{Z}}$ such that $t_0 =$

1, $t_n \rightarrow 0$ as $n \rightarrow -\infty$, $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and

$$\int_{t_n}^{t_{n+1}} w(t) dt = a^n \quad \text{for all } n \in \mathbb{Z}.$$

Then, applying once more [GHS96, Proposition 2.1] (see also [GP03, Lemma 3.2]), for every $f \in \widetilde{L}_p(w)$ we have

$$\begin{aligned} \|f\|_{\widetilde{L}_p(w)}^p &= \int_0^\infty \widetilde{f}(t)^p w(t) dt = \sum_{n \in \mathbb{Z}} \int_{t_n}^{t_{n+1}} \operatorname{ess\,sup}_{s \geq t} |f(s)|^p w(t) dt \\ &\leq \sum_{n \in \mathbb{Z}} \operatorname{ess\,sup}_{s \geq t_n} |f(s)|^p a^n = \sum_{n \in \mathbb{Z}} \sup_{k \geq n} \operatorname{ess\,sup}_{t_k \leq s \leq t_{k+1}} |f(s)|^p a^n \\ &\leq C(a) \sum_{n \in \mathbb{Z}} \|f \chi_{[t_n, t_{n+1}]}\|_{L_\infty}^p a^n = C(a) \|f\|_{\oplus}^p, \end{aligned}$$

where

$$\|f\|_{\oplus} := \left(\sum_{n \in \mathbb{Z}} a^n \|f \chi_{[t_n, t_{n+1}]}\|_{L_\infty}^p \right)^{1/p}$$

and $C(a)$ is some constant depending only on a (and hence on w). On the other hand,

$$\begin{aligned} \|f\|_{\oplus}^p &= a \sum_{n \in \mathbb{Z}} a^{n-1} \|f \chi_{[t_n, t_{n+1}]}\|_{L_\infty}^p = a \sum_{n \in \mathbb{Z}} \int_{t_{n-1}}^{t_n} \operatorname{ess\,sup}_{t_n \leq s \leq t_{n+1}} |f(s)|^p w(t) dt \\ &\leq a \sum_{n \in \mathbb{Z}} \int_{t_{n-1}}^{t_n} \operatorname{ess\,sup}_{s \geq t} |f(s)|^p w(t) dt = a \|f\|_{L_p(w)}^p. \end{aligned}$$

Since the space $L_\infty[a, b]$ is isomorphic to the space $L_\infty[0, 1]$ for every $0 < a < b < \infty$, the result follows. ■

Proof of Theorem 4.1 (i) It is clear that $\widetilde{\Lambda}_\varphi = \widetilde{L}_1(\varphi')$. Therefore, from Proposition 4.2 it follows that $\widetilde{\Lambda}_\varphi \simeq (\oplus_{n \in \mathbb{N}} L_\infty[0, 1])_{\ell_1}$ and, by Bourgain’s result [Bo81, Corollary 7], $\widetilde{\Lambda}_\varphi$ has the Dunford–Pettis property.

(ii) Since $q_\varphi < 1$, then the operator C is bounded in $M_\varphi[0, \infty)$ [KPS82, Theorem 6.6, p. 138] and hence, by Theorem 2.1 (i), the Köthe dual of the space CM_φ coincides with the space $\widetilde{\Lambda}_\psi$, where $\psi(t) = t/\varphi(t)$, $t > 0$. Therefore, applying Proposition 4.2, we are able to get the result, arguing in the same way as in the concluding part of the proof of Theorem 3.8. ■

Theorem 4.3 *The spaces $\operatorname{Ces}_\infty(I)$ and $\widetilde{L}_1(I)$, where $I = [0, \infty)$ or $[0, 1]$, have the Dunford–Pettis property and they are not isomorphic.*

Proof At first, let $I = [0, \infty)$. Since $\widetilde{L}_1 = \widetilde{\Lambda}_{\varphi_1}$, where $\varphi_1(t) = t$, and $\operatorname{Ces}_\infty = CL_\infty = CM_{\varphi_0}$, where $\varphi_0(t) = 1$, by Theorem 4.1, the spaces \widetilde{L}_1 and $\operatorname{Ces}_\infty$ have the Dunford–Pettis property.

Let us show that \widetilde{L}_1 and $\operatorname{Ces}_\infty$ are not isomorphic. By Theorem 4.1, the space \widetilde{L}_1 is isomorphic to the space $(\oplus_{n \in \mathbb{N}} L_\infty[0, 1])_{\ell_1}$ and therefore, according to Pełczyński’s result on isomorphism between L_∞ and ℓ_∞ ([Pe58]; see also [AK06, Theorem 4.3.10]), \widetilde{L}_1 is isomorphic also to $(\oplus_{n \in \mathbb{N}} \ell_\infty)_{\ell_1}$. Since the latter space fails to contain a complemented subspace isomorphic to $L_1[0, 1]$ [CM08, Proposition 3], then \widetilde{L}_1 also does

not. On the other hand, the space Ces_∞ contains such a complemented subspace (see Proposition 2.2), and the result follows.

It is easy to see that the assertion of Proposition 4.2 holds also for weighted L_p -spaces on $[0, 1]$ (with the same proof). Therefore, $\tilde{L}_1[0, 1] \simeq (\bigoplus_{n \in \mathbb{N}} L_\infty[0, 1])_{\ell_1}$ and, since $(Ces_\infty[0, 1])' = \tilde{L}_1[0, 1]$, then $Ces_\infty[0, 1] \simeq (\bigoplus_{n \in \mathbb{N}} L_1[0, 1])_{\ell_\infty}$. Thus, the result can be proved in the same way as in the case of $[0, \infty)$. ■

A similar result can also be deduced from Theorem 4.1 for the separable part of a Marcinkiewicz space M_φ^0 (see also Corollary 3.10). In fact, since the space $C(M_\varphi^0)$ on $[0, \infty)$ is order continuous and the condition $\beta_{M_\varphi^0} = q_\varphi < 1$ implies the boundedness of the operator C on M_φ^0 , by Theorem 2.1 (i), we have $[C(M_\varphi^0)]^* = [C(M_\varphi^0)]' = \widetilde{\Lambda}_\psi$. As a result, applying Theorem 4.1 (i), we get that $C(M_\varphi^0)$ has the Dunford–Pettis property. However, we prefer to give the more direct proof of the latter fact (without exploiting Bourgain’s results [Bo81]) by using the following property of separable Cesáro–Marcinkiewicz spaces.

Proposition 4.4 *Suppose that φ is an increasing concave function on $[0, \infty)$ such that $\lim_{t \rightarrow 0^+} \varphi(t) = 0$ and $q_\varphi < 1$. Let $X = C(M_\varphi^0)$ on $[0, \infty)$ and let $I_n := [a_n, b_n]$ be a sequence of intervals from $[0, \infty)$ such that either*

$$(4.1) \quad b_1 > a_1 > b_2 > a_2 > \dots > 0 \quad \text{and} \quad b_n \rightarrow 0^+ \text{ as } n \rightarrow \infty$$

or

$$(4.2) \quad a_1 < b_1 < a_2 < b_2 < \dots \quad \text{and} \quad a_n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Then there are a subsequence of positive integers $\{n_k\}_{k=1}^\infty$, $n_1 < n_2 < \dots$ and a constant $C > 0$ such that for every sequence $\{x_n\} \subset X$ satisfying the condition $\text{supp } x_n \subset I_n$, $n = 1, 2, \dots$, we have

$$(4.3) \quad \max_{k=1, \dots, m} \|x_{n_k}\|_X \leq \left\| \sum_{k=1}^m x_{n_k} \right\|_X \leq C \max_{k=1, \dots, m} \|x_{n_k}\|_X, \quad m = 1, 2, \dots$$

Proof Since a given sequence $\{x_n\}$ under consideration consists of pairwise disjoint functions, the left inequality in (4.3) holds for an arbitrary subsequence $\{n_k\}_{k=1}^\infty$. So, we need only to prove the reverse inequality. Obviously, we may assume that $x_n \geq 0$ almost everywhere. Since $q_\varphi < 1$, then $\lim_{t \rightarrow 0^+} \frac{t}{\varphi(t)} = 0$. Therefore, in the case (4.1), applying the diagonal procedure, from any given sequence $\{I_n\}$ we can extract a subsequence of intervals (which we will denote still by $I_n = [a_n, b_n]$) such that

$$(4.4) \quad \sum_{k=n+1}^\infty \psi(b_k) \leq \psi(a_n), \quad \text{where } \psi(t) = t/\varphi(t).$$

We claim that the corresponding sequence of functions (still denoting them by $\{x_n\}$ and assuming that $\text{supp } x_n \subset I_n$) satisfies the right-hand inequality in (4.3). For any

$m \in \mathbb{N}$ and $t \in (0, \infty)$, we have

$$\begin{aligned} \frac{1}{t} \int_0^t \left(\sum_{k=1}^m x_k(s) \right) ds &= \frac{1}{t} \sum_{j=2}^m \sum_{i=m-j+2}^m \int_{a_i}^{b_i} x_i(s) ds \chi_{[a_{m-j+1}, b_{m-j+1}]}(t) \\ &\quad + \frac{1}{t} \sum_{j=1}^m \int_{a_{m-j+1}}^t x_{m-j+1}(s) ds \chi_{[a_{m-j+1}, b_{m-j+1}]}(t) \\ &\quad + \frac{1}{t} \sum_{j=1}^m \sum_{i=m-j+1}^m \int_{a_i}^{b_i} x_i(s) ds \chi_{[b_{m-j+1}, a_{m-j}]}(t) \\ &= f_1(t) + f_2(t) + f_3(t), \end{aligned}$$

where $a_0 = \infty$, and

$$\begin{aligned} f_1(t) &:= \frac{1}{t} \sum_{j=2}^m \sum_{i=m-j+2}^m \int_{a_i}^{b_i} x_i(s) ds \chi_{[a_{m-j+1}, a_{m-j}]}(t), \\ f_2(t) &:= \frac{1}{t} \sum_{j=1}^m \int_{a_{m-j+1}}^{b_{m-j+1}} x_{m-j+1}(s) ds \chi_{[b_{m-j+1}, a_{m-j}]}(t), \\ f_3(t) &:= \frac{1}{t} \sum_{j=1}^m \int_{a_{m-j+1}}^t x_{m-j+1}(s) ds \chi_{[a_{m-j+1}, b_{m-j+1}]}(t). \end{aligned}$$

Since $\beta_{M_\varphi^0} = q_\varphi < 1$, the operator C is bounded in M_φ^0 [KPS82, Theorem 6.6], and hence, by Theorem 2.1 (i), $(CX)^* = (CX)' = \widetilde{X}' = \widetilde{\Lambda}_\psi$ with equivalent norms. Thus, by (2.1) and (2.8), for arbitrary $0 < a < b < \infty$ and $x \in X$,

$$(4.5) \quad \int_a^b |x(s)| ds \leq C_1 \|x\|_X \|\chi_{[a,b]}\|_{\widetilde{\Lambda}_\psi} = C_1 \|x\|_X \psi(b).$$

Hence, by (4.4),

$$\begin{aligned} f_1(t) &\leq \frac{C_1}{t} \sum_{j=2}^m \sum_{i=m-j+2}^m \psi(b_i) \|x_i\|_X \chi_{[a_{m-j+1}, a_{m-j}]}(t) \\ &\leq \frac{C_1}{t} \sum_{j=2}^m \psi(a_{m-j+1}) \chi_{[a_{m-j+1}, a_{m-j}]}(t) \cdot \max_{i=1, \dots, m} \|x_i\|_X \\ &\leq \frac{C_1}{\varphi(t)} \sum_{j=2}^m \chi_{[a_{m-j+1}, a_{m-j}]}(t) \cdot \max_{i=1, \dots, m} \|x_i\|_X \leq \frac{C_1}{\varphi(t)} \max_{i=1, \dots, m} \|x_i\|_X, \end{aligned}$$

whence, since $q_\varphi < 1$, by (2.5), it follows that

$$(4.6) \quad \|f_1\|_{M_\varphi} \leq C_1 \frac{\varphi(t)}{t} \int_0^t \frac{1}{\varphi(s)} ds \max_{i=1, \dots, m} \|x_i\|_X \leq C_1 C_2 \max_{i=1, \dots, m} \|x_i\|_X.$$

Similarly, since ψ increases, we have

$$\begin{aligned} f_2(t) &\leq \frac{C_1}{t} \sum_{j=1}^m \psi(b_{m-j+1}) \|x_{m-j+1}\|_X \cdot \chi_{[b_{m-j+1}, a_{m-j}]}(t) \\ &\leq \frac{C_1}{\varphi(t)} \sum_{j=1}^m \chi_{[b_{m-j+1}, a_{m-j}]}(t) \max_{i=1, \dots, m} \|x_i\|_X \leq \frac{C_1}{\varphi(t)} \max_{i=1, \dots, m} \|x_i\|_X, \end{aligned}$$

and again

$$(4.7) \quad \|f_2\|_{M_\varphi} \leq C_1 C_2 \max_{i=1, \dots, m} \|x_i\|_X.$$

Finally, using (4.5) once more, we have

$$\begin{aligned} f_3(t) &\leq \frac{C_1}{t} \sum_{j=1}^m \psi(t) \|x_{m-j+1}\|_X \cdot \chi_{[a_{m-j+1}, b_{m-j+1}]}(t) \\ &\leq \frac{C_1}{\varphi(t)} \sum_{j=1}^m \chi_{[a_{m-j+1}, b_{m-j+1}]}(t) \max_{i=1, \dots, m} \|x_i\|_X \leq \frac{C_1}{\varphi(t)} \max_{i=1, \dots, m} \|x_i\|_X, \end{aligned}$$

whence again

$$(4.8) \quad \|f_3\|_{M_\varphi} \leq C_1 C_2 \max_{i=1, \dots, m} \|x_i\|_X.$$

Thus, from (4.5)–(4.8) it follows that

$$\left\| \sum_{k=1}^m x_k \right\|_X \leq \|f_1\|_{M_\varphi} + \|f_2\|_{M_\varphi} + \|f_3\|_{M_\varphi} \leq C \max_{i=1, \dots, m} \|x_i\|_X,$$

where the constant $C := 3C_1 C_2$ depends only on the function φ .

Regarding the case (4.2), we note that condition $q_\varphi < 1$ implies $\lim_{t \rightarrow \infty} \frac{t}{\varphi(t)} = \infty$. Hence, from any given sequence of intervals we can select a subsequence of intervals (denoted still by $I_n = [a_n, b_n]$) such that

$$(4.9) \quad \sum_{i=1}^{k-1} \psi(b_i) \leq \psi(a_k), \quad k = 2, 3, \dots$$

For arbitrary $m \in \mathbb{N}$ and $t > 0$, we have

$$\begin{aligned} \frac{1}{t} \int_0^t \left(\sum_{k=1}^m x_k(s) \right) ds &= \frac{1}{t} \sum_{k=1}^m \left(\sum_{i=1}^{k-1} \int_{a_i}^{b_i} x_i(s) ds + \int_{a_k}^t x_k(s) ds \right) \chi_{[a_k, b_k]}(t) \\ &\quad + \frac{1}{t} \sum_{k=1}^m \sum_{i=1}^k \int_{a_i}^{b_i} x_i(s) ds \cdot \chi_{[b_k, a_{k+1}]}(t) \\ &= g_1(t) + g_2(t) + g_3(t), \end{aligned}$$

where $a_{m+1} = \infty$, and

$$\begin{aligned} g_1(t) &:= \frac{1}{t} \sum_{k=2}^m \sum_{i=1}^{k-1} \int_{a_i}^{b_i} x_i(s) ds \cdot \chi_{[a_k, a_{k+1}]}(t), \\ g_2(t) &:= \frac{1}{t} \sum_{k=1}^m \int_{a_k}^{b_k} x_i(s) ds \cdot \chi_{[b_k, a_{k+1}]}(t), \\ g_3(t) &:= \sum_{k=1}^m \int_{a_k}^t x_k(s) ds \cdot \chi_{[a_k, b_k]}(t). \end{aligned}$$

First, applying (4.5) and (4.9) we obtain

$$\begin{aligned} g_1(t) &\leq \frac{C_1}{t} \sum_{k=2}^m \sum_{i=1}^{k-1} \psi(b_i) \|x_i\|_X \cdot \chi_{[a_k, a_{k+1}]}(t) \\ &\leq \frac{C_1}{t} \sum_{k=2}^m \psi(a_k) \cdot \chi_{[a_k, a_{k+1}]}(t) \cdot \max_{i=1, \dots, m} \|x_i\|_X \\ &\leq \frac{C_1}{\varphi(t)} \sum_{k=2}^m \chi_{[a_k, a_{k+1}]}(t) \cdot \max_{i=1, \dots, m} \|x_i\|_X \leq \frac{C_1}{\varphi(t)} \max_{i=1, \dots, m} \|x_i\|_X. \end{aligned}$$

Next, similarly,

$$\begin{aligned} g_2(t) &\leq \frac{C_1}{t} \sum_{k=1}^m \psi(b_k) \|x_k\|_X \cdot \chi_{[b_k, a_{k+1}]}(t) \leq \frac{C_1}{\varphi(t)} \max_{i=1, \dots, m} \|x_i\|_X, \\ g_3(t) &\leq \frac{C_1}{t} \sum_{k=1}^m \psi(t) \|x_k\|_X \cdot \chi_{[a_k, b_k]}(t) \leq \frac{C_1}{\varphi(t)} \max_{i=1, \dots, m} \|x_i\|_X. \end{aligned}$$

As a result, using (2.5), we have

$$\left\| \sum_{k=1}^m x_k \right\|_X \leq \|g_1\|_{M_\varphi} + \|g_2\|_{M_\varphi} + \|g_3\|_{M_\varphi} \leq C \max_{i=1, \dots, m} \|x_i\|_X,$$

where the constant $C := 3C_1C_2$ depends only on the function φ . ■

Corollary 4.5 *Let φ satisfy all the conditions of Proposition 4.4 and let $X = C(M_\varphi^0)$ on $[0, \infty)$. Suppose that $I_n := [a_n, b_n], n = 1, 2, \dots$, be a sequence of intervals from $[0, \infty)$ such that either $b_1 > a_1 > b_2 > a_2 > \dots > 0$ and $b_n \rightarrow 0^+$ as $n \rightarrow \infty$ or $a_1 < b_1 < a_2 < b_2 < \dots$ and $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Then every semi-normalized sequence $\{f_n\} \subset X$ such that $\text{supp } f_n \subset I_n, n = 1, 2, \dots$ contains a subsequence $\{f_{n_k}\}$ that is equivalent in X to the canonical basis in c_0 .*

Proof At first, applying Proposition 4.4, we find a subsequence of positive integers $\{n_k\}_{k=1}^\infty, n_1 < n_2 < \dots$ and a constant $C > 0$ such that for every sequence $\{x_n\} \subset X$ with $\text{supp } x_n \subset I_n, n = 1, 2, \dots$, we have

$$\max_{k=1, \dots, m} \|x_{n_k}\|_X \leq \left\| \sum_{k=1}^m x_{n_k} \right\|_X \leq C \max_{k=1, \dots, m} \|x_{n_k}\|_X, \quad m = 1, 2, \dots$$

In particular, setting $x_n = c_k f_n$ if $n_k \leq n < n_{k+1}, k = 1, 2, \dots$, where (c_k) is an arbitrary sequence from c_0 , and assuming that $D^{-1} \leq \|f_n\|_X \leq D, n = 1, 2, \dots$ for all $m \in \mathbb{N}$, we obtain

$$D^{-1} \max_{k=1, \dots, m} |c_k| \leq \left\| \sum_{k=1}^m c_k f_{n_k} \right\|_X \leq CD \max_{k=1, \dots, m} |c_k|.$$

Since $(c_k) \in c_0$, then the series $\sum_{k=1}^\infty c_k f_{n_k}$ converges in X and we have

$$D^{-1} \|(c_k)\|_{c_0} \leq \left\| \sum_{k=1}^\infty c_k f_{n_k} \right\|_X \leq CD \|(c_k)\|_{c_0}. \quad \blacksquare$$

Theorem 4.6 *Let φ be an increasing concave function on $[0, \infty)$ such that*

$$\lim_{t \rightarrow 0^+} \varphi(t) = 0 \quad \text{and} \quad q_\varphi < 1.$$

Then the space $X = C(M_\varphi^0)$ on $[0, \infty)$ has the Dunford–Pettis property.

Proof Assuming to the contrary that X does not have the Dunford–Pettis property, we can find sequences $\{u_n\} \subset X$ such that $\|u_n\|_X = 1, u_n \rightarrow 0$ weakly in X , and $\{v_n\} \subset X^* = X' = \widetilde{\Lambda}_\psi$ such that $\|v_n\|_{X'} = 1, v_n \rightarrow 0$ weakly in X' satisfying the condition

$$(4.10) \quad \langle u_n, v_n \rangle := \int_0^\infty u_n(t)v_n(t) dt \geq \delta,$$

for some $\delta > 0$ and all $n \in \mathbb{N}$. It is easy to see that $u_n\chi_{[a,b]} \rightarrow 0$ weakly in X for every $0 < a < b < \infty$. In fact, if $v \in X'$, then $\langle u_n\chi_{[a,b]}, v \rangle = \langle u_n, v\chi_{[a,b]} \rangle \rightarrow 0$ as $n \rightarrow \infty$, because of $v\chi_{[a,b]} \in X'$. Moreover (see Proposition 2.2),

$$X|_{[a,b]} := \{u \in X : \text{supp } u \subset [a, b]\} = L_1[a, b]$$

with equivalence of norms, and therefore $u_n\chi_{[a,b]} \rightarrow 0$ weakly in $L_1[a, b]$. Setting $\alpha_n(u) := \int_a^b u(t)v_n(t) dt, n \in \mathbb{N}$, we see that $\alpha_n \in (L_1[a, b])^* = L_\infty[a, b] = X'/M$, where $M = \{v \in X' : \langle u, v \rangle = 0 \text{ for all } u \in L_1[a, b]\}$. Then

$$(L_\infty[a, b])^* = (X'/M)^* = \{F \in (X')^* : F(v) = 0 \text{ for all } v \in M\},$$

and therefore $(L_\infty[a, b])^* \subset (X')^*$. Hence, from the fact that $v_n \rightarrow 0$ weakly in X' , it follows that $\alpha_n \rightarrow 0$ weakly in $L_\infty[a, b]$. Since $L_1[a, b]$ has the Dunford–Pettis property, as a result we have

$$\alpha_n(u_n \cdot \chi_{[a,b]}) = \int_a^b u_n(t)v_n(t) dt \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

for every $0 < a < b < \infty$. Thus, taking into account (4.10), we can select subsequences of $\{u_n\}$ and $\{v_n\}$ (we will denote them still by $\{u_n\}$ and $\{v_n\}$) such that at least one of the following conditions holds

- there exists a sequence $\{b_n\}_{n=1}^\infty$ with $b_1 > b_2 > \dots, \lim_{n \rightarrow \infty} b_n = 0$ and

$$\int_0^{b_n} u_n(t)v_n(t) dt \geq \frac{3\delta}{4}, \quad n \in \mathbb{N};$$

- there exists a sequence $\{a_n\}_{n=1}^\infty$ with $a_1 < a_2 < \dots, \lim_{n \rightarrow \infty} a_n = \infty$ and

$$\int_{a_n}^\infty u_n(t)v_n(t) dt \geq \frac{3\delta}{4}, \quad n \in \mathbb{N}.$$

Since $\int_0^\infty |u_n(t)v_n(t)| dt < \infty$ for every $n \in \mathbb{N}$, passing to further subsequences, we can find a sequence of intervals $I_n = [a_n, b_n], n = 1, 2, \dots$ such that either $b_1 > a_1 > b_2 > a_2 > \dots, \lim_{n \rightarrow \infty} b_n = 0$, or $a_1 < b_1 < a_2 < b_2 < \dots, \lim_{n \rightarrow \infty} a_n = \infty$, for which

$$(4.11) \quad \int_{I_n} u_n(t)v_n(t) dt \geq \frac{\delta}{2}, \quad n \in \mathbb{N}.$$

Now we set $f_n := u_n \cdot \chi_{I_n}, n = 1, 2, \dots$. From (4.11) it follows that $\{f_n\}$ is a seminormalized sequence in X . So applying Corollary 4.5, we can extract a subsequence (denoted still by $\{f_n\}$), which is equivalent in X to the canonical basis in c_0 . Therefore, $f_n \rightarrow 0$ weakly in the closed linear span $[f_n]$ (and in X). Clearly, $\theta_n(f) := \int_0^\infty f(t)v_n(t) dt$ is a bounded linear functional on $[f_n]$. As above, $[f_n]** \subset (X')^*$.

Therefore, since $v_n \rightarrow 0$ weakly in X' , we have $\theta_n \rightarrow 0$ weakly in $[f_n]^*$. Noting that the subspace $[f_n]$ is isomorphic to c_0 , which has the Dunford–Pettis property, we obtain

$$\int_{I_n} u_n(t)v_n(t) dt = \theta_n(f_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which contradicts (4.11). Thus, the proof is complete. ■

As we know, the condition $0 < p_\varphi \leq q_\varphi < 1$ guarantees that $C\Lambda_\varphi$ on $[0, \infty)$ is a weighted L_1 -space up to equivalence of norms (see [DS07, Theorem 4.4], [LM15a]). It turns out that a similar result holds also for the Cesàro–Lorentz spaces on $[0, 1]$.

Theorem 4.7 *Let φ be an increasing concave function on $[0, 1]$ such that $0 < p_\varphi^0 \leq q_\varphi^0 < 1$. Then $C\Lambda_\varphi[0, 1] = L_1(w)$, with $w(t) = \int_0^{1-t} \frac{\varphi'(s)}{t+s} ds$.*

Proof By duality and Fubini’s theorem, we have

$$\begin{aligned} \|f\|_{C\Lambda_\varphi} &= \sup_{\|g\|_{\Lambda'_\varphi} \leq 1} \int_0^1 C|f|(x)|g(x)| dx = \sup_{\|g\|_{\Lambda'_\varphi} \leq 1} \int_0^1 |g(x)| \left(\frac{1}{x} \int_0^x |f(t)| dt \right) dx \\ &= \sup_{\|g\|_{\Lambda'_\varphi} \leq 1} \int_0^1 |f(t)| \left(\int_t^1 \frac{|g(x)|}{x} dx \right) dt \leq \int_0^1 |f(t)| \|h_t\|_{\Lambda_\varphi} dt, \end{aligned}$$

where $h_t(x) = \frac{1}{x} \chi_{[t,1]}(x)$. Then

$$\|h_t\|_{\Lambda_\varphi} = \int_0^1 (h_t)^*(s)\varphi'(s) ds = \int_0^{1-t} \frac{\varphi'(s)}{s+t} ds = w(t)$$

and consequently the above inequality means that $L_1(w) \xrightarrow{1} C\Lambda_\varphi$. In view of the conditions imposed on the indices p_φ^0 and q_φ^0 , the operators C and C^* are bounded in Λ_φ (see [KPS82, Chapter II, §8.6]). Therefore, the reverse inclusion is equivalent by duality (see Theorem 2.1 (ii)) to the following one: $L_\infty(1/w) \xrightarrow{1} \widetilde{M_\psi}(v)$, where $\psi(t) = \frac{t}{\varphi(t)}$ and $v(t) = \frac{1}{1-t}$. Thus, it is enough to check that $w \in \widetilde{M_\psi}(v)$, i.e.,

$$\|w\|_{\widetilde{M_\psi}(v)} = \sup_{0 < t \leq 1} \frac{1}{\varphi(t)} \int_0^t (v\widetilde{w})^*(x) dx < \infty.$$

First notice that w is decreasing, so we have $\widetilde{w} = w$. We divide the function $v \cdot w$ into two parts, namely,

$$v(t)w(t) = \frac{w(t)}{1-t} \chi_{[0,1/2]}(t) + \frac{w(t)}{1-t} \chi_{[1/2,1]}(t) = w_0(t) + w_1(t).$$

Thus, we need only to check that w_0 and w_1 belong to the space M_ψ .

By Fubini’s theorem, we have

$$\begin{aligned} \int_0^x w^*(t) dt &= \int_0^x \left(\int_0^{1-t} \frac{\varphi'(s)}{t+s} dt \right) ds \\ &= \int_0^{1-x} \left(\int_0^x \frac{\varphi'(s)}{t+s} dt \right) ds + \int_{1-x}^1 \left(\int_0^{1-s} \frac{\varphi'(s)}{t+s} dt \right) ds \\ &= \int_0^{1-x} \varphi'(s) \ln \frac{x+s}{s} ds + \int_{1-x}^1 \varphi'(s) \ln \frac{1}{s} ds. \end{aligned}$$

Then, since $0 < x < 1/2$, the second summand can be estimated thanks to monotonicity and subadditivity of the concave function φ as follows:

$$\int_{1-x}^1 \varphi'(s) \ln \frac{1}{s} ds \leq \ln 2 \int_{1-x}^1 \varphi'(s) ds = \ln 2[\varphi(1) - \varphi(1-x)] \leq \ln 2\varphi(x).$$

While for the first one, integrating by parts we get

$$\begin{aligned} \int_0^{1-x} \varphi'(s) \ln \frac{x+s}{s} ds &= \varphi(1-x) \ln \frac{1}{1-x} - \lim_{s \rightarrow 0^+} \varphi(s) \ln \frac{x+s}{s} \\ &\quad + x \int_0^{1-x} \frac{\varphi(s)}{(s+x)s} ds \\ &\leq \varphi(1) \frac{x}{1-x} + x \int_0^{1-x} \frac{\varphi(s)}{(s+x)s} ds. \end{aligned}$$

Since $0 < x < 1/2$, then by concavity of φ , we get $\varphi(1) \frac{x}{1-x} \leq 2\varphi(1)x \leq 2\varphi(x)$. Moreover, for some $0 < a < 1$, $A \geq 1$, and all $0 < x < 1$, $t > 0$ we have $\varphi(tx) \leq At^a\varphi(x)$ and consequently, putting $s = tx$, we obtain

$$\begin{aligned} \int_0^{1-x} \frac{\varphi(s)}{(s+x)s} ds &= \frac{1}{x} \int_0^{\frac{1-x}{x}} \frac{\varphi(tx)}{(1+t)t} dt \leq \frac{A}{x} \int_0^{\frac{1-x}{x}} \frac{t^a \varphi(x)}{(1+t)t} dt \\ &\leq A \frac{\varphi(x)}{x} \int_0^\infty \frac{t^{a-1}}{1+t} dt = B \frac{\varphi(x)}{x}. \end{aligned}$$

Thus, for $0 < x < 1/2$,

$$\begin{aligned} \int_0^x (w_0)^*(t) dt &\leq 2 \int_0^x (w\chi_{(0,1/2]})^*(t) dt \leq 2 \int_0^x w^*(t) dt \\ &\leq 2(2 + B + \ln 2) \varphi(x), \end{aligned}$$

whence $w_0 \in M_\psi$.

Let us consider now w_1 . For $1/2 < t \leq 1$ we have

$$w_1(t) = \frac{1}{1-t} \int_0^{1-t} \frac{\varphi'(s)}{t+s} ds \leq \frac{2}{1-t} \int_0^{1-t} \varphi'(s) ds = 2 \frac{\varphi(1-t)}{1-t}.$$

Since the function $\frac{\varphi(1-t)}{1-t}$ is increasing, we conclude that $w_1^*(t) \leq 2 \frac{\varphi(t)}{t}$, $0 < t \leq 1$. In consequence, from (2.5) and the condition $q_\psi^0 = 1 - p_\varphi^0 < 1$, it follows that

$$\int_0^x w_1^*(t) dt \leq 2 \int_0^x \frac{\varphi(t)}{t} dt = 2 \int_0^x \frac{1}{\psi(t)} dt \leq 2C \frac{x}{\psi(x)} = 2C\varphi(x),$$

which finishes the proof. ■

Of course, from Theorem 4.7 it follows that the space $CA_\varphi[0, 1]$ has the Dunford–Pettis property whenever $0 < p_\varphi^0 \leq q_\varphi^0 < 1$. Let us prove an analogous result for separable Cesàro–Marcinkiewicz spaces.

Theorem 4.8 *Let φ be an increasing concave function on $[0, 1]$ such that*

$$\lim_{t \rightarrow 0^+} \varphi(t) = 0 \quad \text{and} \quad q_\varphi^0 < 1.$$

Then the space $C(M_\varphi^0)[0, 1]$ has the Dunford–Pettis property.

Proof For every $k = 2, 3, \dots$, we set

$$X_k := CM_\varphi^0|_{[0, 1-1/k]} = \left\{ f \in CM_\varphi^0 : \text{supp } f \subset \left[0, 1 - \frac{1}{k} \right] \right\}.$$

Since CM_φ^0 is an order continuous space, the union $\bigcup_{k=2}^\infty X_k$ is dense in it. Moreover, from the definition of Cesàro spaces, it follows that, for every $k = 2, 3, \dots$, X_k can be regarded as a complemented subspace of the space $CM_{\varphi_1}^0[0, \infty)$, where φ_1 is a concave extension of the function φ to the semi-axis $[0, \infty)$ such that $q_{\varphi_1} < 1$. (Notice that $CX[0, 1]$ is not a restriction of $CX[0, \infty)$ to the interval $[0, 1]$. More precisely, similarly to [AM09, Remark 5], one can check that $CX[0, \infty)|_{[0, 1]} = CX[0, 1] \cap L_1[0, 1]$). Therefore, an inspection of the proof of Theorem 4.1 shows that

$$X_k \simeq \left(\bigoplus_{n \in \mathbb{N}} L_1[0, 1] \right)_{\ell_\infty},$$

whence the space $(\bigoplus_{k=2}^\infty X_k)_{\ell_\infty}$ is isomorphic to the latter ℓ_∞ -sum as well. Thus, $(\bigoplus_{k=2}^\infty X_k)_{\ell_\infty}$ has the Dunford–Pettis property. Finally, applying [Bo81, Proposition 2], we conclude that CM_φ^0 also possesses the latter property, and the proof is complete. ■

Remark 2 The assertion of Theorem 4.8 cannot be deduced from Theorem 4.1 (i), using Bourgain’s results as above, because of the difference in the duality results for Cesàro spaces for the cases of $[0, 1]$ and $[0, \infty)$ (see Theorem 2.1). We would like to mention here also that we could not identify conditions under which the space $CM_\varphi[0, 1]$ has the Dunford–Pettis property.

Now we present some negative results related to the Dunford–Pettis property of Cesàro and Tandori function spaces.

Theorem 4.9 *Let X be a reflexive symmetric function space on I such that the operator C is bounded on X .*

- (i) *If $I = [0, \infty)$, then the spaces CX and \widetilde{X}' do not have the Dunford–Pettis property.*
- (ii) *If $I = [0, 1]$, X has the Fatou property, and the operator C^* is bounded on X , then the spaces CX and \widetilde{X}' do not have the Dunford–Pettis property.*

Proof (i) The proof is rather similar to the proof in the sequence case (Theorem 3.7). Again it is sufficient to prove that the operator $C: CX \rightarrow X$ is not a Dunford–Pettis operator. Let us show that $x_n = \frac{1}{\varphi_X(1/n)} \chi_{[0, 1/n]}$, $n = 1, 2, \dots$, is a weakly null sequence in CX . Since X is order continuous, it follows that CX is also order continuous and by Theorem 2.1 (i) we obtain $(CX)^* = (CX)' = \widetilde{X}'$. Thus, we need only to check that

$$(4.12) \quad \langle y, x_n \rangle = \frac{1}{\varphi_X(1/n)} \int_0^{1/n} y(t) dt \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

for every decreasing positive function $y \in X'$. Again $X' \subset M_{\varphi_{X'}}$, where $M_{\varphi_{X'}}$ is the Marcinkiewicz function space with the fundamental function $\varphi_{X'}$. By reflexivity of X we have $X' = (X')^0$, and thus

$$X' \subset M_{\varphi_{X'}}^0 \subset \left\{ z = z(t) : \lim_{t \rightarrow 0} \frac{\varphi_{X'}(t)}{t} \int_0^t z^*(s) ds = 0 \right\}.$$

But $\varphi_X(t) = t/\varphi_{X'}(t)$ and (4.12) follows from the above embedding. On the other hand, $Cx_n \geq x_n$ and so $\|Cx_n\|_X \geq \|x_n\|_X = 1$. This means that CX does not have the Dunford–Pettis property. Moreover, since $\widetilde{X}' = (CX)' = (CX)^*$, \widetilde{X}' fails to have the latter property as well.

(ii) The only difference of this case from the case of $[0, \infty)$ is the fact that now $(CX)^* = (CX)' = X'(\overline{1/(1-t)})$. However, “near zero”, the latter space coincides with the space \widetilde{X}' without a weight. Thus, we can repeat the same proof as in (i). ■

As we know [DS07, Theorem 4.4] and Theorem 4.7, the Cesàro–Lorentz spaces may coincide with weighted L_1 -spaces and therefore may be isomorphic to the symmetric space L_1 . At the same time, this is not the case for Cesàro spaces CX when X is reflexive.

Corollary 4.10 *If X is a reflexive symmetric function space on $[0, 1]$ such that the operator C is bounded on X , then CX is not isomorphic to any symmetric space on $[0, 1]$.*

Proof Suppose CX is isomorphic to some symmetric space Y on $[0, 1]$. Hence, by Proposition 2.2, Y contains a complemented copy of $L_1[0, 1]$. On the other hand, as Kalton proved [Ka93, Theorem 7.4], every separable symmetric space on $[0, 1]$ that contains a complemented subspace isomorphic to $L_1[0, 1]$ is isomorphic to $L_1[0, 1]$ itself. Therefore, we conclude that Y is isomorphic to $L_1[0, 1]$. On the other hand, $CX \simeq Y$ cannot be isomorphic to $L_1[0, 1]$, because by Theorem 4.9 that space fails to have the Dunford–Pettis property. ■

5 Isomorphism Between Ces_∞ and ces_∞

In [AM09, Theorem 9] (see also [AM14, Theorem 7.2]) it was proved that the spaces $\text{Ces}_\infty[0, 1]$ and $\text{Ces}_\infty[0, \infty)$ are isomorphic and there the question was raised if the spaces Ces_∞ and ces_∞ are isomorphic [AM09, Problem 1], [AM14, Problem 4]. The following theorem answers this question in the affirmative.

Theorem 5.1 *The spaces Ces_∞ and ces_∞ are isomorphic.*

Proof At first, we recall that, by Corollary 3.5, $\text{ces}_\infty \simeq (\bigoplus_{n=0}^\infty \ell_1^{2^n})_{\ell_\infty}$ and, by Theorem 4.1, $\text{Ces}_\infty \simeq (\bigoplus_{n \in \mathbb{N}} L_1[0, 1])_{\ell_\infty}$. Therefore, $\text{ces}_\infty \simeq \text{ces}_\infty \oplus \text{ces}_\infty$ and $\text{Ces}_\infty \simeq \text{Ces}_\infty \oplus \text{Ces}_\infty$, which shows that we can apply Pełczyński decomposition argument, see [Pe60, Proposition 4] or [AK06, Theorem 2.2.3]. In other words, the proof will be completed once we check that ces_∞ is isomorphic to a complemented subspace of Ces_∞ and vice versa.

Clearly, for every $n = 0, 1, 2, \dots$ the space $\ell_1^{2^n}$ can be complementedly embedded into the space $L_1[0, 1]$. Therefore, the fact that ces_∞ is isomorphic to a complemented subspace of Ces_∞ follows at once from the above isomorphic representations of these spaces.

Just a little more effort is required for the proof of the reverse statement. Let us represent the set $\mathbb{N} \cup \{0\}$ as a union of infinite increasing pairwise disjoint sequences

$(a_n^k)_{n=0}^\infty, k = 1, 2, \dots$. Then we can write

$$(5.1) \quad \text{ces}_\infty \simeq \left(\bigoplus_{k=1}^\infty \left(\bigoplus_{n=0}^\infty \ell_1^{2^{a_n^k}} \right)_{\ell_\infty} \right)_{l_\infty}.$$

Since $n \leq a_n^k$, where $k = 1, 2, \dots$ and $n = 0, 1, 2, \dots$ are arbitrary, the space $\ell_1^{2^n}$ can be considered as a complemented subspace of the space $\ell_1^{2^{a_n^k}}$. Let P_n^k be a respective projection and $P_k = \bigoplus_{n=0}^\infty P_n^k$. Noting that $P_k \left(\left(\bigoplus_{n=0}^\infty \ell_1^{2^{a_n^k}} \right)_{\ell_\infty} \right) = \left(\bigoplus_{n=0}^\infty \ell_1^{2^n} \right)_{\ell_\infty}$, we see that by Theorem 3.7, $L_1[0, 1]$ is complemented in $P_k \left(\left(\bigoplus_{n=0}^\infty \ell_1^{2^{a_n^k}} \right)_{\ell_\infty} \right)$ and hence in the space $\left(\bigoplus_{n=0}^\infty \ell_1^{2^{a_n^k}} \right)_{\ell_\infty}$. Finally, from (5.1) it follows that $\text{Ces}_\infty \simeq \left(\bigoplus_{n \in \mathbb{N}} L_1[0, 1] \right)_{\ell_\infty}$ is isomorphic to a complemented subspace of ces_∞ and the proof is complete. ■

Corollary 5.2 *The space $\text{Ces}_\infty(I)$, where $I = [0, \infty)$ or $[0, 1]$, is isomorphic to a dual space.*

Proof By (2.9) and Theorem 5.1 we have $(\tilde{\ell}_1)^* = (\tilde{\ell}_1)' = \text{ces}_\infty \simeq \text{Ces}_\infty$. ■

In contrast to the latter result, order continuous Cesàro spaces fail to be isomorphic to the dual ones.

Proposition 5.3 *If X is a symmetric function space on $I = [0, 1]$ or $I = [0, \infty)$ such that X is order continuous and C is bounded on X , then CX is not isomorphic to a dual space.*

Proof Suppose that CX is isomorphic to a dual space. By Proposition 2.2, CX contains a complemented subspace isomorphic to $L_1[0, 1]$. Therefore, applying the Hagler–Stegall theorem [HS73, Theorem 1], we see that CX also contains a subspace isomorphic to $C[0, 1]^*$. However, it is impossible, since, by [LM15b, Lemma 1], CX is separable. ■

Let us comment on the latter results. Suppose that X is an ideal Banach function space with the Fatou property such that the separable part of its Köthe dual $(X')^0$ has the same support as X itself. Then an easy argument shows that

$$[(X')^0]^* = [(X')^0]' = X'' = X,$$

i.e., X is a dual space. So the space $(X')^0$ is a natural candidate for being the predual of a dual ideal Banach space X . Moreover, as we have seen, separable CX spaces are not isomorphic to dual ones similarly to L_1 and both of them have Köthe dual without nontrivial absolutely continuous elements. Hence, the following conjecture may arise: an ideal Banach space whose Köthe dual has trivial subspace of order continuous elements is not isomorphic to a dual space. This statement, however, is false; by Corollary 5.2, the Cesàro space Ces_∞ , satisfying $[(\text{Ces}_\infty)']^0 = (\tilde{L}_1)^0 = \{0\}$, is a dual space. In contrast to that, the symmetric space $X = L_1 + L_\infty$ on $[0, \infty)$, for which we also have $(X')^0 = (L_1 \cap L_\infty)^0 = \{0\}$, is not isomorphic to a dual space [AM17, Theorem 4].

It is interesting to observe that the above phenomenon has its counterpart in the general theory of Banach lattices. Let E be a separable Banach lattice satisfying the

Radon–Nikodym property (RNP). Then the set F of all $x^* \in E^*$, such that the interval $[0, |x^*|]$ is weakly compact, is a Banach lattice. Lotz showed (in an unpublished preprint [Lo75]) that if F is big enough, *i.e.*, the topology $\sigma(E, F)$ is Hausdorff, then $E = F^*$. Hence, F is a natural candidate as the predual of E . Talagrand, however, motivated by above results, constructed a separable Banach lattice being a dual space (and hence with RNP) such that for each $x^* \in E^*$, $x^* \geq 0$, the interval $[0, x^*]$ is not weakly compact [Ta81].

To see that the space Ces_∞ may be regarded as a natural “function” counterpart of Talagrand’s example (which seems to be rather artificial) we present the following simple assertion.

Proposition 5.4 *Let X be an ideal Banach space on $[0, 1]$, $x_0 \in X$, $x_0 \geq 0$. Then the interval $[0, x_0]$ is weakly compact in X if and only if $x_0 \in X^0$.*

Proof First let $[0, x_0]$ be weakly compact in X . On the contrary, assume that $x_0 \notin X^0$. Then there is a sequence of sets $\{A_n\}_{n=1}^\infty$, $A_1 \supset A_2 \supset A_3 \supset \dots$, $\bigcap_{n=1}^\infty A_n = \emptyset$, and $\varepsilon > 0$ such that

$$(5.2) \quad \|x_0 \chi_{A_n}\|_X \geq \varepsilon.$$

Since, by hypothesis, $x_0 \chi_{A_n} \in [0, x_0]$, we can find a subsequence $\{x_0 \chi_{A_{n_k}}\}_{k=1}^\infty$ such that $x_0 \chi_{A_{n_k}} \rightarrow y$ weakly in X . Then, by [KA77, Lemma 10.4.1], we get that

$$\|x_0 \chi_{A_{n_k}}\|_X \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This contradicts (5.2).

Conversely, let $x_0 \in X^0$. Clearly, we have $[0, x_0] \subset X^0$. Therefore, by [KA77, Lemma 10.4.2], the interval $[0, x_0]$ is weakly compact in X^0 , *i.e.*, with respect to the topology generated in X^0 by the space $(X^0)^* = X'$. Since $X^* = X' \oplus X'_s$, where X'_s consists of all singular functionals f such that $f|_{X^0} = 0$ [KA77, Theorem 10.3.6], we get that $[0, x_0]$ is weakly compact in X as well. ■

Remark 3 In particular, from Proposition 5.4, it follows that Lotz’s result cannot be applied to Ces_∞ . In fact, $(\text{Ces}_\infty)^* = \widetilde{L}_1 \oplus S$, where S is the space of singular functionals, and, since singular functionals are not comparable with regular ones, each interval $[0, |x^*|] \subset \text{Ces}_\infty^*$ is either non-weakly compact or is of the form $[0, |s|]$ with $s \in S$. Therefore, the set F of all $x^* \in (\text{Ces}_\infty)^*$ with the weakly compact interval $[0, |x^*|]$ is contained in S and the topology $\sigma(\text{Ces}_\infty, F)$ fails to be Hausdorff, because singular functionals vanish on absolutely continuous elements.

One more interesting observation comes from the above considerations.

Remark 4 The space $(\bigoplus_{k=1}^\infty L_1[0, 1])_{\ell_\infty}$ is isomorphic to a dual space, but its unit sphere does not contain extreme points.

Since $\text{Ces}_\infty(I) \simeq X^*$, where X is a Banach space, and it contains a complemented subspace isomorphic to $L_1[0, 1]$, then, according to the above-mentioned Hagler–Stegall result, $\text{Ces}_\infty(I)$ contains a complemented subspace isomorphic to $C[0, 1]^*$, *i.e.*, to the space $\mathcal{M}[0, 1]$ of all regular Borel measures on $[0, 1]$ of finite variation. We

would like to conclude this paper by presenting the following stronger result, which was noticed by Michał Wojciechowski and is included here with his kind permission.

Theorem 5.5 *The spaces $\text{Ces}_\infty(I)$, where $I = [0, 1]$ or $[0, \infty)$, is isomorphic to the space $(\bigoplus_{k=1}^\infty \mathcal{M}[0, 1])_{\ell_\infty}$.*

Proof At first, by Miljutin’s theorem [AK06, p. 94], we know that $C[0, 1] \simeq C(\mathbb{T})$, where \mathbb{T} is the unit circle. Since also $\mathcal{M}[0, 1] \simeq \mathcal{M}(\mathbb{T})$ and $L_1[0, 1] \simeq L_1(\mathbb{T})$, we can regard all spaces on \mathbb{T} instead of $[0, 1]$.

By Theorem 4.1, it is sufficient to prove that the spaces

$$\left(\bigoplus_{k=1}^\infty L_1(\mathbb{T})\right)_{\ell_\infty} \quad \text{and} \quad \left(\bigoplus_{k=1}^\infty \mathcal{M}(\mathbb{T})\right)_{\ell_\infty}$$

are isomorphic. Since both spaces are isomorphic to their squares, we may again apply the Pełczyński decomposition argument. Clearly, $(\bigoplus_{k=1}^\infty L_1(\mathbb{T}))_{\ell_\infty}$ is isomorphic to a complemented subspace of $(\bigoplus_{k=1}^\infty \mathcal{M}(\mathbb{T}))_{\ell_\infty}$. So we only need to check that, conversely, $(\bigoplus_{k=1}^\infty \mathcal{M}(\mathbb{T}))_{\ell_\infty}$ is isomorphic to a complemented subspace of

$$\left(\bigoplus_{k=1}^\infty L_1(\mathbb{T})\right)_{\ell_\infty}.$$

Let $\{K_n\}_{n=1}^\infty$ be the Fejer kernel and let $\{N_i\}_{i=1}^\infty$ be a sequence of pairwise disjoint infinite subsets of positive integers such that $\sum_{i=1}^\infty N_i = \mathbb{N}$. For every $i = 1, 2, \dots$, define the operator $K^i: \mathcal{M}(\mathbb{T}) \rightarrow (\bigoplus_{k=1}^\infty L_1(\mathbb{T}))_{\ell_\infty}$ as follows:

$$K^i(\mu) := (K_n * \mu)_{n \in N_i} \quad \text{for every } \mu \in \mathcal{M}(\mathbb{T}).$$

Then $\|K^i\| = 1$, and if $N_i = \{n_j^i\}_{j=1}^\infty$, $n_1^i < n_2^i < \dots$, then $K_{n_j^i} * \mu \rightarrow \mu$ as $j \rightarrow \infty$ weakly* in $\mathcal{M}(\mathbb{T})$ for each $i = 1, 2, \dots$. Hence, $K := \bigoplus_{i=1}^\infty K^i$ is an injective operator from $(\bigoplus_{i=1}^\infty \mathcal{M}(\mathbb{T}))_{\ell_\infty}$ into the space

$$\left(\bigoplus_{k=1}^\infty L_1(\mathbb{T})\right)_{\ell_\infty} \simeq \left(\bigoplus_{i=1}^\infty \left(\bigoplus_{n \in N_i} L_1(\mathbb{T})\right)_{\ell_\infty}\right)_{\ell_\infty}.$$

Denoting by Y the image of K , we prove that it is complemented in the latter space.

Let \mathcal{U} be a free ultrafilter. For a given $\{f_k\} \in (\bigoplus_{k=1}^\infty L_1(\mathbb{T}))_{\ell_\infty}$ and any $i = 1, 2, \dots$ define the functional $g_i^* \in C(\mathbb{T})^*$ by $\langle g_i^*, g \rangle := \lim_{\mathcal{U}} \langle f_{n_j^i}, g \rangle$, $g \in C(\mathbb{T})$. Since $\|\{f_k\}\| = \sup_{k \in \mathbb{N}} \|f_k\|_{L_1} < \infty$, then g_i^* is a well-defined linear and bounded functional. Therefore, by the Riesz representation theorem, for every $i = 1, 2, \dots$, there is a measure $\mu_i \in \mathcal{M}(\mathbb{T})$ such that $\langle g_i^*, g \rangle = \langle \mu_i, g \rangle$ for each $g \in C(\mathbb{T})$. Setting $P(\{f_k\}) := \{\mu_i\}$, we see that P is a linear bounded operator from $(\bigoplus_{k=1}^\infty L_1(\mathbb{T}))_{\ell_\infty}$ into $(\bigoplus_{k=1}^\infty \mathcal{M}(\mathbb{T}))_{\ell_\infty}$. It remains only to show that the composition KP is a projection from $(\bigoplus_{k=1}^\infty L_1(\mathbb{T}))_{\ell_\infty}$ onto Y . In fact, suppose that $\{f_k\} \subset Y$. Then $f_{n_j^i} = K_{n_j^i} * \mu_i$, $i, j = 1, 2, \dots$, where $\{\mu_i\} \in (\bigoplus_{i=1}^\infty \mathcal{M}(\mathbb{T}))_{\ell_\infty}$, and we have

$$\lim_{\mathcal{U}} \langle f_{n_j^i}, g \rangle = \lim_{\mathcal{U}} \langle K_{n_j^i} * \mu_i, g \rangle = \lim_{\mathcal{U}} \langle \mu_i, K_{n_j^i} * g \rangle = \lim_{j \rightarrow \infty} \langle \mu_i, K_{n_j^i} * g \rangle = \langle \mu_i, g \rangle,$$

for every $g \in C(\mathbb{T})$. Thus, $KP\{f_k\} = \{\mu_i\}$ if $\{f_k\} \in Y$, and the proof is complete. ■

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Department of Mathematics and Mechanics, Samara State University, Acad. Pavlova 1, 443011 Samara, Russia, and Samara State Aerospace University (SSAU), Moskovskoye shosse 34, 443086, Samara, Russia
e-mail: astash56@mail.ru

Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3a, 60-965 Poznań, Poland
e-mail: klesnik@vp.pl

Department of Engineering Sciences and Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden
e-mail: lech.maligranda@ltu.se