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A NOTE ON ASYMPTOTIC INFERENCE FOR AN ASYMMETRIC ISING MODEL AROUND A TORUS

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Abstract

Limiting distribution results are obtained for the sufficient statistics of an asymmetric Ising model on a torus. Applications are discussed.

1. Introduction

Consider a rectangular lattice with $M \times N$ pixels. Let $X = (x_{ij})$ denote the vector variable with each $x_{ij} \in \{-1, +1\}$. We assume a periodic boundary condition which is equivalent to wrapping the lattice around a torus. The distribution function for the periodic boundary asymmetric Ising model, which involves two parameters, is

$$P(X \mid \alpha, \beta) = \frac{1}{C(\alpha, \beta)} \exp\left\{\alpha \sum_{i=1}^{M} \sum_{j=1}^{N} x_{ij}x_{i,j+1} + \beta \sum_{i=1}^{M} \sum_{j=1}^{N} x_{ij}x_{i+1,j}\right\} \tag{1.1}$$

where $x_{i,N+1} = x_{i1}$ and $x_{M+1,j} = x_{1j}$, and $C(\alpha, \beta)$ is the normalizing factor known as the partition function. Define $Q = (Q_1, Q_2)' = (\sum\sum x_{ij}x_{i,j+1}, \sum\sum x_{ij}x_{i+1,j})'$. For convenience, we only consider the region $R_+^2 = \{0 \leqq \alpha, \beta < \infty\}$. The results can also be extended to other areas by symmetric extension.

2. The limiting results of Q

The moment generating function of Q is $C(\alpha + t_1, \beta + t_2)/C(\alpha, \beta)$. The asymptotic properties of Q depend upon those of $C(\alpha, \beta)$. There is a function, $B(\alpha, \beta)$, which is an approximation to $(MN)^{-1} \log C(\alpha, \beta)$ and defined by the following Riemann integral. Detailed proof of the validity of the approximation is not given here, but the approach is similar to that of Pickard (1976):

$$B(\alpha, \beta) = (4\pi)^{-1} \int_0^{2\pi} \psi(\alpha, \beta, \omega)\, d\omega,$$

where

$$\psi(\alpha, \beta\, \omega) = \log\{\cosh 2\alpha \cosh 2\beta - \sinh 2\alpha \cos \omega + [(\cosh 2\alpha \cosh 2\beta - \sinh 2\alpha \cos \omega)^2 - (\sinh 2\beta)^2]^{\frac{1}{2}}\}.$$

It is not difficult to show that there is a line, called the critical line, and defined by

$$L = \{(\alpha, \beta)' : \sinh 2\alpha \sinh 2\beta = 1\}$$

such that, although the first-order partial derivatives of B are continuous on R_+^2, the high-order partial derivatives are not defined on L. By using Kaufmann's (1949) exact representation for $C(\alpha, \beta)$, Pickard (1976) obtained the limiting distribution properties of $Q_1 + Q_2$, but only for the symmetric case $\alpha = \beta$. Following almost the same method and notations as those of Pickard (1976) to analyse $C(\alpha, \beta)$, and then using the moment generating functions of $(MN)^{-1}Q$ and $(MN)^{\frac{1}{2}}[(MN)^{-1}Q - \nabla B(\alpha, \beta)]$, we can prove the following theorem.

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Theorem. Suppose $(\alpha, \beta)' \notin L$. Then as $M, N \to \infty$, and provided $N^{\theta_1} \leqq M \leqq N^{\theta}$ for any fixed θ_1 and θ with $0 < \theta_1 < \theta < \infty$,

1. $(MN)^{-1}Q \xrightarrow{\text{Pr}} \nabla B(\alpha, \beta)$

2. $(MN)^{\frac{1}{2}}[(MN)^{-1}Q - \nabla B(\alpha, \beta)] \xrightarrow{\text{D}} \text{N}(0, \nabla^2 B(\alpha, \beta))$

where ∇ denotes the first-order derivative vector, and ∇^2 the second-order derivative matrix.

For the case of free boundary condition (no interaction between x_{i1} and x_{iN}, and between x_{1j} and x_{Mj}), it is almost impossible to analyse the partition function exactly in order to obtain rate of convergence of $(MN)^{-1}EQ$ to $\nabla B(\alpha, \beta)$. Pickard (1977) showed only that

$$(MN)^{-\frac{1}{2}}[Q - EQ] \xrightarrow{\text{D}} \text{N}(0, \nabla^2 B(\alpha, \beta)).$$

3. Applications

Since the number of possible terms of X is 2^{MN}, it is not feasible to carry out an exact calculation for $C(\alpha, \beta)$ and $\nabla C(\alpha, \beta)$, except when M and N are small. The maximization of the log-likelihood is therefore infeasible. Monte Carlo methods are not a completely satisfactory approach, since the author believes the computational burden of simulating the field to be very heavy. Note that $MNB(\alpha, \beta)$ is an approximation to $\log C(\alpha, \beta)$, an obvious alternative way of estimating parameters is to maximize an asymptotic likelihood, namely, $AL(Q \mid \alpha, \beta) = \alpha Q_1 + \beta Q_2 - MNB(\alpha, \beta)$, or to solve the following corresponding asymptotic normal equation:

$$(MN)^{-1}Q = \nabla B(\alpha, \beta).$$

Suppose $(\alpha_0, \beta_0)' \notin L$ are the true values of the parameters and denote by $(\hat{\alpha}, \hat{\beta})'$ the solution of the above equation. Standard methods therefore show that as $M, N \to \infty$, and provided $N^{\theta_1} \leqq M \leqq N^{\theta}$ with $0 < \theta_1 < \theta < \infty$,

$$(MN)^{-1}\begin{pmatrix} \hat{\alpha} - \alpha_0 \\ \hat{\beta} - \beta_0 \end{pmatrix} \xrightarrow{\text{D}} \text{N}(0, (\nabla^2 B(\alpha, \beta))^{-1}).$$

This demonstrates consistency of $(\hat{\alpha}, \hat{\beta})'$. Another application of the results is in testing the null hypothesis:

$$\text{H}_0\text{: } \alpha = \beta$$

against the general alternative hypothesis of 'not H_0'. We can naturally attack this test problem by an asymptotic likelihood-ratio approach. Define the test statistic

$$\Lambda(Q) = \sup_{\alpha,\beta} AL(Q \mid \alpha, \beta) - \sup_{\alpha} AL(Q \mid \alpha, \alpha).$$

The upper δ-point q_δ of the distribution can then be defined by

$$\max_{(\alpha,\alpha)\notin L} \lim_{M,N\to\infty} \Pr[\Lambda(Q) \geqq q_\delta \mid \alpha, \alpha] = \delta.$$

It can be shown that, when the null hypothesis holds, $\Lambda(Q)$ follows, asymptotically, a $\chi^2(1)$ distribution, while, when the alternative hypothesis holds, $\Lambda(Q)$ has. in probability, the same order as MN. For the free boundary condition case, the asymptotic normality of maximum asymptotic likelihood estimators and the properties of above test statistic have not been established.

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