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ALGEBRAS OF ANALYTIC OPERATORS ASSOCIATED WITH A PERIODIC FLOW ON A VON NEUMANN ALGEBRA

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1. Introduction. Let M be a σ-finite von Neumann algebra and $\{\alpha_t\}_{t\in\mathbf{T}}$ be a σ-weakly continuous representation of the unit circle, $\mathbf{T}$, as $*$-automorphisms of M. Let $H^{\infty}(\alpha)$ be the set of all $x \in M$ such that

$$sp_{\alpha}(x) \subseteq \{n \in \mathbf{Z}:n \geqq 0\}.$$

The structure of $H^{\infty}(\alpha)$ was studied by several authors (see [**2-13**]).

The main object of this paper is to study the σ-weakly closed subalgebras of M that contain $H^{\infty}(\alpha)$. In [**12**] this was done for the special case where $H^{\infty}(\alpha)$ is a nonselfadjoint crossed product.

Let M_n, for $n \in \mathbf{Z}$, be the set of all $x \in M$ such that

$$sp_{\alpha}(x) = \{n\}.$$

With a projection e in the centre of M_0 (the fixed point algebra with respect to α) we associate projections $\{e(n)\}_{n=-\infty}^{\infty}$ by defining

$$e(n) = I \text{ for } n \geqq 0 \quad \text{and}$$

$$e(n) = \Lambda\{1 - \beta_m(e):n \leqq m \leqq -1\} \text{ for } n < 0$$

(see Section 2 for the definition of β_m). We prove (Theorem 3.6) that for each σ-weakly closed subalgebra B that contains $H^{\infty}(\alpha)$ there is a projection e in the centre of M_0 such that B is generated by $\cup\ \{e(n)M_n: n \in \mathbf{Z}\}$ (as a σ-weakly closed linear subspace of M). We also show (Theorem 3.9) that each such subalgebra is $H^{\infty}(\gamma)$ for some periodic flow γ on M. As a corollary we prove that if $\mathscr{A}$ is a nest subalgebra associated with a nest $\{0, \dots, P_{-1}, P_0, P_1, \dots, I\} \subseteq M$ and B is a σ-weakly closed subalgebra of M that contains $\mathscr{A}$ then B is a nest subalgebra.

2. Preliminaries. Let M be a σ-finite von Neumann algebra acting on a Hilbert space H and let $\{\alpha_t\}_{t\in\mathbf{R}}$ be a periodic σ-weakly continuous representation of $\mathbf{R}$ as $*$-automorphisms of M. We assume that the period is 2π and write $\mathbf{T}$ for the interval $[0, 2\pi]$ identified with the unit circle. For each $n \in \mathbf{Z}$ we define a σ-weakly continuous linear map ϵ_n, on M, by

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$$\epsilon_n(x) = \int_0^{2\pi} e^{-itn}\alpha_t(x)d\mu(t), \quad x \in M,$$

where $d\mu$ is the normalized Lebesgue measure on $\mathbf{T}$. Let M_n be $\epsilon_n(M)$. Then it is clear that

$$M_n = \{x \in M:\alpha_t(x) = e^{int}x, \, t \in \mathbf{T}\}.$$

Whenever $\{\gamma_t\}_{t\in\mathbf{T}}$ is a σ-weakly continuous representation of $\mathbf{T}$ as *-automorphisms of M we let $sp_\gamma(x)$ denote the Arveson's spectrum of $x \in M$ with respect to $\{\gamma_t\}$ (see [**1**]). For a subset $S \subseteq \mathbf{Z}$, $M^\gamma(S)$ will denote the spectral subspace associated with S; i.e.,

$$M^\gamma(S) = \{x \in M:sp_\gamma(x) \subseteq S\}.$$

If $S = \{n \in \mathbf{Z}:n \geqq 0\}$ we write $H^\infty(\gamma)$ for $M^\gamma(S)$. It is known ([**3**]) that $H^\infty(\gamma)$ is a σ-weakly closed subalgebra of M which is a finite maximal subdiagonal algebra (with respect to the map

$$\epsilon_0 = \int_0^{2\pi} \alpha_t d\mu(t)).$$

When $\gamma = \alpha$ we have $M_n = M^\alpha(\{n\})$, $n \in \mathbf{Z}$ and

$$sp_\alpha(x) = \{n \in \mathbf{Z}:\epsilon_n(x) \neq 0\} \quad \text{for } x \in M.$$

Since M is $\mathbf{T}$-finite (i.e., there is a faithful expectation ϵ_0 from M onto M_0 such that $\epsilon_0 \circ \alpha_t = \epsilon_0$ for all $t \in \mathbf{T}$) and σ-finite, there exists a faithful normal $\{\alpha_t\}$-invariant state ϕ on M. Considering the Gelfand-Naimark-Segal construction of ϕ, we may suppose that M has a separating and cyclic vector $\xi_0 \in H$ such that $\phi(x) = \langle x\xi_0, \xi_0\rangle$ is an $\{\alpha_t\}$-invariant state on M.

Remark 2.1. Suppose $\{\gamma_t\}_{t\in\mathbf{T}}$ is a σ-weakly continuous representation as above and $a \in M$ such that, for each $t \in \mathbf{T}$, $\gamma_t(a) = e^{itb}a$ for some self adjoint operator b in the centre of M_0 with $\sigma(b) \subseteq \mathbf{Z}$ (where $\sigma(b)$ is the spectrum of b as an operator). Then

$$sp_\gamma(a) \subseteq \sigma(b).$$

In fact, assume that there is some $n \in sp_\gamma(a)$, $n \notin \sigma(b)$. Then

$$\int_0^{2\pi} e^{-itn}e^{itb}d\mu(t) = 0 \quad (\text{as } n \notin \sigma(b));$$

but $n \in sp_\gamma(a)$ hence

$$1 = \int_0^{2\pi} e^{-itn}e^{itn}d\mu(t) = 0.$$

The contradiction shows that $sp_\gamma(a) \subseteq \sigma(b)$.

For each $n \in \mathbf{Z}$ define projections e_n, f_n by

$$e_n = \sup\{u^*u:u \text{ is a partial isometry in } M_n\}$$

$$f_n = \sup\{uu^*:u \text{ is a partial isometry in } M_n\}.$$

Then, by [**11**, Lemma 2.2], e_n and f_n lie in $Z(M_0)$ (the centre of M_0). The following lemma appears in [**11**].

LEMMA 2.2. (1) *For every* $n, m \in \mathbf{Z}$, $M_n M_m \subseteq M_{n+m}$ *and* $M_n^* = M_{-n}$.
(2) *Let* $x \in M_n$ *and let* $x = v|x|$ *be the polar decomposition of* x. *Then* $v \in M_n$ *and* $|x| \in M_0$.

The following result can be found in [**13**, Proposition 2.3 and Theorem 2.4]. Although it was assumed there that the algebra M is finite, this assumption was not used in the proof of the following proposition.

PROPOSITION 2.3. *Fix* $n \in \mathbf{Z}$. *Then there is a sequence* $\{v_{n,m}\}_{m=1}^{\infty}$ *of partial isometries in* M_n *with the following properties*:

(1) $v_{n,m}^* v_{n,j} = 0$ *if* $m \neq j$.

(2) $\displaystyle\sum_{m=1}^{\infty} v_{n,m} v_{n,m}^* = f_n$.

(3) $\displaystyle M_n = \sum_{m=1}^{\infty} v_{n,m} M_0$;

i.e., each $x \in M_n$ *can be written as*

$$\sum_{m=1}^{\infty} v_{n,m} x_m \quad \text{for some } x_m \in M_0$$

where the sum converges in the σ*-weak operator topology.*

For each $\rho \in M_*$ there are sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ in H satisfying

$$\sum \|x_n\|^2 < \infty \quad \text{and} \quad \sum \|y_n\|^2 < \infty,$$

such that

$$\rho(a) = \sum_{n=1}^{\infty} \langle a x_n, y_n \rangle.$$

Let $\widetilde{H}$ be the space $H \otimes K$ (for some separable infinite dimensional subspace K with an orthogonal basis $\{g_n\}_{n=1}^{\infty}$). Write $\widetilde{a}$ for the operator $a \otimes I_k$ and then

$$\rho(a) = \langle \widetilde{a} x, y \rangle$$

where

$$x = \sum_{n=1}^{\infty} x_n \otimes g_n \in \widetilde{H} \quad \text{and} \quad y = \sum_{n=1}^{\infty} y_n \otimes g_n \in \widetilde{H}.$$

Let $\widetilde{M}$ be $\{\widetilde{a}:a \in M\}$ and then $\widetilde{M}$ is *-isomorphic to M and $\xi = \xi_0 \otimes g_1$ is a separating vector for $\widetilde{M}$.

Replacing M by $\widetilde{M}$ and H by $\widetilde{H}$ we assume that M has a separating vector $\xi \in H$ and each $\xi \in M_*$ is of the form $w_{x,y}$ for some $x, y \in H$. Also $\phi(a) = \langle a\xi, \xi\rangle$ is a faithful normal $\{\alpha_t\}$-invariant state on M.

The following result appears in [**11**, Theorem 2.4].

PROPOSITION 2.4. (1) $H^\infty(\alpha) = \{x \in M:\epsilon_n(x) = 0$ *for each* $n < 0\}$

(2) $H^\infty(\alpha)$ *is the* σ*-weakly closed subalgebra of* M *which is generated by* M_0 *and all partial isometries in* M_n ($n \in \mathbf{Z}$, $n > 0$).

With the partial isometries $\{v_{n,m}:n \in \mathbf{Z},\ m \geqq 1\}$ defined as in Proposition 2.3, we can define maps $\{\beta_n\}_{n\in\mathbf{Z}}$ on M_0' by the formula

$$\beta_n(T) = \sum_{m=1}^{\infty} v_{n,m}Tv_{n,m}^*.$$

Let us denote the orthogonal projection onto the subspace $[M_n\xi]$ (the closure, in H, of $\{a\xi:a \in M_n\}$) by E_n, $n \in \mathbf{Z}$.

LEMMA 2.4. (1) β_n *is a well defined homomorphism from* M_0' *onto* f_nM_0'.

(2) *For a projection* $Q \in M_0'$,

$$\beta_n(Q) = V\{uQu^*:u \text{ is a partial isometry in } M_n\},$$

hence $\beta_n(Q)$ *is a projection.*

(3) *For each* $n, m \in \mathbf{Z}$, $T \in M_0'$,

$$\beta_{n+m}(f_{-m}T) = \beta_n\beta_m(T) = f_n\beta_{n+m}(T).$$

(4) β_n *is a* *-*isomorphism from* e_nM_0' *onto* f_nM_0'.

(5) *For* $T \in M_0'$, $T \in M'$ *if and only if* $\beta_n(T) = f_nT$ *for each* $n \in \mathbf{Z}$. *If* T *is a projection then* $T \in M'$ *if and only if* $\beta_n(T) \leqq T$ *for each* $n \in \mathbf{Z}$.

(6) *If* $T \in M_0'$ *and* $\sum_{m=-\infty}^{\infty} \beta_m(T)$ *is a well defined bounded operator in* M_0' *then* $\sum_{m=-\infty}^{\infty} \beta_m(T) \in M'$ (*where the sum converges in the strong operator topology.*)

(7) *For each* $n \in \mathbf{Z}$, $\beta_n(E_0) = E_n$.

(8) *Suppose* Q_1 *and* Q_2 *are projections in* M_0' *and* $Q_1 \sim Q_2$ (*with respect to the equivalence relation in* M_0'), *then*

$$\beta_n(Q_1) \sim \beta_n(Q_2) \text{ for each } n \in \mathbf{Z}.$$

Proof. (1) Fix $T \in M_0'$. Since the range projections of $\{v_{n,m}\}_{m=1}^{\infty}$ are mutually orthogonal, $\beta_n(T)$ is a linear bounded operator. Now fix a unitary operator $u \in M_0$ and $m \geqq 1$. Then

$$uv_{n,m} = \sum_j v_{n,j}x_j \quad \text{for some } x_j \in M_0 \quad \text{and,}$$

$$\begin{aligned} uv_{n,m}Tv^*_{n,m}u^* &= \left(\sum_j v_{n,j}x_j\right)T\left(\sum_i x^*_i v^*_{n,i}\right) \\ &= \sum_{i,j} v_{n,j}Tv^*_{n,j}v_{n,j}x_j x^*_i v^*_{n,i} \\ &= \sum_{i,j} v_{n,j}Tv^*_{n,j}\left(\sum_r v_{n,r}x_r\right)x^*_i v^*_{n,i} \\ &= \sum_j v_{n,j}Tv_{n,j}uv_{n,m}v^*_{n,m}u^* \\ &= \beta_n(T)uv_{n,m}v^*_{n,m}u^*. \end{aligned}$$

Summing over all $m \geqq 1$ we have

$$u\beta_n(T)u^* = \beta_n(T)f_n.$$

Since, clearly $\beta_n(T) = \beta_n(T)f_n$,

$$\beta_n(T) \in M'_0 f_n, \quad n \in \mathbf{Z}.$$

To show that β_n is multiplicative let S, T lie in M'_0. Then

$$\begin{aligned} \beta_n(S)\beta_n(T) &= \left(\sum_m v_{n,m}Sv^*_{n,m}\right)\left(\sum_j v_{n,j}Tv^*_{n,j}\right) \\ &= \sum_{m,j} v_{n,m}Sv^*_{n,m}v_{n,j}Tv^*_{n,j} \\ &= \sum_m v_{n,m}STv^*_{n,m} = \beta_n(ST). \end{aligned}$$

Linearity of β_n is obvious. The fact that $\beta_n(M'_0) = f_n M'_0$ will follow from (3), since

$$\beta_n\beta_{-n}(T) = f_n\beta_0(T) = f_nT = T \text{ for each } T \in f_nM'_0.$$

This, in fact, shows that

$$\beta_n(f_{-n}M'_0) = M'_0.$$

(2) This is proved in [**13**, Lemma 3.1(1)].

(3) This is proved in [**13**, Lemma 3.1(2)] for the case where $T \in M'_0$ is a projection. The linearity and continuity, in the strong operator topology, of β_n proves it for any $T \in M'_0$.

(4) Since $\beta_{-n}\beta_n(e_nT) = f_{-n}e_nT = e_nT$ (note that $e_n = f_{-n}$, $n \in \mathbf{Z}$), β_n is one-to-one on $e_nM'_0$. The rest follows from (1) (with the observation that

$$\beta_n(e_nM'_0) = \beta_n(f_{-n}M'_0) = f_nM'_0,$$

as noted above).

(5) If $T \in M'$ then obviously $\beta_n(T) = f_nT$. Conversely, if $\beta_n(T) = f_nT$ for each $n \in \mathbf{Z}$, then, for each $m \geqq 1$,

$$\begin{aligned} v_{n,m}T - Tv_{n,m} &= v_{n,m}v_{n,m}^*v_{n,m}T - Tv_{n,m}v_{n,m}^*v_{n,m} \\ &= (v_{n,m}T - Tv_{n,m})v_{n,m}^*v_{n,m} \\ &= (v_{n,m}Tv_{n,m}^* - Tv_{n,m}v_{n,m}^*)v_{n,m} \\ &= \beta_n(T)v_{n,m} - Tf_nv_{n,m} = 0. \end{aligned}$$

Since M_0 together with $\{v_{n,m}\}_{m \geqq 1, n \in \mathbf{Z}}$ span M, $T \in M'$.

(6) Let S be $\sum_{m=-\infty}^{\infty} \beta_n(T)$ then

$$\beta_n(S) = \sum_{m=-\infty}^{\infty} \beta_n\beta_m(T) = \sum_{m=-\infty}^{\infty} f_n\beta_{n+m}(T) = f_nS.$$

Hence, by (5), $S \in M'$.

(7) Recall that E_n is the projection onto $[M_n\xi]$. Hence, for $m \geqq 1$, $n \in \mathbf{Z}$, $v_{n,m}E_0v_{n,m}^*$ is the projection onto $[v_{n,m}M_0\xi]$ and $\beta_n(E_0)$ is the projection onto

$$\textstyle\sum_m [v_{n,m}M_0\xi] = [M_n\xi].$$

Hence $\beta_n(E_0) = E_n$.

(8) Suppose W is a partial isometry in M_0' such that $WW^* = Q_1$ and $W^*W = Q_2$. Then

$$\beta_n(W)\beta_n(W^*) = \beta_n(Q_1) \text{ and } \beta_n(W^*)\beta_n(W) = \beta_n(Q_2).$$

Since $\beta_n(W) \in M_0'$ and $\beta_n(W^*) = \beta_n(W)^*$,

$$\beta_n(Q_1) \sim \beta_n(Q_2).$$

The following notations and definitions will be used later:

1. A projection $Q \in M_0'$ is said to be a *wandering projection* if, for each $n \in \mathbf{Z}$, $Q\beta_n(Q) = 0$ (note that this implies that, for $n \neq m$, $\beta_n(Q)\beta_m(Q) = 0$). The set of all the wandering projections in M_0' will be denoted by $\mathscr{P}_1$.

2. For $Q \in \mathscr{P}_1$ we let $\sigma(Q)$ be $\sum_{n=0}^{\infty} \beta_n(Q)$.

3. A closed subspace $\mathscr{M}$ of H is called *invariant* if for each $a \in H^\infty(\alpha)$ and $x \in \mathscr{M}$, $ax \in \mathscr{M}$. Let us denote by $\mathscr{P}_2$ the set of all orthogonal projections whose range is an invariant subspace. Note that

$$\mathscr{P}_2 = \{P \in M_0' : \beta_n(P) \leqq P \text{ for each } n \geqq 0\}.$$

(Since $[M_nP(H)] = \beta_n(P)(H)$ for each $n \in \mathbf{Z}$ and $\bigcup_{n \geqq 0} M_n$ span $H^\infty(\alpha)$).

4. For $P \in \mathscr{P}_2$ let $\delta(P)$ be $P - V\{\beta_n(P):n > 0\}$.

The following lemma can be found in [**13**].

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LEMMA 2.5. *If* $P \in \mathscr{P}_2$ *then* $\delta(P) \in \mathscr{P}_1$,

$$P = \sigma(\delta(P)) + \bigwedge_{n>0} \bigvee_{m \geqq n} \beta_m(P) \quad \textit{and}$$

$$\bigwedge_{n>0} \bigvee_{m \geqq n} \beta_m(P) \in M'.$$

3. Subalgebras of M. Let $\mathscr{C}$ be the collection of all σ-weakly closed subalgebras of M that contain I. For each $y \in H$ and $B \in \mathscr{C}$ we define

$$B_y = \{a \in M : a[By] \subseteq [By]\}.$$

Then B_y is a σ-weakly closed subalgebra of M that contains B. In particular $B_y \in \mathscr{C}$.

LEMMA 3.1. *For each* $B \in \mathscr{C}$ *and* $y \in H$,

$$[By] = [B_y y].$$

Proof. Since $B \subseteq B_y$, $[By] \subseteq [B_y y]$. For the other inclusion, suppose a is in B_y. Then, since $y \in [By]$, $ay \in [By]$; hence $[B_y y] \subseteq [By]$.

LEMMA 3.2. *Suppose* B, C *lie in* $\mathscr{C}$ *and* $B \neq C$. *Then there is some* $y \in H$ *such that* $B_y \neq C_y$.

Proof. Since $B \neq C$ we can assume that there is some $a \in B$, $a \notin C$. (The case $B \subset C$ can be handled similarly.) Since C is σ-weakly closed there is some $\rho \in M_*$ such that $\rho(c) = 0$ for each $c \in C$ and $\rho(a) \neq 0$. Since M has a separating vector, there are vectors x, $y \in H$ such that $\rho(b) = \langle by, x \rangle$ for all $b \in M$. Hence x is orthogonal to $[Cy]$ but not to $[By]$. Since

$$[C_y y] = [Cy] \neq [By] = [B_y y],$$

$B_y \neq C_y$.

LEMMA 3.3. *For each* $B \in \mathscr{C}$, $B = \cap \{B_y : y \in H\}$.

Proof. Clearly B is contained in the algebra on the right (which we now denote by $\widetilde{B}$). For each $z \in H$, $B \subseteq \widetilde{B} \subseteq B_z$ and, by Lemma 3.1, $[Bz] = [B_z z]$. Hence, for each $z \in H$, $[Bz] = [\widetilde{B}z]$ and, therefore,

$$B_z = \{a \in M : a[Bz] \subseteq [Bz]\} = \{a \in M : a[\widetilde{B}z] \subseteq [\widetilde{B}z]\} = \widetilde{B}_z.$$

By the previous lemma $B = \widetilde{B}$.

Suppose $\mathscr{M}$ is an invariant subspace of H and P is the orthogonal projection onto $\mathscr{M}$. Then we let $B(\mathscr{M})$ be the algebra

$$\{a \in M : a\mathscr{M} \subseteq \mathscr{M}\} = \{a \in M : aP = PaP\}.$$

Clearly $H^\infty(\alpha) \subseteq B(\mathscr{M})$ for each invariant subspace $\mathscr{M}$.

For a projection $Q \in M_0'$ we let $c(Q)$ be the central support of Q.

LEMMA 3.4. *Let $\mathcal{M}_i$, $i = 1, 2$, be an invariant subspace in H with corresponding projection $P_i \in \mathcal{P}_2$ such that*

$$c(\delta(P_1)) = c(\delta(P_2)).$$

Then $B(\mathcal{M}_1) = B(\mathcal{M}_2)$.

Proof. By symmetry it suffices to show that each $a \in B(\mathcal{M}_1)$ lies in $B(\mathcal{M}_2)$. Let Q_i denote $\delta(P_i)$, $i = 1, 2$. Let $\{q_\gamma\}_{\gamma \in \Gamma}$ be a maximal orthogonal family of subprojections of Q_2 in M_0' with the property that q_γ is equivalent to a subprojection of Q_1 (to be denoted p_γ) for each $\gamma \in \Gamma$. Let q be $\sum_{\gamma \in \Gamma} q_\gamma$. Then, by the maximality of $\{q_\gamma\}_{\gamma \in \Gamma}$, no subprojection of $Q_2 - q$ (in M_0') is equivalent to a subprojection of Q_1. This implies that

$$c(Q_2 - q)c(Q_1) = 0.$$

But

$$c(Q_2 - q) \leqq c(Q_2) = c(Q_1);$$

thus

$$Q_2 = q = \Sigma q_\gamma.$$

By Lemma 2.5, $P_2 = \sigma(Q_2) + R$ where R is some projection in M'. Hence

$$P_2 = \sum_{\gamma \in \Gamma} \sigma(q_\gamma) + R.$$

In order to show that $a \in B(\mathcal{M}_2)$ it will suffice to show that, for each $\gamma \in \Gamma$, a maps $\sigma(q_\gamma)(H)$ into itself.

Now fix $\gamma \in \Gamma$ and let $v \in M_0'$ be a partial isometry in M_0' such that $vv^* = q_\gamma$ and $v^*v = p_\gamma \leqq Q_1$. Let $R(v)$ be the partial isometry $\sum_{m=-\infty}^{\infty} \beta_n(v) \in M'$ (see Lemma 2.4(6)). The initial projection of $R(v)$ is $\sum_{m=-\infty}^{\infty} \beta_m(p_\gamma)$ and its final projection is $\sum_{m=-\infty}^{\infty} \beta_m(q_\gamma)$.

Now fix $n \geqq 0$, and then

$$\begin{aligned} a\beta_n(q_\gamma) &= aR(v)R(v)^*\beta_n(q_\gamma) \\ &= R(v)aR(v)^*\beta_n(q_\gamma) = R(v)a\beta_n(p_\gamma)R(v)^*. \end{aligned}$$

Since a maps $\sigma(p_\gamma)$ into P_1,

$$\begin{aligned} a\sigma(p_\gamma) &= P_1 a\sigma(p_\gamma) \\ &= P_1 aR(v)^*R(v)\sigma(p_\gamma) \\ &= P_1\left(\sum_{m=-\infty}^{\infty} \beta_m(p_\gamma)\right)a\sigma(p_\gamma). \end{aligned}$$

But

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$$p_\gamma \leqq \delta(P_1) = P_1 - V\{\beta_m(P_1):m > 0\};$$

thus $\beta_m(p_\gamma)P_1 = 0$ for each $m < 0$ and we have

$$a\sigma(p_\gamma) = \sigma(p_\gamma)a\sigma(p_\gamma).$$

Therefore,

$$\begin{aligned} a\beta_n(q_\gamma) &= R(v)\sigma(p_\gamma)a\beta_n(p_\gamma)R(v^*) \\ &= R(v)\sigma(p_\gamma)aR(v)^*R(v)\beta_n(p_\gamma)R(v)^* \\ &= R(v)\sigma(p_\gamma)R(v)^*aR(v)\beta_n(p_\gamma)R(v)^* \\ &= \sigma(q_\gamma)a\beta_n(q_\gamma). \end{aligned}$$

Thus

$$\sigma(q_\gamma)a\sigma(q_\gamma) = a\sigma(q_\gamma)$$

and this implies that a lies in $B(\mathscr{M}_2)$.

For a projection e in $Z(M_0)$ and $n > 0$ we write $e(-n)$ for the projection $\Lambda\{1 - \beta_{-m}(e):1 \leqq m \leqq n\}$.

PROPOSITION 3.5. *Let $\mathscr{M}$ be an invariant subspace with P the orthogonal projection onto it. Let e be $c(\delta(P))$. Then*

$$B(\mathscr{M}) = \{a \in M:\epsilon_{-n}(a) \in e(-n)M_{-n} \text{ for each } n > 0\}.$$

Proof. Let $\mathscr{M}_0$ be the invariant subspace $\sum_{n=0}^{\infty} \beta_n(e)E_n(H)$. Then the projection P_0 onto $\mathscr{M}_0$ is

$$\sum_{n=0}^{\infty} \beta_n(e)E_n = \sum_{n=0}^{\infty} \beta_n(eE_0)$$

and

$$\delta(P_0) = eE_0.$$

If z is a nonzero projection in $Z(M_0)$ then $z^2\xi = z\xi \neq 0$ and $z\xi \in E_0$ (as $z \in M_0$). Hence $zE_0 \neq 0$ for each nonzero projection $z \in Z(M_0)$. This implies that $c(E_0) = I$ and that

$$c(eE_0) = ec(E_0) = e.$$

Therefore

$$c(\delta(P_0)) = c(\delta(P))$$

and, by the previous lemma, $B(\mathscr{M}) = B(\mathscr{M}_0)$.

For $t \in \mathbf{T}$ let W_t be the linear operator that maps $x\xi(x \in M)$ into $\alpha_t(x)\xi$. Since

$$\begin{aligned} \langle \alpha_t(x)\xi, \alpha_t(x)\xi \rangle &= \langle \alpha_t(x^*x)\xi, \xi \rangle \\ &= \phi(\alpha_t(x^*x)) = \phi(x^*x) = \langle x\xi, x\xi \rangle, \end{aligned}$$

W_t can be extended to a unitary operator on H. For $n \in \mathbf{Z}$, $x \in M_n$ and $a \in M$,

$$\alpha_t(a)\beta_n(e)x\xi = \alpha_t(a\beta_n(e)\alpha_{-t}(x))\xi$$
$$= W_t a\beta_n(e)\alpha_{-t}(x)\xi \in W_t a[\beta_n(e)M_n\xi]$$
$$= W_t a\beta_n(e)E_n(H).$$

If $a \in B(\mathcal{M}_0)$ then

$$\alpha_t(a)\beta_n(e)x\xi \in W_t P_0(H) \quad \text{for all } n \in \mathbf{Z},\ x \in M_n,\ t \in \mathbf{T}.$$

Hence

$$\alpha_t(a)P_0(H) \subseteq W_t P_0(H),\ t \in \mathbf{T}.$$

But

$$W_t\beta_n(e)x\xi = \alpha_t(\beta_n(e)x)\xi$$
$$= \beta_n(e)\alpha_t(x)\xi \in P_0(H) \text{ for } n \geqq 0,\ x \in M_n,\ t \in \mathbf{T}.$$

Hence

$$\alpha_t(a)P_0(H) \subseteq W_t P_0(H) \subseteq P_0(H).$$

Therefore $\alpha_t(B(\mathcal{M}_0)) = B(\mathcal{M}_0)$. Since

$$\epsilon_n = \int_0^{2\pi} e^{-int}\alpha_t d\mu(t),$$

$\epsilon_n(B(\mathcal{M}_0)) \subseteq B(\mathcal{M}_0)$, for all $n \in \mathbf{Z}$. Using [**7**, Theorem 1] we have

$$B(\mathcal{M}_0) = \{a \in M : \epsilon_n(a) \in B(\mathcal{M}_0) \text{ for each } n \in \mathbf{Z}\}.$$

For each $n \in \mathbf{Z}$ we denote the set $\{a \in M_n : a \in B(\mathcal{M}_0)\}$ by L_n. Then

$$B(\mathcal{M}_0) = \{a \in M : \epsilon_n(a) \in L_n \text{ for each } n \in \mathbf{Z}\}.$$

Since $H^\infty(\alpha) \subseteq B(\mathcal{M}_0)$, $L_n = M_n$ for $n \geqq 0$.

Now fix $n > 0$. We claim that $L_{-n} = e(-n)M_{-n}$. Suppose $x \in e(-n)M_{-n}$, then

$$x = \sum_{j=1}^{\infty} v_{-n,j}x_j \quad \text{for some } x_j \in M_0.$$

Then, for $m \geqq 0$,

$$\beta_{-n}(\beta_m(e))x = \sum_{i,j=1}^{\infty} v_{-n,j}\beta_m(e)v^*_{-n,j}v_{-n,i}x_i$$

$$= \sum_{j=1}^{\infty} v_{-nj}\beta_m(e)v^*_{-n,j}v_{-nj}x_j$$

$$(*) \qquad = \sum_{j=1}^{\infty} v_{-n,j}v^*_{-n,j}\beta_m(e)x_j = x\beta_m(e).$$

Hence, for each $y \in M_m$,

$$x\beta_m(e)y\xi = \beta_{-n}(\beta_m(e))xy\xi \in \beta_{-n}\beta_m(e)E_{m-n}(H)$$
$$\subseteq \beta_{m-n}(e)E_{m-n}(H).$$

Thus x maps $\sum_{m=n}^{\infty} \beta_m(e)E_m(H)$ into $\mathcal{M}_0$. For $0 \leqq m < n$ and $y \in M_m$,

$$x\beta_m(e)y\xi = (1 - \beta_{m-n}(e))x\beta_m(e)y\xi$$
$$= (1 - \beta_{m-n}(e))\beta_{m-n}(e)x\beta_m(e)y\xi = 0.$$

(The first equality holds because $x \in e(-n)M_{-n}$.) Hence

$$x\mathcal{M}_0 \subseteq \mathcal{M}_0.$$

This proves that $e(-n)M_{-n} \subseteq L_{-n}$.

Now suppose $x \in L_{-n}$. Since $x \in M_{-n}$,

$$x\beta_m(e)y\xi \in \beta_{m-n}(e)E_{m-n}(H)$$

for each $m \geqq 0$ and $y \in M_m$. Hence, for $0 \leqq m < n$,

$$x\beta_m(e) = x\beta_m(e)f_m = 0$$

(since for each $j \geqq 1$,

$$x\beta_m(e)v_{m,j}v^*_{m,j} = (x\beta_m(e)v_{m,j})v^*_{m,j} = 0).$$

But (*) implies that

$$\beta_{-n}\beta_m(e)x = x\beta_m(e) = 0.$$

Thus

$$x \in (1 - \beta_{-n}(\beta_m(e)))M_{-n} = (1 - f_{-n}\beta_{m-n}(e))M_{-n}$$
$$= (1 - \beta_{m-n}(e))M_{-n}.$$

Since this holds for each $0 \leqq m < n$, $x \in e(-n)M_{-n}$.

For a projection $e \in Z(M_0)$ let us denote by $B(e)$ the set

$$\{a \in M : \epsilon_{-n}(a) \in e(-n)M_{-n} \text{ for each } n > 0\}.$$

THEOREM 3.6. *For each σ-weakly closed subalgebra B of M that contains $H^{\infty}(\alpha)$ there is a projection $e \in Z(M_0)$ such that $B = B(e)$. Conversely, for each projection $e \in Z(M_0)$, $B(e)$ is a σ-weakly closed subalgebra of M that contains $H^{\infty}(\alpha)$.*

Proof. Suppose B is a σ-weakly closed subalgebra of M that contains $H^{\infty}(\alpha)$. By Lemma 3.3 we can write B as $\cap \{B_y : y \in H\}$. Hence

$$B = \{a \in M{:}a[By] \subseteq [By] \text{ for each } y \in H\}.$$

Since $[By]$ is an invariant subspace of H (as $H^{\infty}(\alpha) \subseteq B$), it follows from Proposition 3.5 that

$$B_y = B(e(y)) \text{ for some projection } e(y) \in Z(M_0).$$

Thus, clearly, $B = B(e)$ where $e = V\{e(y){:}y \in H\}$.

For the converse just note that the set $B(e)$ was shown, in the proof of Proposition 3.5, to be $B(\mathcal{M}_0)$ for some invariant subspace $\mathcal{M}_0$. Therefore $B(e)$ is a σ-weakly closed subalgebra of M that contains $H^{\infty}(\alpha)$.

Recall that W_t, $t \in \mathbf{T}$ is the unitary operator defined by

$$W_t a\xi = \alpha_t(a)\xi, \; a \in M$$

and E_n is the orthogonal projection onto $[M_n\xi]$. It is easy to check that the spectral decomposition of W_t is given by:

$$W_t = \sum_{n=-\infty}^{\infty} e^{int}E_n, \quad t \in \mathbf{T}.$$

Let us now fix a projection $e \in Z(M_0)$ and define, for each $n \in \mathbf{Z}$,

$$c_n = \begin{cases} f_n \sum_{k=0}^{n-1} \beta_k(e) & n > 0 \\ 0 & n = 0 \\ -f_n \sum_{k=n}^{-1} \beta_k(e)(= -\beta_n(c_{-n})) & n < 0. \end{cases}$$

For $t \in \mathbf{T}$ let the operator U_t be $\sum_{n=-\infty}^{\infty} \exp(itc_n)E_n$. Then U_t is a unitary operator and the map $t \to U_t$ is continuous in the strong operator topology. We now let γ_t be the *-automorphism of M implemented by U_t (i.e., $\gamma_t(a) = U_t a U_t^*$, $a \in M$). The map

$$t \to \gamma_t(a)$$

is continuous in the σ-weak operator topology and

$$\gamma_{t+s} = \gamma_t\gamma_s \quad \text{for } t, s \in \mathbf{T}.$$

Our next object is to show that the algebra $B(e)$ is $H^{\infty}(\gamma)$. This will prove that every σ-weakly closed subalgebra of M that contains $H^{\infty}(\alpha)$ is $H^{\infty}(\gamma)$ for some flow γ as described above.

LEMMA 3.7. *For each* $n, k \in \mathbf{Z}$,

$$f_{n+k}f_n c_{n+k} = f_{n+k}c_n + f_{n+k}\beta_n(c_k).$$

Proof. If $n = 0$ or $k = 0$ the equality above follows trivially. If $n > 0$ and $k > 0$,

$$f_{n+k}f_nc_{n+k} = f_{n+k}f_n \sum_{i=0}^{n+k-1} \beta_i(e)$$

$$= f_{n+k}f_n \sum_{i=0}^{n-1} \beta_i(e) + f_{n+k}f_n \sum_{i=0}^{k-1} \beta_{n+i}(e)$$

$$= f_{n+k}f_nc_n + f_{n+k} \sum_{i=0}^{k-1} \beta_n(\beta_i(e))$$

$$= f_{n+k}c_n + f_{n+k}\beta_n(c_k).$$

If $n > 0$, $k < 0$ and $n + k > 0$,

$$f_{n+k}f_nc_{n+k} = f_{n+k}f_n \sum_{i=0}^{n+k-1} \beta_i(e)$$

$$= f_{n+k}f_n \sum_{i=0}^{n-1} \beta_i(e) - f_{n+k}f_n \sum_{i=k}^{-1} \beta_{n+i}(e)$$

$$= f_{n+k}c_n - f_{n+k}\beta_n\left(\sum_{i=k}^{-1} \beta_i(e)\right)$$

$$= f_{n+k}c_n - f_{n+k}\beta_n(f_k)\beta_n\left(\sum_{i=k}^{-1} \beta_i(e)\right)$$

$$= f_{n+k}c_n - f_{n+k}\beta_n\left(\sum_{i=k}^{-1} f_k\beta_i(e)\right)$$

$$= f_{n+k}c_n - f_{n+k}\beta_n\left(\beta_k\left(\sum_{i=0}^{-k-1} \beta_i(e)\right)\right)$$

$$= f_{n+k}c_n + f_{n+k}\beta_n(c_k).$$

The other possible choices for n and k can be handled similarly.

LEMMA 3.8. *For each* $t \in \mathbf{T}$ *and* $n \in \mathbf{Z}$,

$$\gamma_t(a) = \exp(itc_n)a.$$

Proof. Fix $t \in \mathbf{T}$, $n \in \mathbf{Z}$, $a \in M_n$ and $k \in \mathbf{Z}$. Then

$$\gamma_t(a)E_k = U_taU_t^*E_k = U_ta \exp(-itc_k)E_k.$$

Since a lies in M_n,

$$a = \sum_{j=1}^{\infty} v_{n,j}a_j \quad \text{(for some } a_j \in M_0) \quad \text{and}$$

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$$a \exp(-itc_k)E_k \subseteq E_{k+n}.$$

Thus

$$\begin{aligned}\gamma_t(a)E_k &= \exp(itc_{n+k})\left(\sum_{j=1}^{\infty} v_{n,j}a_j\right)\exp(-itc_k)E_k \\ &= \exp(itc_{n+k})\sum_j v_{n,j}\exp(-itc_k)v_{n,j}{}^*v_{n,j}a_jE_k \\ &= \exp(itc_{n+k})\beta_n(\exp(-itc_k))aE_k \\ &= \exp(itc_{n+k}f_n)\beta_n(\exp(-itc_k))f_{n+k}aE_k.\end{aligned}$$

By the previous lemma we now have

$$\begin{aligned}\gamma_t(a)E_k &= \exp(itf_{n+k}c_n)\exp(itf_{n+k}\beta_n(c_k))\exp(-it\beta_n(c_k)f_{n+k})aE_k \\ &= \exp(itf_{n+k}c_n)aE_k = \exp(itc_n)aE_k.\end{aligned}$$

Since this holds for each $k \in \mathbf{Z}$ and $\sum_{k=-\infty}^{\infty} E_k = I$, we are done.

THEOREM 3.9. *Let e be a projection in* $Z(M_0)$ *and* γ_t *be the flow associated with e, as defined in the discussion preceding Lemma* 3.7. *Then* $H^\infty(\gamma) = B(e)$, *where* $B(e)$ *is the algebra*

$$\{a \in M:\epsilon_{-n}(a) \in e(-n)M_{-n} \text{ for each } n > 0\}.$$

(*Recall that*

$$e(-n) = \Lambda\{1 - \beta_{-k}(e):1 \leqq k \leqq n\}.)$$

Hence every σ*-weakly closed subalgebra of M that contains* $H^\infty(\alpha)$ *is* $H^\infty(\gamma)$ *for some flow* γ *associated with a projection* $e \in Z(M_0)$.

Proof. Since for $n \geqq 0$, $c_n \geqq 0$ it follows from Remark 2.1 that

$$H^\infty(\alpha) \subseteq H^\infty(\gamma).$$

As $H^\infty(\gamma)$ is a σ-weakly closed subalgebra of M, $H^\infty(\gamma) = B(f)$ for some projection $f \in Z(M_0)$. We can also conclude from the proof of Theorem 3.6 (the fact that $B(e)$ is determined by $\epsilon_n(B(e))$, $n < 0$) that in order to prove that $B(e) = B(f)$ it suffices to show that for each $n > 0$, $\epsilon_{-n}(B(e))(= B(e) \cap M_{-n})$ equals $\epsilon_{-n}(B(f))(= H^\infty(\gamma) \cap M_{-n})$.

For $a \in M_{-n} \cap B(e)$, $a\beta_k(e) = 0$ for each $0 < k \leqq n$; hence

$$c_{-n}a = \sum_{k=0}^{n-1} f_{-n}\beta_{k-n}(e)a = 0 \quad \text{and}$$

$$\gamma_t(a) = \exp(itc_{-n})a = a.$$

Thus

$$sp_\gamma(a) = \{0\} \quad \text{and} \quad a \in M_{-n} \cap H^\infty(\gamma).$$

Suppose that $B(e) \cap M_{-n}$ is strictly smaller than

$$H^\infty(\gamma) \cap M_{-n} = B(f) \cap M_{-n}.$$

Then, if we let $f(-n)$ be

$$\Lambda\{1 - \beta_{-k}(f):1 \leqq k \leqq n\}$$

(and, hence, $M_{-n} \cap B(f) = f(-n)M_{-n}$), we have

$$f(-n) \geqq e(-n) \quad \text{and} \quad f(-n) \neq e(-n).$$

Therefore there is some $a \in (f(-n) - e(-n))M_{-n}$ and it satisfies: $e(-n)a = 0$ and $a \in B(f)$ (i.e., $sp_\gamma(a) \subseteq \mathbf{Z}_+$). Since $e(-n)a = 0$ we have, for $t \in \mathbf{T}$,

$$\begin{aligned}\gamma_t(a) &= \exp(itc_{-n})a = \exp(itc_n - ite(-n))a \\ &= \exp(it(-f_n \sum_{k=1}^{n} \beta_{-k}(e) - e(-n)))a \\ &= \exp\left(it\left(-\sum_{k=1}^{n} \beta_{-k}(e) - e(-n)\right)\right)a.\end{aligned}$$

But clearly

$$-\sum_{k=1}^{n} \beta_{-k}(e) - e(-n) \leqq -I.$$

Hence it follows from Remark 2.1 that

$$sp_\gamma(a) \subseteq \{n \in \mathbf{Z}:n \leqq -1\}$$

contradicting our assumption that $a \in B(f) = H^\infty(\gamma)$. This contradiction completes the proof that

$$B(e) \cap M_{-n} = H^\infty(\gamma) \cap M_{-n}.$$

Since this holds for each $n \in \mathbf{Z}$, $B(e) = H^\infty(\gamma)$.

COROLLARY 3.10. *Suppose M is a σ-finite von Neumann algebra and $\mathcal{N} = \{0, \ldots, P_{-1} < P_0 < P_1 < P_2, \ldots, I\}$ is a nest of projections in M with*

$$\wedge\{P_n:n \in \mathbf{Z}\} = 0 \text{ and } \vee\{P_n:n \in \mathbf{Z}\} = I.$$

Let $\mathcal{A}$ be the associated nest subalgebra of M (i.e., $\mathcal{A} = M \cap \mathrm{Alg}\,\mathcal{N}$). Then every σ-weakly closed subalgebra of M that contains $\mathcal{A}$ is also a nest subalgebra of M.

Proof. We will use the characterization of nest subalgebras as algebras of the form $H^{\infty}(\gamma)$ for an inner flow γ. (For details see [**3**].) We define a spectral measure P on $\mathbf{R}$ by $P(t, \infty) = P_{[t]}$ (where $[t]$ denotes the integral part of t), and, for $t \in \mathbf{T}$ let V_t be the unitary operator $\int_{\mathbf{R}} e^{its}dP(s)$. We now let α_t be the *-automorphism on M that is implemented by V_t; i.e.,

$$\alpha_t(x) = V_t x V_t^*, \quad x \in M, t \in \mathbf{T}.$$

The map $t \to \alpha_t$ is a homomorphism of $\mathbf{T}$ into the group of inner *-automorphisms on M. By [**3**, Corollary 2.14 and Theorem 4.2.3] $\mathscr{A} = H^{\infty}(\alpha)$. As in the discussion preceding Lemma 3.7 we associate with α unitary operators $\{W_t : t \in \mathbf{T}\}$ and projections $\{E_n : n \in \mathbf{Z}\}$ such that the spectral decomposition of W_t is given by

$$W_t = \sum_{n=-\infty}^{\infty} e^{int}E_n, \quad t \in \mathbf{T}.$$

We have

$$\alpha_t(x) = W_t x W_t^*, \ x \in M, t \in \mathbf{T};$$

hence, for $t \in \mathbf{T}$, $W_t V_t^* \in M'$.

Now let B be a σ-weakly closed subalgebra of M that contains $\mathscr{A}$. We know that $B = H^{\infty}(\gamma)$ and $\gamma_t(x) = U_t x U_t^*$, $x \in M$, $t \in \mathbf{T}$ is a flow associated with some projection $e \in Z(M_0)$ as in the discussion preceding Lemma 3.7. Hence

$$U_t = \sum_{n=-\infty}^{\infty} e^{itc_n}E_n$$

where c_n are the elements of $Z(M_0)$ associated with the projection e.

Now let Q_j be $P_j - P_{j-1}$ for all $j \in \mathbf{Z}$ and then

$$V_t = \sum_{m=-\infty}^{\infty} e^{itm}Q_m \quad \text{and}$$

$$V_t W_t^* = \sum_{m,j=-\infty}^{\infty} e^{itm}Q_m e^{-itj}E_j$$

$$= \sum_{n=-\infty}^{\infty} e^{itn}\left(\sum_{m=-\infty}^{\infty} Q_{n+m}E_m\right).$$

Since, for each $t \in \mathbf{T}$, $V_t W_t^* \in M'$, the projection $\sum_{m=-\infty}^{\infty} Q_{n+m}E_m$ (to be denoted by G_n) also lies in M' for each $n \in \mathbf{Z}$. We have, for each $n, m \in \mathbf{Z}$,

$$G_nE_m = Q_{n+m}E_m = Q_{n+m}G_n = G_nQ_{n+m}$$
$$= (Q_{n+m}G_n)^* = E_mG_n = E_mQ_{n+m}.$$

Fix now $n \in \mathbf{Z}$ and let $T_t^{(n)}$ be $\sum_{j=-\infty}^{\infty} e^{itc_j}f_jQ_{j+n}$, $t \in \mathbf{T}$.

$$T_t^{(n)}U_t^* = \sum_{j,m=-\infty}^{\infty} e^{itc_{j+m}}f_{j+m}Q_{j+m+n}E_me^{-itc_m}$$

$$= \sum_{j,m=-\infty}^{\infty} e^{itf_{j+m}c_m}e^{itf_{j+m}\beta_m(c_j)}Q_{m+j+n}f_mE_mf_{m+j}e^{-itf_{m+j}c_m}.$$

Since $M_0 = \mathscr{A} \cap \mathscr{A}^* = \{P_j:j \in \mathbf{Z}\}'$, $Q_j \in M_0'$ for each $j \in \mathbf{Z}$. We have, therefore,

$$T_t^{(n)}U_t^* = \sum_{j,m=-\infty}^{\infty} e^{itf_{j+m}\beta_m(c_j)}Q_{m+j+n}E_n f_{m+j}$$

$$= \sum_{m,j=-\infty}^{\infty} \beta_m(e^{itc_n})Q_{m+j+n}E_mf_{m+j}$$

$$= \sum_{m,j=-\infty}^{\infty} G_{j+n}\beta_m(e^{itc_j})E_m\beta_m(f_j)$$

$$= \sum_{j=-\infty}^{\infty} G_{j+n}\left(\sum_{m=-\infty}^{\infty} \beta_m(e^{itc_j}f_jE_0)\right).$$

But $\sum_{m=-\infty}^{\infty} \beta_m(e^{itc_j}f_jE_0)$ lies in M' (see Lemma 2.4 (6)). Hence

$$T_t^{(n)}U_t^* \in M' \text{ for each } n \in \mathbf{Z} \text{ and } t \in \mathbf{T}.$$

Let us denote by F_n the projection $\sum_{j=-\infty}^{\infty} f_jQ_{j+n}$. Then it is easy to check that

$$T_t^{(n)*}T_t^{(n)} = T_t^{(n)}T_t^{(n)*} = F_n \text{ for } n \in \mathbf{Z},\ t \in \mathbf{T}.$$

Hence

$$F_n = T_t^{(n)}T_t^{(n)*} = (T_t^{(n)}U_t^*)(T_t^{(n)}U_t^*)^* \in M'.$$

Since, for j, $n \in \mathbf{Z}$, f_j and Q_{j+n} lie in M, $F_n \in M \cap M'$. For each $n \in \mathbf{Z}$,

$$F_n \geqq Q_n \quad \text{and} \quad \sum_{n=-\infty}^{\infty} Q_n = I.$$

Thus $V\{F_n:n \in \mathbf{Z}\} = I$ and we can find a sequence $\{\widetilde{F}_n:n \in \mathbf{Z}\}$ of projections in $M \cap M'$ such that $\widetilde{F}_n\widetilde{F}_m = 0$ for $n \neq m$, $\sum \widetilde{F}_n = I$ and

$\widetilde{F}_n \leqq F_n$.

We now set

$$T_t = \sum_{n=-\infty}^{\infty} T_t^{(n)} \widetilde{F}_n.$$

Then

$$T_t U_t^* = \sum_{n=-\infty}^{\infty} T_t^{(n)} \widetilde{F}_n U_t^* = \sum_{n=-\infty}^{\infty} \widetilde{F}_n T_t^{(n)} U_t^* \in M', \text{ for each } t \in \mathbf{T}.$$

Also, for $t \in \mathbf{T}$,

$$T_t T_t^* = \sum_{n,m=-\infty}^{\infty} T_t^{(n)} \widetilde{F}_n \widetilde{F}_m T_t^{(m)*}$$

$$= \sum_{n=-\infty}^{\infty} T_t^{(n)} \widetilde{F}_n T_t^{(n)*} = \sum_{n=-\infty}^{\infty} \widetilde{F}_n T_t^{(n)} T_t^{(n)*}$$

$$= \sum_{n=-\infty}^{\infty} \widetilde{F}_n F_n = \sum_{n=-\infty}^{\infty} \widetilde{F}_n = I.$$

Similarly $T_t^* T_t = I$ for each $t \in \mathbf{T}$. Hence $\{T_t : t \in \mathbf{T}\}$ is a unitary group of operators ($T_t T_s = T_{t+s}$ for each $t, s \in \mathbf{T}$ since it holds for $\{T_t^{(n)}\}$ for each $n \in \mathbf{Z}$). Also, for $t \in \mathbf{T}$, $x \in M$,

$$\gamma_t(x) = U_t \times U_t^* = T_t \times T_t^*$$

(as $T_t U_t^* \in M'$).

Since $\{T_t : t \in \mathbf{T}\} \subseteq M$ this implies that $H^\infty(\gamma)$ is a nest subalgebra of M. In fact, let $\sum_{m=-\infty}^{\infty} e^{itm} \widetilde{Q}_m$ be the spectral decomposition of T_t and let $\widetilde{P}_n$ be the projection $\sum_{m \leqq n} \widetilde{Q}_m (\in M)$. Then $B = M \cap \operatorname{alg} \widetilde{\mathcal{N}}$ where $\widetilde{\mathcal{N}}$ is the nest $\{0, I\} \cup \{\widetilde{Q}_n : n \in \mathbf{Z}\}$.

Let us denote by $f(\alpha)$ the projection $V\{f_n : n > 0\}$ and by $e(\alpha)$ the projection $V\{e_n : n > 0\} = V\{f_n : n < 0\}$ (cf. [**11**, Proposition 2.7]). Note that

$$(1 - f(\alpha))H^\infty(\alpha) = (1 - f(\alpha))M_0 \quad \text{and}$$
$$H^\infty(\alpha)(1 - e(\alpha)) = M_0(1 - e(\alpha)).$$

LEMMA 3.11. *For projections e, f in $Z(M_0)$, $B(e) = B(f)$ if and only if*

$$(e - ef) \vee (f - ef) \leqq 1 - f(\alpha).$$

In particular, $B(e) = H^\infty(\alpha)$ if and only if $e \geqq f(\alpha)$ and $B(e) = M$ if and only if $e \leqq 1 - f(\alpha)$.

Proof. Since

$$B(e \vee f) = B(e) \cap B(f) \quad \text{and} \quad (e - ef) \vee (f - ef) \leqq 1 - f(\alpha)$$

if and only if $e \vee f - e \leqq 1 - f(\alpha)$ and $e \vee f - f \leqq 1 - f(\alpha)$, we can replace e by $e \vee f$, hence assume that $e \leqq f$. We now have to show $B(e) = B(f)$ if and only if $e - f \leqq 1 - f(\alpha)$ (where $e \geqq f$).

From the definition of $B(e)$ (and $B(f)$) it follows that $B(e) = B(f)$ if and only if, for each $n > 0$,

$$(1) \quad f_{-n}(\Lambda\{1 - \beta_{-m}(e):1 \leqq m \leqq n\}) \\ = f_{-n}(\Lambda\{1 - \beta_{-m}(f):1 \leqq m \leqq n\}).$$

Suppose now that $e - f \leqq 1 - f(\alpha)$, then for each $m > 0$,

$$e - f \leqq 1 - f_m = 1 - e_{-m}.$$

Hence, for $m > 0$, $\beta_{-m}(e - f) = 0$ and (1) follows for each $n > 0$.

For the other direction, suppose that (1) holds for each $n > 0$ and that $e - f \nleqq 1 - f(\alpha)$. Then there is a positive integer j such that $(e - f)f_j \neq 0$ and $(e - f)f_m = 0$ for each $0 < m < j$. Since $(e - f)f_m = 0$,

$$\beta_{-m}(e - f) = 0 \quad \text{for } 0 < m < j.$$

Hence

$$\Lambda\{1 - \beta_{-m}(f):1 \leqq m < j\} \\ = f_{-j}(1 - \beta_{-j}((f))(\Lambda\{1 - \beta_{-m}(e):1 \leqq m < j\})$$

and (1) implies that

$$f_{-j}(1 - \beta_{-j}(e))(\Lambda\{1 - \beta_{-m}(e):1 \leqq m < j\}) \\ = f_{-j}(1 - \beta_{-j}(f))(\Lambda\{1 - \beta_{-m}(e):1 \leqq m < j\}).$$

Therefore

$$\begin{aligned} \beta_{-j}(e - f) &= \beta_{-j}(1 - f) - \beta_{-j}(1 - e) \\ &\leqq 1 - \Lambda\{1 - \beta_{-m}(e):1 \leqq m < j\} \\ &= V\{\beta_{-m}(e):1 \leqq m < j\}. \end{aligned}$$

But

$$\beta_{-j}(e - f)\beta_{-m}(e) = \beta_{-j}[(e - f)\beta_{j-m}(e)] \\ \leqq \beta_{-j}[(e - f)f_{j-m}] = 0$$

(as $(e - f)f_m = 0$ for $0 < m < j$) for $0 < m < j$. Thus

$$\beta_{-j}(e - f) = 0 \quad \text{and} \quad f_j(e - f) = \beta_j(\beta_{-j}(e - f)) = 0,$$

contradicting our assumption. Hence it follows from (1) that

$$e - f \leqq 1 - f(\alpha).$$

The last assertion of the lemma follows from the fact that $H^{\infty}(\alpha) = B(I)$ and $M = B(0)$.

COROLLARY 3.12. *Let e be a projection in $Z(M_0)$. Then $B(e)$ is a maximal σ-weakly closed subalgebra of M if and only if $ef(\alpha)M_0$ is a factor (or $ef(\alpha)M_0 = \{0\}$).*

In particular, $H^{\infty}(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if $f(\alpha)M_0$ is a factor.

Proof. Suppose $ef(\alpha)M_0$ is a factor or $ef(\alpha) = 0$. Then each projection $z \in Z(M_0)$ that satisfies $z \leqq ef(\alpha)$ is either 0 or $ef(\alpha)$. Hence, for each such z, $B(z) = M$ (if $z = 0$) or $B(z) = B(e)$ (if $z = ef(\alpha)$, as

$$e - z = e(1 - f(\alpha)) \leqq 1 - f(\alpha)).$$

If there is some projection $f \in Z(M_0)$ such that $B(f) \supseteq B(e)$ then $B(f) = B(ff(\alpha))$ (by the previous lemma) and

$$B(fef(\alpha)) = B(f) \cup B(e) = B(f) \supseteq B(e).$$

But $fef(\alpha) \leqq ef(\alpha)$; hence $B(f) = B(e)$ or $B(f) = M$.

Now suppose that $B(e)$ is a maximal σ-weakly closed subalgebra of M. If $ef(\alpha)M_0$ is not a factor and $ef(\alpha) \neq 0$ then there is some projection $q \leqq ef(\alpha)$ in $Z(M_0)$ such that $q \neq 0$ and $q \neq ef(\alpha)$. It follows that

$$q \nleqq 1 - f(\alpha) \text{ and } e - q \nleqq 1 - f(\alpha).$$

Hence (by the previous lemma) $B(q) \neq M$ and $B(q) \neq B(e)$. Since $B(e)$ is a maximal σ-weakly closed subalgebra this cannot occur and, hence, $ef(\alpha)M_0$ is a factor or $ef(\alpha) = 0$.

The last assertion follows immediately.

For analytic crossed products it was proved in [**4**] that the maximality of H^{∞} is equivalent to M_0 being a factor. The next corollary also extends a result that was known for analytic crossed products (see [**5**]).

COROLLARY 3.13. *The following conditions are equivalent:*

(1) *For each σ-weakly closed subalgebra B of M that contains $H^{\infty}(\alpha)$ there is a projection $q \in Z(M_0)$ such that*

$$B = qM + (1 - q)H^{\infty}(\alpha).$$

(2) $f(\alpha)e(\alpha)Z(M_0) \subseteq Z(M)$.

Proof. (1) implies (2): Let e be a projection in $f(\alpha)e(\alpha)Z(M_0)$ and suppose that $j > 0$ is such that

$$\beta_{-m}(e) \leqq e \quad \text{for each } 0 \leqq m < j.$$

Let p be the projection $e\beta_j(1 - e)$. Then p satisfies the following properties:

(i) For each $m \in \mathbf{Z}$,

$$\beta_{j+m}(p)\beta_m(p) = 0.$$

(ii) For each $0 < m < j$ and $n \in \mathbf{Z}$,

$$f_n\beta_{n-m}(p) = 0.$$

In particular $\beta_{-m}(p) = 0$.

(iii) For each $m \in \mathbf{Z}$,

$$\beta_m(p) \leqq f_{m+j}.$$

Indeed, to prove (i) note that

$$\beta_{j+m}(p) \leqq \beta_{j+m}(e) \quad \text{and}$$

$$\beta_m(p) \leqq \beta_m(\beta_j(1 - e)) \leqq \beta_{m+j}(1 - e).$$

We assumed that $\beta_{-m}(e) \leqq e$ for $0 < m < j$. Hence

$$f_{m-j}\beta_{-j}(e) = \beta_{m-j}(\beta_{-m}(e)) \leqq \beta_{m-j}(e) \leqq e \quad \text{for } 0 < m < j$$

and it follows that

$$f_{m-j}\beta_{-j}(p) = f_{m-j}\beta_{-j}(e)(1 - e) = 0.$$

Thus

$$f_m p = f_m f_j p = \beta_j(f_{m-j}\beta_{-j}(p)) = 0$$

and consequently

$$\beta_{-m}(p) \leqq \beta_{-m}(1 - f_m) = 0.$$

Property (ii) follows by applying β_n to $\beta_{-m}(p) = 0$. Property (iii) is an immediate consequence of the fact that $p \leqq f_j$.

Consider now the algebra $B(1 - p)$. By (1) there is a projection $q \in Z(M_0)$ such that

$$B(1 - p) = qM + (1 - q)H^{\infty}(\alpha).$$

This implies that for each $n > 0$,

$$qf_{-n} = f_{-n}(\wedge\{1 - \beta_{-m}(1 - p):0 < m \leqq n\}).$$

But then

$$qf_{-n} = \beta_{-n}(p)(\wedge\{1 - f_{-m} + \beta_{-m}(p):0 < m < n\}).$$

By (ii) $f_{-m}\beta_{-n}(p) = 0$ for $0 < m < n \leqq j$. Hence

$$qf_{-n} = \beta_{-n}(p) \quad \text{for } n \leqq j$$

(in fact, for $0 \leqq n < j$, $qf_{-n} = \beta_{-n}(p) = 0$ by (ii)).

If $n > j$ then

$$qf_{-n} \leqq \beta_{-n}(p)(1 - f_{-n+j} + \beta_{-n+j}(p)) = 0$$

(applying (i) and (iii)). It follows that, for $n > j$,

$$f_{-n}\beta_{-j}(p) = f_{-n}f_{-j}q = 0$$

and consequently

$$\beta_{-j}(p) \leqq 1 - f_{-n} \quad \text{and}$$

$$p = f_j p = \beta_j(\beta_{-}j(p)) \leqq \beta_{-j}(1 - f_{-n}) \leqq 1 - f_{j-n}$$

for each $n > j$.

Hence $p \leqq 1 - e(\alpha)$. But $p \leqq e \leqq e(\alpha)$ and thus

$$0 = p = e\beta_j(1 - e)$$

and, by applying β_{-j},

$$\beta_{-j}(e)(1 - e) = 0.$$

Hence $\beta_{-j}(e) \leqq e$. By induction we find that for each projection $e \in e(\alpha)f(\alpha)Z(M_0)$ and each $j > 0$, $\beta_{-j}(e) \leqq e$.

Fix now a projection $e \in e(\alpha)f(\alpha)Z(M_0)$ and suppose that $j > 0$ is such that for each $0 \leqq m < j$, $\beta_m(e) \leqq e$. We will show that $\beta_j(e) \leqq e$ and this induction argument will imply that $\beta_n(e) \leqq e$ for each $n \in \mathbf{Z}$ and, hence, that e lies in $Z(M)$ (by Lemma 2.4(5)).

Let p be the projection $e\beta_{-j}(1 - e)$. Then for $n > 0$,

$$\beta_{-n}(p) \leqq p \leqq f_{-j}$$

(since $p \leqq e \leqq e(\alpha)f(\alpha)$). Also

$$\beta_j(p) = \beta_j(e)(1 - e) \leqq 1 - e \leqq 1 - p \quad \text{and}$$

$$f_j p = \beta_j(\beta_{-j}(p)) \leqq \beta_j(p) \leqq 1 - p.$$

Hence $f_j p = 0$ and consequently $\beta_{-j}(p) = 0$. Consider now the algebra $B(1 - p)$. Then there is a projection $q \in Z(M_0)$ such that

$$B(1 - p) = qM + (1 - q)H^{\infty}(\alpha).$$

Hence, for $n > 0$.

$$qf_{-n} = \beta_{-n}(p)(\Lambda\{1 - f_{-m} + \beta_{-m}(p):0 < m < n\}).$$

For $n = j$,

$$\beta_{-n}(p) = \beta_{-j}(p) = 0,$$

hence $qf_{-j} = 0$. For $n \neq j$

$$qf_{-n} \leqq \beta_{-n}(p) \leqq f_{-j}.$$

Thus

$$qf_{-n} = qf_{-n}f_{-j} \leqq qf_{-j} = 0.$$

This implies that

$$B(1 - p) = qM + (1 - q)H^{\infty}(\alpha) = H^{\infty}(\alpha)$$

and, by Lemma 3.11,

$$p \leqq 1 - f(\alpha).$$

But $p \leqq e \leqq f(\alpha)$ and consequently $p = 0$. Since $p = e\beta_{-j}(1 - e)$,

$$0 = \beta_j(e)(1 - e) \quad \text{and} \quad \beta_j(e) \leqq e.$$

This completes the proof that

$$e(\alpha)f(\alpha)Z(M_0) \subseteq Z(M).$$

(2) implies (1): Suppose that

$$e(\alpha)f(\alpha)Z(M_0) \subseteq Z(M).$$

Let e be a projection in $Z(M_0)$ and write $e = p_1 + p_2 + p_3$ where

$$p_1 = ee(\alpha)f(\alpha), p_2 = ee(\alpha)(1 - f(\alpha)) \quad \text{and} \quad p_3 = e(1 - e(\alpha)).$$

Then $B(1 - p_2)$ is $H^{\infty}(\alpha)$ (by Lemma 3.11). We now show that $B(1 - p_1)$ and $B(1 - p_3)$ have the property described in (1).

For each $n > 0, f_{-n}p_3 = 0$ hence $\beta_n(p_3) = 0$. But then, for $m \in \mathbf{Z}$ and $n > 0$,

$$f_m\beta_{m+n}(p_3) = \beta_m(\beta_n(p_3)) = 0.$$

Hence

$$\beta_m(p_3)\beta_n(p_3) = 0 \quad \text{for } n \neq m \text{ in } \mathbf{Z}.$$

For each $n > 0$ let $z(-n)$ be the projection in $Z(M_0)$ that satisfies

$$B(1 - p_3) \cap M_{-n} = z(-n)M_{-n}.$$

Then

$$z(-n) = \beta_{-n}(p_3)(\Lambda\{1 - f_{-m} + \beta_{-m}(p_3):0 < m < n\}).$$

Since $\beta_{-n}(p_3)\beta_{-m}(p_3)$ whenever $n \neq m$,

$$z(-n) = \beta_{-n}(p_3)(\Lambda\{1 - f_{-m}:0 < m < n\}) \quad \text{and}$$

$$z(-n)z(-j) = 0 \quad \text{if } n \neq j.$$

Let q_3 be $\sum_{n=1}^{\infty} z(-n)$. If $0 < m < n$ then

$$z(-n) \leqq 1 - f_{-m}.$$

If $m > n > 0$ then

$$f_{-m}\beta_{-n}(p_3) = 0$$

(because $f_m\beta_{m+n}(p_3) = 0$ for $m \in \mathbf{Z}$, $n > 0$) and consequently $f_{-m}z(-n) = 0$. We see, therefore, that

$$z(-n)f_{-m} = 0 \quad \text{for all } n \neq m,\ n,\ m > 0.$$

It follows from this that

$$q_3 f_{-m} = z(-m) \quad \text{for each } m > 0.$$

Hence

$$B(1 - p_3) = q_3 M + (1 - q_3)H^{\infty}(\alpha).$$

Now consider the algebra $B(1 - p_1)$ and write $z(-n)$ for

$$\beta_{-n}(p_1)(\Lambda\{1 - f_{-m} + \beta_{-m}(p_1):0 < m < n\})$$

(such that $B(1 - p_1) \cap M_{-n} = z(-n)M_{-n}$) for each $n > 0$. But

$$p_1 \in e(\alpha)f(\alpha)Z(M_0) \subseteq Z(M).$$

Hence

$$1 - p_1 \in Z(M) \quad \text{and}$$

$$\beta_{-m}(1 - p_1) = f_{-m}(1 - p_1),\ m \in \mathbf{Z}.$$

Consequently

$$z(-n) = f_{-n}p_1(\Lambda\{1 - f_{-m}(1 - p_1):0 < m < n\} = f_{-n}p_1.$$

Therefore

$$B(1 - e) = V\{B(1 - p_i):i = 1, 2, 3\}$$
$$= (q_3 + p_1)M + (1 - q_3 - p_1)H^{\infty}(\alpha).$$

Since any σ-weakly closed subalgebra of M that contains $H^{\infty}(\alpha)$ is $B(1 - e)$ for some projection $e \in Z(M_0)$, (1) follows.

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