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Peirce Domains

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Abstract. A theorem of Korányi and Wolf displays any Hermitian symmetric domain as a Siegel domain of the third kind over any of its boundary components. In this paper we give a simple proof that an analogous realization holds for any self-adjoint homogeneous cone.

1 Introduction

Suppose D is a Hermitian symmetric domain and F is a boundary component of D. Then the pair (D, F) admits a realization as a *Siegel domain of the third kind* [WK], [PS], [Sa]: there exists a real vectorspace U, a self-adjoint homogeneous cone $C_F \subset U$, a family of bilinear symmetric forms $h_t \colon \mathbb{C}^k \times \mathbb{C}^k \to U$ and an embedding $D \to F \times \mathbb{C}^k \times U(\mathbb{C})$ whose image is defined by the well-known inequality

$$D = \{(t, w, z) \in F \times \mathbb{C}^k \times U(\mathbb{C}) \mid \operatorname{Im}(z) - h_t(w, w) \in C_F\}.$$

In this short note we give a surprisingly simple proof that an analogous realization holds for any pair (C, C_1) where C is a self adjoint homogeneous cone and C_1 is a boundary component of C. Just as the Siegel domain realization of a Hermitian symmetric space D is used to describe the geometry of compactifications of arithmetic quotients $\Gamma \setminus D$ (where Γ is an arithmetic group) [AMRT], [BB], [Sa], the "Peirce domain" realization of the cone C, which we describe in this paper, is likely to be useful in describing compactifications of arithmetic quotients $\Gamma \setminus C$.

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2 Statement of results

2.1

Let $C \subset V$ denote a self adjoint homogeneous (open) cone in a real vectorspace V and let $G = \operatorname{Aut}^0(C, V) \subset GL(V)$ denote the connected component of the group of (linear) automorphisms of C. Fix a basepoint $e \in C$. Then V admits the structure of a Euclidean (= formally real) Jordan algebra with identity element e, and

$$C = \{x^2 \mid x \in V \text{ is invertible}\}$$

may naturally be identified with the symmetric space G/K (where $K = \operatorname{Stab}_G(e)$). Moreover, the cone C, the Jordan algebra V, and the group G determine each other. For any

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$v \in V$ let $T(v)\colon V \to V$ denote Jordan multiplication by v and let $P(v)\colon V \to V$ be the linear mapping given by

$$P(v)(t) = 2v(vt) - v^2 t$$

for any $t \in V$. Then P is called the "quadratic representation" and its polarization $P(u,v) = \frac{1}{2}\big(P(u+v) - P(u) - P(v)\big)$ determines a parametrized family of V-valued bilinear forms

$$h_t(u,v) = P(u,v)t^{-1}$$

for $t \in V$, so that $h_t(v,v) = P(v)(t^{-1})$.

Throughout this paper we fix an idempotent $\epsilon_1 \in V$ and let

$$V = V_1(\epsilon_1) \oplus V_{\frac{1}{2}}(\epsilon_1) \oplus V_0(\epsilon_1)$$

be the resulting Peirce decomposition into the 1, $\frac{1}{2}$, and 0-eigenspaces of $T(\epsilon_1)$, which we will often abbreviate as V_1, $V_{\frac{1}{2}}$, and V_0 whenever there is no danger of confusion. Set $\epsilon_0 = e - \epsilon_1$, and let C_1 (resp. C_0) denote the projection of the cone C to $V_1(\epsilon_1)$ (resp. to $V_0(\epsilon_1) = V_1(\epsilon_0)$). Then $C_1 \subset V_1$ is a self adjoint homogeneous cone with basepoint ϵ_1 and similarly for C_0.

Theorem 2.2 *The cone C is given by*

$$\begin{aligned} C &= \{(t,w,z) \mid t \in C_1 \text{ and } z - h_t(w,w) \in C_0\} \\ &= \{(t,w,z) \mid z \in C_0 \text{ and } t - h_z(w,w) \in C_1\}. \end{aligned}$$

3 Preliminaries

3.1

Let us first recall some standard facts about Jordan algebras, most of which may be found in [FK] and [AMRT, Section II]. The cone $C \subset V$ is self-adjoint with respect to some inner product $\langle\, ,\rangle$ on V, which may be taken to be given by $\langle x,y\rangle = \operatorname{tr}\big(T(xy)\big)$ [FK, III.4.1]. This determines an involution $g \mapsto {}^t g$ on G by $\langle gx, y\rangle = \langle x, {}^t g y\rangle$. Moreover, $\theta(g) = {}^t g^{-1}$ is the Cartan involution on G with respect to the choice $e \in C$ of basepoint [AMRT, II Section 3.1] and for all $g \in G$ we have

$$\theta(g)(e) = {}^t g^{-1}(e) = (ge)^{-1} \qquad (3.1.1)$$

(the latter inverse taken in the Jordan algebra). For all $g \in G$ and all $v \in V$ we have [FK, III.5.2],

$$P(gv) = gP(v)^t g. \qquad (3.1.2)$$

Let $Q = \operatorname{Norm}\big(C_1(\epsilon_1)\big) \subset G$ denote the parabolic subgroup which normalizes $C_1(\epsilon_1)$. Let $\mathcal{U}(Q)$ denote its unipotent radical and $L(Q) = Q/\mathcal{U}(Q)$ its Levi quotient. Then the choice $e \in C$ of basepoint determines a canonical lift [BS] $L(Q) \subset G$ of the Levi quotient; it is the subgroup of G which normalizes both $C_1(\epsilon_1)$ and $C_0(\epsilon_1)$.

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Lemma 3.2 *In these coordinates, the action of Q on V is given by*

$$g.v = \begin{pmatrix} A & M & N \\ 0 & C & D \\ 0 & 0 & B \end{pmatrix} \begin{pmatrix} v_1 \\ v_{\frac{1}{2}} \\ v_0 \end{pmatrix}$$

(where A, M, N, C, D and B are linear mappings which depend on g). Furthermore,

1. *$g \in L(Q)$ iff $M = 0$, $N = 0$, $D = 0$.*
2. *If $g \in \mathcal{U}(Q)$ then $A = I$ and $B = I$.*
3. *The orbit of the basepoint $e = (\epsilon_1, 0, \epsilon_0)$ under $L(Q)$ is the product $C_1 \times \{0\} \times C_0$.*

If $x = (x_1, x_{\frac{1}{2}}, x_0)$ satisfies $x_{\frac{1}{2}} = 0$ then $x \in C$ iff $x_1 \in C_1$ and $x_0 \in C_0$.

The Jordan product satisfies: $V_1V_1 \subset V_1$, $V_0V_0 \subset V_0$, $V_1V_0 = \{0\}$, $V_{\frac{1}{2}}V_{\frac{1}{2}} \subset V_1 \oplus V_0$, $V_1V_{\frac{1}{2}} \subset V_{\frac{1}{2}}$, and $V_0V_{\frac{1}{2}} \subset V_{\frac{1}{2}}$. If $x = (x_1, x_{\frac{1}{2}}, x_0)$ let $x' = (x_1, -x_{\frac{1}{2}}, x_0)$. Then $(xy)' = x'y'$ for all $x, y \in V$, as may be seen by setting $y = (y_1, y_{\frac{1}{2}}, y_0)$ and multilying out both sides. Hence, $x \in V$ is invertible iff x' is invertible, and in this case $(x')^{-1} = (x^{-1})'$. Similarly, $x = u^2$ for some $u \in V$ iff $x' = (u')^2$. From this it follows that

$$x \in C \iff x' \in C. \tag{3.2.1}$$

Every $x \in V$ has an eigenvalue decomposition $x = \sum_{i=1}^{r} \lambda_i f_i$ where the f_i form a Jordan frame, (*i.e.*, they are orthogonal idempotents, $f_1 + \cdots + f_r = e$, and $r = \dim_{\mathbb{R}}(V)$) and where $\{\lambda_i\} \subset \mathbb{R}$ are the eigenvalues of the linear transformation $T(x) \colon V \to V$. Moreover $x \in C$ iff $\lambda_i > 0$ for $i = 1, 2, \ldots, r$.

Lemma 3.3 *Let $b \in V_{\frac{1}{2}}$. Then $e + b \in C \iff e - b^2 \in C$.*

Proof 3.4 Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ denote the eigenvalues of b. Then the eigenvalues of $e - b$ are $\{1 - \lambda_i\}$ and the eigenvalues of $e - b^2$ are $\{1 - \lambda_i^2\}$. By (3.2.1) we see that $e + b \in C$ iff $(e + b)' = e - b \in C$ iff $e \pm b \in C$ iff $1 \pm \lambda_i > 0$ iff $1 - \lambda_i^2 > 0$ iff $e - b^2 \in C$. ∎

Lemma 3.5 *Let $b \in V_{\frac{1}{2}}$ and set $b^2 = (y_1, 0, y_0)$. Then $y_1 = \epsilon_1 b^2$, $y_0 = \epsilon_0 b^2$, and $by_1 = by_0$.*

Proof 3.6 Compute $\epsilon_1 b^2 = \epsilon_1(y_1 + y_0) = \epsilon_1 y_1 = y_1$ and similarly $\epsilon_0 b^2 = y_0$. Hence $by_1 = b(b^2\epsilon_1) = b^2(b\epsilon_1) = b^2\frac{1}{2}b$. But a similar calculation gives $by_0 = \frac{1}{2}b^3$. ∎

Proposition 3.7 *Let $b \in V_{\frac{1}{2}}$ and write $b^2 = (y_1, 0, y_0)$. Then $e + b \in C$ iff $\epsilon_1 - y_1 \in C_1$ iff $\epsilon_0 - y_0 \in C_0$.*

Proof 3.8 By Lemma 3.3, $e + b \in C$ iff $e - b^2 = (\epsilon_1 - y_1, 0, \epsilon_0 - y_0) \in C$ iff $\epsilon_1 - y_1 \in C_1$ and $\epsilon_0 - y_0 \in C_0$. So it suffices to show that $\epsilon_1 - y_1 \in C_1$ iff $\epsilon_0 - y_0 \in C_0$. Find Jordan frames for $y_1 \in V_1$ and $y_0 \in V_0$, say $y_1 = \sum_{i=1}^{m} \lambda_i c_i$ and $y_0 = \sum_{j=1}^{n} \mu_j d_j$ with $\sum c_i = \epsilon_1$ and $\sum d_j = \epsilon_0$, so that

$$b^2 = \sum_{i=1}^{m} \lambda_i c_i + \sum_{j=1}^{n} \mu_j d_j. \tag{3.8.1}$$

We claim that (for any $b \in V_{\frac{1}{2}}$), the set of nonzero eigenvalues $\{1 - \lambda_i\}$ of $e_1 - y_1$ coincides with the set of nonzero eigenvalues $\{1 - \mu_j\}$ of $e_0 - y_0$.

The vectorspace $V_{\frac{1}{2}}$ decomposes,

$$V_{\frac{1}{2}} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} V_{\frac{1}{2}}(c_i) \cap V_{\frac{1}{2}}(d_j)$$

with respect to which we may write $b = \sum_{i,j} b_{ij}$. So

$$by_1 = \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}\Big)\Big(\sum_{k=1}^{m} \lambda_k c_k\Big) = \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda_k \frac{1}{2} b_{kj}$$

since $c_k b_{ij} = 0$ for $k \neq i$ and $c_k b_{kj} = \frac{1}{2} b_{kj}$. Similarly,

$$by_0 = \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}\Big)\Big(\sum_{\ell=1}^{n} \mu_\ell d_\ell\Big) = \sum_{\ell=1}^{n} \frac{1}{2} \sum_{i=1}^{m} \mu_\ell b_{i\ell}.$$

By Lemma 3.5 these are equal, hence equating components we obtain:

(3.8.2) $$\text{if } b_{ij} \neq 0 \quad \text{then} \quad \lambda_i = \mu_j.$$

Therefore it suffices it show that if $\lambda_i \neq 0$ (with $1 \leq i \leq m$) there exists a j such that $b_{ij} \neq 0$, (and that if $\mu_j \neq 0$ (with $1 \leq j \leq n$) there exists an i such that $b_{ij} \neq 0$).

To see this, first compute $b^2 = (\sum_{i,j} b_{ij})(\sum_{k,\ell} b_{k\ell})$. Then we find

1. $b_{ij} b_{k\ell} = 0$ if $i \neq k$ and $j \neq \ell$
2. $b_{ij} b_{kj} \in V_{\frac{1}{2}}(c_i) \cap V_{\frac{1}{2}}(c_k)$ for $i \neq k$
3. $b_{ij} b_{i\ell} \in V_{\frac{1}{2}}(d_j) \cap V_{\frac{1}{2}}(d_\ell)$ for $j \neq \ell$
4. $b_{ij} b_{ij} = \frac{1}{2} \|b_{ij}\|^2 (c_i + d_j)$.

But $b^2 \in \bigoplus_{i=1}^{m} V_1(c_i) \bigoplus_{j=1}^{n} V_1(d_j)$ so terms of type (2) and (3) (namely $\sum_{j=1}^{n} b_{ij} b_{kj}$ for $i \neq k$ and $\sum_{i=1}^{m} b_{ij} b_{i\ell}$ for $j \neq \ell$) must vanish, *i.e.*,

$$b^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{2} \|b_{ij}\|^2 (c_i + d_j).$$

Comparing this with (3.8.1) we obtain

$$\lambda_i = \frac{1}{2} \sum_{j=1}^{n} \|b_{ij}\|^2 \quad \text{and} \quad \mu_j = \frac{1}{2} \sum_{i=1}^{m} \|b_{ij}\|^2.$$

It follows that if $\lambda_i \neq 0$ then $b_{ij} \neq 0$ for some j (in which case (3.8.2) implies that $\lambda_i = \mu_j$), and similarly if $\mu_j \neq 0$ then $b_{ij} \neq 0$ for some i (in which case (3.8.2) implies that $\mu_j = \lambda_i$). This concludes the proof that $\epsilon_1 - y_1$ has the same eigenvalues as $\epsilon_0 - y_0$ so it also concludes the proof of Proposition 3.7. ■

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Lemma 3.9 *Let $b \in V_{\frac{1}{2}}$. Set $b^2 = (y_1, 0, y_0)$ as above. Then $P(b)(\epsilon_0) = y_1$ and $P(b)(\epsilon_1) = y_0$. Moreover, if $v_0 \in C_0$ then $P(b)(v_0) \in \overline{C_1}$. If $v_1 \in C_1$ then $P(b)(v_1) \in \overline{C_0}$.*

Proof 3.10 Compute $P(b)(\epsilon_1) = 2b(b\epsilon_1) - b^2\epsilon_1 = 2b(\frac{1}{2}b) - b^2\epsilon_1 = b^2(e - \epsilon_1) = \epsilon_0 b^2 = y_0$. Similarly for $P(b)(\epsilon_1)$. Now let $v_0 \in C_0$. Let $L(Q)$ be the Levi subgroup of G which preserves both $C_1(\epsilon_1)$ and $C_0(\epsilon_1)$. Then there exists $g \in L(Q)$ so that $g\epsilon_0 = v_0$. By (3.1.2) we see that $P({}^t gb) = {}^t gP(b)g$ hence

$$P(b)v_0 = P(b)(g\epsilon_0) = ({}^t g)^{-1} P({}^t gb)(\epsilon_0).$$

By Lemma 3.2, the element $\tilde{b} = {}^t gb \in V_{\frac{1}{2}}$ so by the first part of Lemma 3.9, $P(\tilde{b})(\epsilon_0) \in \overline{C_1}$ hence the same is true for $({}^t g)^{-1} P(\tilde{b})(\epsilon_0)$. Similar remarks apply to $P(b)(v_1)$. ■

4 Proof of Theorem 2.2

Let $x = (t, w, z) \in V$ and suppose that $t \in C_1$ and $z - P(w)t^{-1} \in C_0$. By Lemma 3.9, $P(w)t^{-1} \in \overline{C_0}$ hence $z \in C_0$. So there exists $g \in L(Q)$ in so that $gt = \epsilon_1$ and $gz = \epsilon_0$. Set $b = g(w) \in V_{\frac{1}{2}}$. Then $g\big(z - P(w)t^{-1}\big) \in C_0$, which is

$$\begin{aligned} g\big(z - P(w)t^{-1}\big) &= \epsilon_0 - P(gw)({}^t g)^{-1} t^{-1} \quad \text{by (3.1.2)} \\ &= \epsilon_0 - P(b)\epsilon_1 \quad \text{by (3.1.1)} \\ &= \epsilon_0 - y_0 \quad \text{by 3.9} \end{aligned}$$

where $b^2 = (y_1, 0, y_0)$. By Proposition 3.7, $e + b = (\epsilon_1, b, \epsilon_0) \in C$, hence $g^{-1}(e + b) = (t, w, z) \in C$.

The reverse implication is similar: if $x = (t, w, z) \in V$ then $t \in C_1$ and $z \in C_0$ hence there exists $g \in L(Q)$ so that $gt = \epsilon_1$, $gz = \epsilon_0$ and we set $b = gw \in V_{\frac{1}{2}}$. Hence $e + b = gx \in C$ so by Proposition 3.7, $\epsilon_0 - y_0 \in C_0$. Running the above equalities backwards, we find $g\big(z - P(w)t^{-1}\big) \in C_0$ hence also $z - P(w)t^{-1} \in C_0$. ■

References

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y.-S. Tai, *Smooth Compactification of Locally Symmetric Varieties*. Math. Sci. Press, Brookline, MA, 1975.

[BB] W. Baily and A. Borel, *Compactifications of arithmetic quotients of bounded symmetric domains*. Ann. Math. **84**(1966), 442–528.

[BS] A. Borel and J.-P. Serre, *Corners and arithmetic groups*. Comment. Math. Helv. **48**(1973), 436–491.

[FK] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*. Oxford University Press, Oxford, 1994.

[PS] I. I. Piatetskii-Shapiro, *Automorphic Functions and the Geometry of Classical Domains*. Gordon and Breach, NY, 1969. (Russian edition published earlier.)

[Sa] I. Satake, *Algebraic Structures of Symmetric Domains*. Princeton University Press, Princeton, NJ, 1980.

[WK] J. Wolf and A. Korányi, *Generalized Cayley transformations of bounded symmetric domains*. Amer. J. Math. **87**(1965), 899–939.

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