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Large Sieve Inequalities via Subharmonic Methods and the Mahler Measure of the Fekete Polynomials

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Abstract. We investigate large sieve inequalities such as

$$\frac{1}{m}\sum_{j=1}^{m}\psi(\log|P(e^{i\tau_j})|)\leq\frac{C}{2\pi}\int_0^{2\pi}\psi\left(\log[e|P(e^{i\tau})|]\right)\,d\tau,$$

where ψ is convex and increasing, P is a polynomial or an exponential of a potential, and the constant C depends on the degree of P, and the distribution of the points $0\leq\tau_1<\tau_2<\cdots<\tau_m\leq 2\pi$. The method allows greater generality and is in some ways simpler than earlier ones. We apply our results to estimate the Mahler measure of Fekete polynomials.

1 Results

The large sieve of number theory [14, p. 559] asserts that if

$$P(z)=\sum_{k=-n}^{n}a_kz^k$$

is a trigonometric polyonomial of degree $\leq n$,

$$0\leq\tau_1<\tau_2<\cdots<\tau_m\leq 2\pi,$$

and

$$\delta:=\min\{\tau_2-\tau_1,\tau_3-\tau_2,\ldots,\tau_m-\tau_{m-1},2\pi-(\tau_m-\tau_1)\},$$

then

$$\sum_{j=1}^{m}|P(e^{i\tau_j})|^2\leq\left(\frac{n}{2\pi}+\delta^{-1}\right)\int_0^{2\pi}|P(e^{i\tau})|^2\,d\tau. \tag{1}$$

There are numerous extensions of this to L_p norms, or involving $\psi(|P(e^{i\tau})|^p)$, where ψ is a convex function and $p>0$ [8, 12]. There are versions that estimate Riemann sums, for example,

$$\sum_{j=1}^{m}|P(e^{i\tau_j})|^2(\tau_j-\tau_{j-1})\leq C\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\tau})|^2d\tau, \tag{2}$$

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with C independent of n, P, $\{\tau_1, \tau_2, \dots, \tau_m\}$. These are often called forward Marcinkiewicz–Zygmund inequalities. Converse Marcinkiewicz–Zygmund inequalities provide estimates for the integrals above in terms of the sums on the left-hand side [11, 13, 16].

A particularly interesting case is that of the L_0 norm. A result of the first author asserts that if $\{z_1, z_2, \dots, z_n\}$ are the n-th roots of unity and P is a polynomial of degree $\leq n$, then

$$\prod_{j=1}^{n} |P(z_j)|^{1/n} \leq 2M_0(P), \tag{3}$$

where

$$M_0(P) := \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{it})|\,dt\Big)$$

is the Mahler measure of P.

The focus of this paper is to show that methods of subharmonic function theory provide a simple and direct way to generalize previous results. We also extend (3) to points other than the roots of unity. Given $c \geq 0$, $\kappa \in [0, \infty)$, and a positive measure ν of compact support and total mass at most $\kappa \geq 0$ on the plane, we define the associated exponential of its potential by

$$P(z) = c\exp\Big(\int \log|z-t|\,d\nu(t)\Big).$$

We say that this is an *exponential of a potential of mass* $\leq \kappa$, and that its *degree* is $\leq \kappa$. The set of all such functions is denoted by $\mathbb{P}_\kappa$. Note that if P is a polynomial of degree $\leq n$, then $|P| \in \mathbb{P}_n$. More generally, the generalized polynomials studied by several authors [3, 7] also lie in $\mathbb{P}_\kappa$, for an appropriate κ. We prove the following.

Theorem 1.1 *Let $\psi\colon \mathbb{R} \to [0, \infty)$ be nondecreasing and convex. Let $m \geq 1$, $\kappa > 0$, $\alpha > 0$, and $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq 2\pi$. Let $w_j \geq 0$, $1 \leq j \leq m$, with*

$$\sum_{j=1}^{m} w_j = 1.$$

Let μ_m denote the corresponding Riemann–Stieltjes measure, defined for $\theta \in [0, 2\pi]$ by

$$\mu_m([0,\theta]) := \sum_{j:\tau_j \leq \theta} w_j.$$

Let

$$\Delta := \sup\Big\{\Big|\mu_m([0,\theta]) - \frac{\theta}{2\pi}\Big| : \theta \in [0, 2\pi]\Big\} \tag{4}$$

denote the discrepancy of μ_m. Then for $P \in \mathbb{P}_\kappa$,

$$\sum_{j=1}^{m} w_j\psi(\log P(e^{i\tau_j})) \leq \Big(1 + \frac{8}{\alpha}\kappa\Delta\Big)\frac{1}{2\pi}\int_0^{2\pi} \psi(\log[e^{\alpha}P(e^{i\theta})])\,d\theta. \tag{5}$$

Example 1 Let us choose all equal weights,

$$w_j = \frac{1}{m}, \quad 1 \le j \le m.$$

Then μ_m is counting measure,

$$\mu_m([0,\theta]) = \frac{1}{m}\#\{j : \tau_j \in [0,\theta]\}.$$

If we take $\psi(t) = \max\{0,t\}$, and $\alpha = 1$, and use the notation $\log^+ t = \max\{0, \log t\}$, we obtain

$$(6) \qquad \frac{1}{m}\sum_{j=1}^{m} \log^+ P(e^{i\tau_j}) \le (1 + 8\kappa\Delta)\frac{1}{2\pi}\int_0^{2\pi} \log^+[eP(e^{i\theta})]\, d\theta.$$

This result is new. Previous inequalities have been limited to sums involving $\psi(P(e^{i\tau_j})^p)$ for some $p > 0$. If we let $p > 0$, $\psi(t) = e^{pt}$, and $\alpha = \frac{1}{p}$, then (5) becomes

$$(7) \qquad \frac{1}{m}\sum_{j=1}^{m} P(e^{i\tau_j})^p \le (1 + 8p\kappa\Delta)\frac{e}{2\pi}\int_0^{2\pi} P(e^{i\theta})^p\, d\theta.$$

This choice of α is not optimal. The optimal choice is

$$\alpha = 4\kappa\Delta\left[-1 + \sqrt{1 + \frac{1}{2p\kappa\Delta}}\,\right],$$

but one needs further information on the size of $p\kappa\Delta$ to exploit this. For example, if $p\kappa\Delta \le 1$, the optimal choice is of order $\sqrt{\frac{\kappa\Delta}{p}}$, and choosing this α in (5), we obtain

$$(8) \qquad \frac{1}{m}\sum_{j=1}^{m} P(e^{i\tau_j})^p \le \left(1 + C\sqrt{p\kappa\Delta}\right)\frac{1}{2\pi}\int_0^{2\pi} P(e^{i\theta})^p\, d\theta,$$

where C is an absolute constant.

For well-distributed $\{\tau_1, \tau_2, \dots, \tau_m\}$, Δ is of order $\frac{1}{m}$. In particular, when these points are equally spaced and include 2π, but not 0, so that

$$\tau_j = \frac{2j\pi}{m}, \quad 1 \le j \le m,$$

we have $\Delta = \frac{2\pi}{m}$, and (7) becomes

$$(9) \qquad \frac{1}{m}\sum_{j=1}^{m} P(e^{i\tau_j})^p \le \left(1 + \frac{16\pi p\kappa}{m}\right)\frac{e}{2\pi}\int_0^{2\pi} P(e^{i\theta})^p\, d\theta.$$

Example 2 Another important choice of the weights w_j is

$$w_j = \frac{\tau_j - \tau_{j-1}}{2\pi}, \quad 1 \le j \le m,$$

where now we assume $\tau_0 = 0$ and $\tau_m = 2\pi$. For this case (5) becomes an estimate for Riemann sums,

$$\frac{1}{2\pi}\sum_{j=1}^{m}(\tau_j - \tau_{j-1})\psi(\log P(e^{i\tau_j})) \le \Bigl(1+\frac{8}{\alpha}\kappa\Delta\Bigr)\frac{1}{2\pi}\int_0^{2\pi}\psi(\log[(e^{\alpha}P(e^{i\theta})])\,d\theta. \tag{10}$$

The discrepancy Δ in this case is

$$\Delta = \sup_j \frac{\tau_j - \tau_{j-1}}{2\pi}.$$

Remarks

(a) In many ways, the approach of this paper is simpler than that in [12] where Dirichlet kernels were used, or that of [8], where Carleson measures were used. The main idea is to use the Poisson integral inequality for subharmonic functions.

(b) We can reformulate (5) as

$$\int_0^{2\pi}\psi(\log|P(e^{i\tau})|)\,d\mu_m(\tau) \le \Bigl(1+\frac{8}{\alpha}\kappa\Delta\Bigr)\frac{1}{2\pi}\int_0^{2\pi}\psi(\log[e^{\alpha}P(e^{i\theta})])\,d\theta.$$

In fact this estimate holds for any probability measure μ_m on $[0, 2\pi]$, not just the pure jump measures above.

(c) The one severe restriction above is that ψ is nonnegative.

In particular, this excludes $\psi(x) = x$. For that case, we prove two different results.

Theorem 1.2 *Assume that m, κ, $\{\tau_1, \tau_2, \dots, \tau_m\}$ and $\{w_1, w_2, \dots, w_m\}$ are as in Theorem* 1.1. *Let*

$$Q(z) = \prod_{j=1}^{m}|z - e^{i\tau_j}|^{w_j}. \tag{11}$$

Then for $P \in \mathbb{P}_\kappa$,

$$\sum_{j=1}^{m} w_j \log P(e^{i\tau_j}) \le \frac{1}{2\pi}\int_0^{2\pi}\log P(e^{i\theta})\,d\theta + \kappa\log\|Q\|_{L_\infty(|z|=1)}. \tag{12}$$

Remarks If we choose all $w_j = \frac{1}{m}$, this yields

$$\prod_{j=1}^{m} P(e^{i\tau_j})^{1/m} \le \|Q\|^{\kappa}_{L_\infty(|z|=1)}\exp\Bigl(\frac{1}{2\pi}\int_0^{2\pi}\log P(e^{i\theta})\,d\theta\Bigr). \tag{13}$$

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If we take $\{e^{i\tau_1}, e^{i\tau_2}, \dots, e^{i\tau_m}\}$ to be the m-th roots of unity, then $Q(z) = |z^m - 1|^{1/m}$ and (13) becomes

$$\prod_{j=1}^{m} P(e^{i\tau_j})^{1/m} \le 2^{\kappa/m} \exp\Big(\frac{1}{2\pi} \int_0^{2\pi} \log P(e^{i\theta})\, d\theta \Big). \tag{14}$$

In the case $\kappa = m = n$, this gives the first author's inequality (3). In general, however, it is not easy to bound $\|Q\|_{L_\infty(|z|=1)}$. Using an alternative method, we can avoid the term involving Q when the spacing between successive τ_j is $O(\kappa^{-1})$.

Theorem 1.3 *Assume that m, κ and $\{\tau_1, \tau_2, \dots, \tau_m\}$ are as in Theorem* 1.1. *Let $\tau_0 := \tau_m - 2\pi$ and $\tau_{m+1} := \tau_1 + 2\pi$. Let*

$$\delta := \max\{\tau_1 - \tau_0, \tau_2 - \tau_1, \dots, \tau_m - \tau_{m-1}\}.$$

Let $A > 0$. There exists $B > 0$ such that if $\kappa \ge 1$ and $\delta \le A\kappa^{-1}$, then for all $P \in \mathbb{P}_\kappa$,

$$\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} \log P(e^{i\tau_j}) \le \int_0^{2\pi} \log P(e^{i\theta})\, d\theta + B. \tag{15}$$

One application of Theorem 1.2 is to the estimation of Mahler measure. Recall that for a bounded measurable function Q on $[0, 2\pi]$, its Mahler measure is

$$M_0(Q) = \exp\Big(\frac{1}{2\pi} \int_0^{2\pi} \log |Q(e^{i\theta})|\, d\theta \Big).$$

It is well known that $M_0(Q) = \lim_{p\to 0+} M_p(Q)$, where for $p > 0$,

$$M_p(Q) := \|Q\|_p := \Big(\frac{1}{2\pi} \int_0^{2\pi} |Q(e^{i\theta})|^p\, d\theta \Big)^{1/p}.$$

It is a simple consequence of Jensen's formula that if

$$Q(z) = c \prod_{k=1}^{n} (z - z_k)$$

is a polynomial, then

$$M_0(Q) = |c| \prod_{k=1}^{n} \max\{1, |z_k|\}.$$

The construction of polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The Littlewood polynomials,

$$L_n := \Big\{ p : p(z) = \sum_{k=0}^{n} \alpha_k z^k,\ \alpha_k \in \{-1, 1\} \Big\},$$

which have coefficients ± 1, and the unimodular polynomials,

$$K_n := \left\{ p : p(z) = \sum_{k=0}^{n} \alpha_k z^k,\ |\alpha_k| = 1 \right\},$$

are two of the most important classes considered. Beller and Newman [1] constructed unimodular polynomials of degree n whose Mahler measure is at least $\sqrt{n} - c/\log n$. Here we show that for Littlewood polynomials, we can achieve almost $\frac{1}{2}\sqrt{n}$ by considering the Fekete polynomials.

For a prime number p, the p-th Fekete polynomial is

$$f_p(z) = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k,$$

where

$$\left(\frac{k}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv k \pmod{p} \text{ has a non-zero solution } x, \\ 0 & \text{if } p \text{ divides } k, \\ -1 & \text{otherwise.} \end{cases}$$

Since f_p has constant coefficient 0, it is not a Littlewood polynomial, but

$$g_p(z) = f_p(z)/z$$

is a Littlewood polynomial which has the same Mahler measure as f_p. Fekete polynomials are examined in detail in [2, pp. 37–42].

Theorem 1.4 *Let $\varepsilon > 0$. For large enough prime p, we have*

$$M_0(f_p) = M_0(g_p) \geq \left(\frac{1}{2} - \varepsilon\right)\sqrt{p}. \tag{16}$$

Remarks From Jensen's inequality, $M_0(f_p) \leq \|f_p\|_2 = \sqrt{p-1}$. However $\frac{1}{2} - \varepsilon$ in Theorem 1.4 cannot be replaced by $1 - \varepsilon$. Indeed if p is prime, and we write $p = 4m + 1$, then g_p is self-reciprocal, that is, $z^{p-1} g_p\left(\frac{1}{z}\right) = g_p(z)$, and hence

$$g_p(e^{2it}) = e^{i(p-2)t} \sum_{k=0}^{(p-3)/2} a_k \cos((2k+1)t), \quad a_k \in \{-2, 2\}.$$

A result of Littlewood [10, Theorem 2] implies that

$$M_0(f_p) = M_0(g_p) \leq \frac{1}{2\pi} \int_0^{2\pi} |g_p(e^{2it})|\, dt \leq (1 - \varepsilon_0)\sqrt{p-1},$$

for some absolute constant $\varepsilon_0 > 0$. It is an interesting question whether there is a sequence of Littlewood polynomials (f_n) with $f_n \in L_n$ such that, for an arbitrary $\varepsilon > 0$ and n large enough, $M_0(f_n) \geq (1 - \varepsilon)\sqrt{n}$.

The results are proved in the next section.

2 Proofs

We assume the notation of Theorem 1.1. We let

$$r = 1 + \frac{\alpha}{\kappa}, \tag{17}$$

and define the Poisson kernel for the ball $|z| \le r$ (*cf.* [15, p. 8]),

$$\mathcal{P}_r(se^{i\theta}, re^{it}) = \frac{r^2 - s^2}{r^2 - 2rs\cos(t-\theta) + s^2},$$

where $0 \le s < r$ and $t, \theta \in \mathbb{R}$.

Proof of Theorem 1.1

Step 1: The Basic Inequality Let $P \in \mathbb{P}_\kappa \setminus \{0\}$, so that for some $c > 0$ and some measure ν with total mass $\le \kappa$ and compact support,

$$\log P(z) = \log c + \int \log |z - t|\, d\nu(t).$$

As $\log P$ is subharmonic, and as ψ is convex and increasing, $\psi(\log P)$ is subharmonic [15, Theorem 2.6.3, p. 43]. Then we have, for $|z| < r$, the inequality [15, Theorem 2.4.1, p. 35]

$$\psi(\log P(z)) \le \frac{1}{2\pi} \int_0^{2\pi} \psi(\log P(re^{it}))\mathcal{P}_r(z, re^{it})\, dt.$$

Choosing $z = e^{i\tau_j}$, multiplying by w_j, and summing over j gives

$$\sum_{j=1}^{m} w_j \psi(\log P(e^{i\tau_j})) - \frac{1}{2\pi}\int_0^{2\pi} \psi(\log P(re^{it}))\, dt \le \frac{1}{2\pi}\int_0^{2\pi} \psi(\log P(re^{it}))\mathcal{H}(t)\, dt, \tag{18}$$

where

$$\mathcal{H}(t) := \sum_{j=1}^{m} w_j \mathcal{P}_r(e^{i\tau_j}, re^{it}) - 1 = \int_0^{2\pi} \mathcal{P}_r(e^{i\tau}, re^{it})\, d\Big(\mu_m(\tau) - \frac{\tau}{2\pi}\Big).$$

Here we have used the elementary property of the Poisson kernel, that it integrates to 1 over any circle with center 0 inside its ball of definition.

Step 2: Estimating $\mathcal{H}$ We integrate this relation by parts, and note that both

$$\mu_m[0,0] = 0 \quad \text{and} \quad \mu_m[0, 2\pi] = 1.$$

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This gives

$$\mathcal{H}(t) = -\int_0^{2\pi} \Big(\frac{\partial}{\partial\tau}\mathcal{P}_r(e^{i\tau}, re^{it})\Big)\Big(\mu_m([0,\tau]) - \frac{\tau}{2\pi}\Big)\, d\tau,$$

and hence

$$|\mathcal{H}(t)| \le \Delta \int_0^{2\pi} \Big|\frac{\partial}{\partial\tau}\mathcal{P}_r(e^{i\tau}, re^{it})\Big|\, d\tau. \tag{19}$$

Now

$$\frac{\partial}{\partial\tau}\mathcal{P}_r(e^{i\tau}, re^{it}) = \frac{(r^2-1)2r\sin(t-\tau)}{(r^2-2r\cos(t-\tau)+1)^2},$$

so a substitution $s = t - \tau$ and 2π-periodicity give

$$\begin{aligned}\int_0^{2\pi} \Big|\frac{\partial}{\partial\tau}\mathcal{P}_r(e^{i\tau}, re^{it})\Big|\, d\tau &= \int_{-\pi}^{\pi} \Big|\frac{\partial}{\partial s}\mathcal{P}_r(e^{is}, r)\Big|\, ds \\ &= -2\int_0^{\pi} \frac{\partial}{\partial s}\mathcal{P}_r(e^{is}, r)\, ds \\ &= -2[\mathcal{P}_r(e^{i\pi}, r) - \mathcal{P}_r(1, r)] = \frac{8r}{r^2-1}.\end{aligned} \tag{20}$$

Combining (18)–(20), gives

$$\sum_{j=1}^{m} w_j \psi(\log P(e^{i\tau_j})) \le \Big(1 + \Delta\frac{8r}{r^2-1}\Big)\frac{1}{2\pi}\int_0^{2\pi} \psi(\log P(re^{it}))\, dt. \tag{21}$$

Step 3: Return to the Unit Circle Next, we estimate the integral on the right-hand side in terms of an integral over the unit circle. Let us assume that ν has total mass $\lambda(\le \kappa)$. Let $S(z) = |z|^{\lambda} P(\frac{r}{z})$, so that $\log S(z) = \log c + \int \log|r - tz|\, d\nu(t)$, a function subharmonic in $\mathbb{C}$. Then the same is true of $\psi(\log S)$, so its integrals over circles with centre 0 increase with the radius [15, Theorem 2.6.8, p. 46]. In particular, recalling our choice (17) of r,

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(\log S(e^{i\theta}))\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \psi(\log S(re^{i\theta}))\, d\theta,$$

and a substitution $\theta \to -\theta$ gives

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \psi(\log P(re^{i\theta}))\, d\theta &\le \frac{1}{2\pi}\int_0^{2\pi} \psi(\lambda\log r + \log P(e^{i\theta}))\, d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \psi(\kappa\log r + \log P(e^{i\theta}))\, d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \psi(\alpha + \log P(e^{i\theta}))\, d\theta.\end{aligned}$$

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Then (21) becomes

$$\sum_{j=1}^{m} w_j \psi(\log P(e^{i\tau_j})) \leq \Big(1 + \Delta \frac{8r}{r^2-1}\Big) \frac{1}{2\pi} \int_0^{2\pi} \psi(\log[e^{\alpha} P(e^{i\theta})])\, d\theta$$
$$\leq \Big(1 + 8\Delta \frac{\kappa}{\alpha}\Big) \frac{1}{2\pi} \int_0^{2\pi} \psi(\log[e^{\alpha} P(e^{i\theta})])\, d\theta. \qquad \blacksquare$$

Proof of Theorem 1.2 Write $\log P(z) = \log c + \int \log |z - t|\, d\nu(t)$, so (recall (11)),

$$\sum_{j=1}^{m} w_j \log P(e^{i\tau_j}) = \log c + \int \Big(\sum_{j=1}^{m} w_j \log |e^{i\tau_j} - t|\Big)\, d\nu(t) \tag{22}$$
$$= \log c + \int \log Q(t)\, d\nu(t).$$

Now as all zeros of Q are on the unit circle,

$$g(u) := \log Q(u) - \log \|Q\|_{L_\infty(|z|=1)} - \log |u|$$

is harmonic in the exterior $\{u : |u| > 1\}$ of the unit ball, with finite limit at ∞, and with $g(u) \leq 0$ for $|u| = 1$. By the maximum principle for subharmonic functions,

$$g(u) \leq 0, \quad |u| > 1.$$

We deduce that for $|u| > 1$, $\log Q(u) \leq \log \|Q\|_{L_\infty(|z|=1)} + \log^+ |u|$. Moreover, inside the unit ball, we can regard Q as the absolute value of a function analytic there (with any choice of branches). So the last inequality holds for all $u \in \mathbb{C}$. Then, assuming (as above) that ν has total mass $\lambda \leq \kappa$,

$$\int \log Q(t) d\nu(t) \leq \lambda \log \|Q\|_{L_\infty(|z|=1)} + \int \log^+ |t|\, d\nu(t) \tag{23}$$
$$= \lambda \log \|Q\|_{L_\infty(|z|=1)} + \int \Big(\frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - t| d\theta\Big)\, d\nu(t)$$
$$\leq \kappa \log \|Q\|_{L_\infty(|z|=1)} + \frac{1}{2\pi} \int_0^{2\pi} \Big(\int \log |e^{i\theta} - t|\, d\nu(t)\Big)\, d\theta.$$

In the second line we used a well-known identity [15, Exercise 2.2, p. 29], and in the last line we used the fact that the sup norm of Q on the unit circle is larger than 1. This is true because

$$\frac{1}{2\pi} \int_0^{2\pi} \log Q(e^{i\theta})\, d\theta = \sum_{j=1}^{m} w_j \frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\tau_j} - e^{i\theta}|\, d\theta = 0,$$

while $\log Q < 0$ in a neighborhood of each τ_j, so that $\log Q(e^{i\theta}) > 0$ on a set of θ of positive measure. Substituting (23) into (22) gives

$$\sum_{j=1}^{m} w_j \log P(e^{i\tau_j}) \leq \kappa \log \|Q\|_{L_\infty(|z|=1)} + \frac{1}{2\pi}\int_0^{2\pi} \log |P(e^{i\theta})|\, d\theta. \qquad \blacksquare$$

Proof of Theorem 1.3 Note first that our choice of τ_0, τ_{m+1} gives

$$\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} = 2\pi.$$

It suffices to prove that for every $a \in \mathbb{C}$,

$$\begin{aligned}\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} \log |e^{i\tau_j} - a| &\leq \int_0^{2\pi} \log |e^{it} - a| dt + B\kappa^{-1} \\ &= 2\pi \log^+ |a| + B\kappa^{-1},\end{aligned} \tag{24}$$

for we can integrate this against the measure $d\nu(a)$ that appears in the representation of $P \in \mathbb{P}_\kappa$. Since

$$\log |e^{i\tau} - a| = \log |e^{i\tau} - \overline{a}^{-1}| + \log |a|$$

for $\tau \in \mathbb{R}$ and $|a| < 1$, we can assume that $|a| \geq 1$. Moreover, it is sufficient to prove (24) in the case $|a| \geq 1 + \kappa^{-1}$. Indeed the case $|a| \in [1, 1 + \kappa^{-1}]$ follows easily from the case $|a| = 1 + \kappa^{-1}$ and the fact that the left-hand and right-hand sides in (24) increase as we increase $|a|$, while keeping $\arg(a)$ fixed. We may also assume that $a \in [1 + \kappa^{-1}, \infty)$ (simply rotate the unit circle). To prove (24), we use the integral form of the error for the trapezoidal rule [6, p. 288, (4.3.16)]: if f'' exists and is integrable in $[\alpha, \beta]$,

$$\int_\alpha^\beta f(t)\, dt - \frac{\beta - \alpha}{2}(f(\alpha) + f(\beta)) = \frac{1}{2}\int_\alpha^\beta f''(t)(\alpha - t)(\beta - t)\, dt.$$

From this we deduce that if f'' does not change sign on $[\alpha, \beta]$, then

$$\Big| \int_\alpha^\beta f(t)\, dt - \frac{\beta - \alpha}{2}(f(\alpha) + f(\beta)) \Big| \leq \frac{(\beta - \alpha)^2}{2} |f'(\beta) - f'(\alpha)|. \tag{25}$$

Moreover, if f'' changes sign at most twice, then

$$\Big| \int_\alpha^\beta f(t)\, dt - \frac{\beta - \alpha}{2}(f(\alpha) + f(\beta)) \Big| \leq 3(\beta - \alpha)^2 \max_{t \in [\alpha, \beta]} |f'(t)|. \tag{26}$$

Now let $f(t) := \log |e^{it} - a|$. Then

$$f'(t) = \frac{a \sin t}{1 + a^2 - 2a \cos t} \quad \text{and} \quad f''(t) = \frac{-2a^2 + (1 + a^2) a \cos t}{(1 + a^2 - 2a \cos t)^2}.$$

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Elementary calculus shows that $|f'|$ achieves its maximum on $[0, 2\pi]$ when $\cos t = \frac{2a}{1+a^2}$. Then $|\sin t| = \frac{a^2-1}{a^2+1}$. Hence, as $a \geq 1 + \kappa^{-1}$, and $\kappa \geq 1$,

$$|f'(t)| \leq (a - a^{-1})^{-1} \leq \kappa, \quad t \in \mathbb{R}. \tag{27}$$

Also, since f'' has at most two zeros in the period, the total variation $V_0^{2\pi} f'$ on $[0, 2\pi]$ satisfies

$$V_0^{2\pi} f' \leq 6 \max_{[0,2\pi]} |f'| \leq 6\kappa. \tag{28}$$

Now we apply (25)–(28) to the interval $[\alpha, \beta] = [\tau_{j-1}, \tau_j]$ and sum over j. We also use our conventions on τ_{m+1} and τ_m. Then

$$\begin{aligned}\Big| \int_0^{2\pi} f(t)\, dt - \sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} f(\tau_j) \Big| \\ &= \Big| \sum_{j=1}^{m} \Big(\int_{\tau_{j-1}}^{\tau_j} f(t)\, dt - \frac{\tau_j - \tau_{j-1}}{2} [f(\tau_{j-1}) + f(\tau_j)] \Big) \Big| \\ &\leq \frac{1}{2} \delta^2 V_0^{2\pi} f' + 6\delta^2 \kappa \leq 9A^2 \kappa^{-1},\end{aligned}$$

so we have (24) with $B = 9A^2$. ∎

Proof of Theorem 1.4 We begin by recalling two facts about zeros of Littlewood and unimodular polynomials:

(I) There exists $c > 0$ such that every unimodular polynomial of degree $\leq n$ has at most $c\sqrt{n}$ real zeros [4].

(II) There exists $c > 0$ such that every Littlewood polynomial of degree $\leq n$ has at most $c \log^2 n / \log\log n$ zeros at 1 [5].

Now suppose that 1 is a zero of f_p with multiplicity $m = m(p)$. By (I) or (II), $m = O(p^{1/2})$. Let $h_m(z) = (z-1)^m$ and $F_p(z) = f_p(z)/h_m(z)$. Note that all coefficients of F_p are integers (as $1/h_m(z)$ has Maclaurin series with integer coefficients), so $F_p(1)$ is a non-zero integer. Also h_m is monic and has all zeros on the unit circle, so its Mahler measure is 1. Then as Mahler measure is multiplicative,

$$M_0(f_p) = M_0(F_p) M_0(h_m) = M_0(F_p).$$

Let $z_p = \exp\big(\frac{2\pi i}{p}\big)$. The special case (3) of Theorem 1.2 gives

$$\begin{aligned}M_0(f_p) &\geq \frac{1}{2} \Big(|F_p(1)| \prod_{k=1}^{p-1} |F_p(z_p^k)| \Big)^{1/p} \\ &\geq \frac{1}{2} \Big(1 \cdot \prod_{k=1}^{p-1} \Big| \frac{f_p(z_p^k)}{(z_p^k - 1)^m} \Big| \Big)^{1/p}.\end{aligned}$$

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It is known [2, § 5] that for $1 \le k \le p-1$,

$$f_p(z_p^k) = \sqrt{\left(\frac{-1}{p}\right)p}.$$

Then

$$M_0(f_p) \ge \frac{1}{2}\left(\frac{\sqrt{p}^{p-1}}{p^m}\right)^{1/p} = \frac{1}{2}\sqrt{p}\,p^{-\left(\frac{1}{2}+m\right)/p}.$$

Since $m = O(p^{1/2})$, the bound (16) follows for large p. ■

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