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ON THE EMBEDDING OF PROCESSES IN BROWNIAN MOTION AND THE LAW OF THE ITERATED LOGARITHM FOR REVERSE MARTINGALES

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Techniques from martingale theory are used to obtain the Skorokhod embedding of reverse martingales in Brownian motion. This result is then used to obtain a functional law of the iterated logarithm for reverse martingales.

1. Introduction

It is well known that for ordinary discrete time square integrable martingales there exists a sequence of stopping times on Brownian motion such that the stopped Brownian motion has the same distribution as the martingale. Also for continuous path square integrable martingales there exists a Brownian motion and a time change process such that the martingale is almost surely the same as the time changed Brownian motion; see Kunita and Watanabe [8], Knight [7]. In the first section of this paper we use a theorem of Heath [4] which enables us to exploit the continuous time results to obtain an embedding of reverse martingales in Brownian motion.

A previous attempt to obtain a reverse martingale embedding was made by Loynes [9] who considered the stopping times of Root [12] and attempted to construct stopping times in the following manner.

We may consider a reversed martingale $\{S_n, F_n; n \geq 1\}$ as a martingale indexed by the negative integers, that is, set $S'_n = S_{-n}$,

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$F'_n = F_{-n}$, $n \leq -1$. Therefore for each $n \leq -1$ the process $\{S'_j, F'_j, n \leq j \leq -1\}$ is a martingale which may be embedded in a Brownian motion via a finite sequence of stopping times $\left\{T_j^{(n)}; n \leq j \leq -1\right\}$. A quick consideration shows that we do not necessarily have $T_j^{(n)} = T_j^{(m)}$, $n \neq m$. A counter example may be easily provided by showing that the sum of two stopping times which are defined as the first hitting times of barriers, that is, Roots stopping times, need not be the first hitting time of a barrier. Thus we may take $T_{-1}^{(-1)}$ to be the first hitting time of a barrier whilst $T_{-1}^{(-2)}$ need not be such a hitting time,hence $T_{-1}^{(-1)} \neq T_{-1}^{(-2)}$.

We have therefore a triangular array of stopping times

$$\begin{array}{ccc} & & T_{-1}^{(-1)} \\ & T_{-2}^{(-2)} & T_{-1}^{(-2)} \\ T_n^{(n)} & \cdots\cdots & T_{-1}^{(n)} \\ \cdot^{\cdot^{\cdot}} & & \vdots \end{array}$$

To obtain the reverse martingale embedding from this approach we thus require that $T_j^{(n)} \xrightarrow{P} T_j$ as $n \to -\infty$ for each $j \leq -1$. All that is known about these stopping times is that they exist and have the same first moment. The obvious, and seemingly only possible, way of proceeding is to try to show compactness of an appropriate space of stopping times and the results of Root [12] give some hope of this. The problem therefore reduces to considering a subspace S^K of the space S of all stopping times on our Brownian motion where $S^K = \{\tau \in S : E\tau \leq K\}$ equipped with the metric $d(\tau, T) = \inf\{\varepsilon : P(|\tau - T| \geq \varepsilon) < \varepsilon\}$, that is the metric of convergence in probability. To prove the reverse martingale embedding we require that this space be compact and it does not seem possible to prove this as results which give necessary and sufficient conditions for convergence are inapplicable in this case. See for example, Theorem 18, p. 297, of Dunford and Schwartz [2].

This and Loynes' unsuccessful attempt lead us to conclude that the usual construction of stopping times in the martingale embedding does not provide enough structure to yield the necessary compactness results and a stronger form of the martingale embedding is required. Such an embedding is the subject of the next section of this article.

For discrete time martingales previous embedding results have been used to obtain various limit results, for example, Strassen [16], Scott [13], Heyde and Scott [6], Hall and Heyde [3]. The connection between the original martingale and the stopping times is made via conditional moment inequalities which we obtain here from a theorem of Millar [11]. The conditional moment inequalities given here are more general than those usually associated with the Skorokhod embedding as they hold for any σ-fields with respect to which the process is a martingale (or a reverse martingale) instead of those generated by the process. The proof given here is quite straightforward and fills a noticeable gap in the literature concerning the Skorokhod embedding.

The basic statement of the continuous time embedding result, which is contained in Theorem 3.1 of Kunita and Watanabe [8] and Theorem 1 of Knight [7] is as follows.

THEOREM A. *Let* $\{Y_t, G_t;\ t \ge 0\}$ *be a square integrable martingale, that is,* $EY_t^2 < \infty$ *for all* t *, defined on a complete probability space* (Ω, F, P) . *Suppose* $\{Y_t;\ t \ge 0\}$ *has continuous paths and the family* $\{G_t;\ t \ge 0\}$ *of* σ-*fields is right continuous. Then there exists a Brownian motion* $\{B(u), F_u^*;\ u \ge 0\}$ *and a process* $\tau = \{\tau_u;\ u \ge 0\}$ *satisfying*

(i) *for each* $u \in [0, \infty)$, τ_u *is a stopping time with respect to the family of* σ-*fields* $\{F_u^*;\ u \ge 0\}$ *and* $\tau_u < \infty$,

(ii) *for almost all* $\omega \in \Omega$, $u \in [0, \infty) \to \tau_u \in [0, \infty)$ *is a continuous and non-decreasing function,*

(iii) $Y_t = B(\tau_t)$ *almost surely,* $G_t \subset F^*_\tau$ *, and*

(iv) $\{Y_t^2 - \tau_t, G_t;\ t \geq 0\}$ *is a martingale.*

The original result of Kunita and Watanabe was restricted to martingales without intervals of constancy however this restriction was removed by Knight. As observed by Meyer [10], p. 92, the process $\tau = \{\tau_u;\ u \geq 0\}$ is just the quadratic variation of the martingale $\{Y_t\}$. Note that the quadratic variation of a square integrable martingale is defined as the limit of the sums

$$\sum_{i=0}^{n-1} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2$$

over partitions $(t_0, t_1, \ldots, t_n)$ of $[0, t]$ which become arbitrarily fine. We shall denote the quadratic variation by Q_t and note that Q_t is purely a function of the paths of the process. The above embedding in the continuous case differs from the usual discrete time versions in that the stopping times are natural, being the quadratic variation of the martingale, and the Brownian motion is defined in terms of these stopping times and the original martingale whereas in the usual discrete time case the stopping times are constructed.

We exploit the fact that our stopping times in continuous time correspond to the quadratic variation of a martingale via the following theorem due to Millar [11].

THEOREM B. *Let* $\{Y_t, G_t;\ t \geq 0\}$ *be a square integrable martingale with almost surely continuous paths. Then for* $1 < p < \infty$ *there exist positive constants* M_p *and* N_p *depending only on* p *such that*

$$M_p E\left(Q_t^{p/2}\right) \leq E|Y_t|^p \leq N_p E\left(Q_t^{p/2}\right) .$$

The result of Heath [4] which is the key part of our embeddings is as follows.

THEOREM C. *If* $\{S_n, F_n;\ n \geq 1\}$ *is a martingale on* (Ω, F, P) *there is then (on a possibly enlarged version of* (Ω, F, P) *) a martingale*

$\{Y_t, G_t;\ t \geq 0\}$ such that Y_t has continuous paths and for $n \geq 1$, $Y_n = S_n$ and $F_n \subseteq G_n$.

The proof of this theorem depends on constructing σ-fields

$$G_n = F_n \vee \sigma\{B^j(s);\ 0 \leq s < \infty,\ j \geq n\} \quad \text{if} \quad n = 1, 2, \ldots$$
$$= F_n \vee A_n,$$

and

$$G_t = G_{[t]} \vee \sigma\left\{S_{[t]+1} + B^{[t]+1}(\phi(s));\ 0 \leq s \leq t-[t]\right\}, \quad t \text{ not an integer},$$

where $\{B^k(t);\ k \geq 1\}$ is a countable family of Brownian motions independent of each other and of $F_\infty = \bigvee_{n\geq 1} F_n$. The function ϕ is any continuously differentiable function on $(0, 1]$ for which $\phi(0+) = +\infty$ and $\phi(1) = 0$ with $\phi' < 0$. The martingale Y_t is constructed as a separable version of the process given by $Y_t = E\{S_{[t]+1} | G_t\}$. See Heath [4] for details of the proof.

2. Discrete time embedding results

We now utilise the theorems in our introduction to obtain embeddings for martingales, reverse martingales and doubly infinite martingales. We take (Ω, F, P) to be a complete probability space. The definition relating to martingales, reverse martingales, stopping times and Brownian motion are sufficiently well known for us to need not mention them here.

THEOREM 1. *If $\{S_n, F_n;\ n \geq 1\}$ is a square integrable martingale, that is, $ES_n^2 < \infty$ for each n, then (on a possibly enlarged version of (Ω, F, P)) there exists a Brownian motion $\{B(u), F_u^*;\ u \geq 0\}$ and a non-decreasing sequence of stopping times $\{\tau_n;\ n \geq 1\}$ such that $F_n \subset F^*_{\tau_n}$ and $B(\tau_n) = S_n$ almost surely. Furthermore there exists an increasing family of σ-fields G_n such that τ_n is G_n measurable and the following results hold:*

(2.1) $$E\{\tau_n-\tau_{n-1}|G_{n-1}\} = E\left\{(S_n-S_{n-1})^2|F_{n-1}\right\} \text{ almost surely,}$$

and for $1 < p < \infty$ *there exist positive constants* M_p *and* N_p *depending only on* p *such that*

(2.2) $$M_pE\left\{(\tau_n-\tau_{n-1})^{p/2}|G_{n-1}\right\} \le E\left\{|S_n-S_{n-1}|^p|F_{n-1}\right\}$$
$$\le N_pE\left\{(\tau_n-\tau_{n-1})^{p/2}|G_{n-1}\right\} \text{ almost surely.}$$

Proof. The actual embedding follows directly from Theorem C and Theorem A, as noted in Heath [4], once it is observed that we may take the σ-fields $\{G_t\}$ associated with Theorem C to be right continuous. To do this, replace G_t by $G_{t+} = \bigcap_{s>t} G_s$. It is then easy to show that $E\{Y_s|G_{t+}\} = Y_t$ almost surely; that is, Y_t is still a martingale with respect to the σ-fields G_{t+}.

To obtain the conditional moment results first note that *(iv)* of Theorem A implies that, as $Y_n = S_n$,

$$E\left\{S_n^2-\tau_n|G_{n-1}\right\} = E\left\{E\left\{S_n^2-\tau_n|G_{(n-1)+}\right\}|G_{n-1}\right\}$$
$$= S_{n-1}^2 - \tau_{n-1} \quad \text{almost surely}$$

so that

$$E\left\{S_n^2-S_{n-1}^2|G_{n-1}\right\} = E\{\tau_n-\tau_{n-1}|G_{n-1}\} \quad \text{almost surely.}$$

Using the martingale property and the fact that by construction $G_{n-1} = F_{n-1} \vee A_{n-1}$ where A_{n-1} is independent of the σ-field generated by S_n, S_{n-1} and F_{n-1} the equality (2.1) follows from the above.

The condition moment inequalities, (2.2), are obtained in the following manner. Consider the martingale $\{Y_t, G_t;\ t \ge 0\}$. For any fixed n and any $A \in G_{n-1}$ the process $I(A)(Y_t-Y_{n-1})$, $t \ge n-1$, is still a martingale as, for $t \ge s \ge n$,

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$$E\{I(A)(Y_t-Y_{n-1})|G_s\} = I(A)E\{Y_t-Y_{n-1}|G_s\}$$
$$= I(A)(Y_s-Y_{n-1}) .$$

The quadratic variation of this martingale is just $I(A)(Q_t-Q_{n-1})$. Therefore from Theorem B we have that, considering the left hand inequality, for any $A \in G_{n-1}$ and $t \geq n-1$,

$$M_pE\left\{I(A)|Q_t-Q_{n-1}|^{p/2}\right\} \leq E\left\{I(A)|Y_t-Y_{n-1}|^p\right\} .$$

Hence using the definition of conditional expectation, for any $A \in G_{n-1}$ we have

$$\int_A M_pE\left\{|Q_t-Q_{n-1}|^{p/2}|G_{n-1}\right\}dP \leq \int_A E\left\{|Y_t-Y_{n-1}|^p|G_{n-1}\right\}dP .$$

We may rewrite this as, for any $A \in G_{n-1}$,

$$\text{(2.4)} \qquad \int_A \left[E\left\{|Y_t-Y_{n-1}|^p|G_{n-1}\right\}-M_pE\left\{|Q_t-Q_{n-1}|^{p/2}|G_{n-1}\right\}\right]dP \geq 0 .$$

By choosing

$$A = \left\{E\left\{|Y_t-Y_{n-1}|^p|G_{n-1}\right\}-M_pE\left\{|Q_t-Q_{n-1}|^{p/2}|G_{n-1}\right\} < 0\right\}$$

it follows immediately from (2.4) that $P(A) = 0$. Hence

$$\text{(2.5)} \qquad M_pE\left\{|Q_t-Q_{n-1}|^{p/2}|G_{n-1}\right\} \leq E\left\{|Y_t-Y_{n-1}|^p|G_{n-1}\right\} \text{ almost surely.}$$

As $S_n = Y_n$, $S_{n-1} = Y_{n-1}$, $\tau_n = Q_n$ and $\tau_{n-1} = Q_{n-1}$ we have

$$M_pE\left\{|\tau_n-\tau_{n-1}|^{p/2}|G_{n-1}\right\} \leq E\left\{|S_n-S_{n-1}|^p|G_{n-1}\right\} \text{ almost surely.}$$

Similarly we have for the other part

$$E\left\{|S_n-S_{n-1}|^p|G_{n-1}\right\} \leq N_pE\left\{|\tau_n-\tau_{n-1}|^{p/2}|G_{n-1}\right\} .$$

To complete the proof of (2.2) we have only to note that as above $G_{n-1} = F_{n-1} \vee A_{n-1}$ where A_{n-1} is independent of the σ-field generated by S_n, S_{n-1} and F_{n-1} so that $E\left\{|S_n-S_{n-1}|^p|G_{n-1}\right\} = E\left\{|S_n-S_{n-1}|^p|F_{n-1}\right\}$.

This method of proof extends quite easily to reversed martingales.

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THEOREM 2. *Let $\{S_n, F_n; n \geq 1\}$ be a reverse martingale with $ES_1^2 < \infty$ and suppose without loss of generality that $S_\infty = 0$ (if not consider the reverse martingale $S_n - S_\infty$). Then (on a possible enlarged version of (Ω, F, P)) there exists a Brownian motion $\{B(u), F_u^*; u \geq 0\}$ and a non-increasing sequence of stopping times $\{\tau_n; n \geq 1\}$ such that $F_n \subset F_{\tau_n}^*$, $B(\tau_n) = S_n$ almost surely. Furthermore there exists a decreasing family of σ-fields G_n such that τ_n is G_n measurable and the following results hold:*

$$(2.6) \qquad E\{\tau_n - \tau_{n+1} | G_{n+1}\} = E\left\{(S_n - S_{n+1})^2 | F_{n+1}\right\} \text{ almost surely}$$

and for $1 < p < \infty$ there exist positive constants M_p and N_p depending only on p such that

$$(2.7) \qquad M_p E\left\{(\tau_n - \tau_{n+1})^{p/2} | G_{n+1}\right\} \leq E\left\{|S_n - S_{n+1}|^p | F_{n+1}\right\} \leq N_p E\left\{(\tau_n - \tau_{n+1})^{p/2} | G_{n+1}\right\} \text{ almost surely.}$$

Proof. Firstly note that we may consider a reverse martingale as a martingale indexed by the negative integers.

LEMMA 1. *If $\{S_n, F_n; n \geq 1\}$ is a reverse martingale and we set, for $n \leq -1$, $S'_n = S_{-n}$, $F'_n = F_{-n}$ there is then (on a possibly enlarged version of (Ω, F, P)) a martingale $\{Y_t, G_t; t \leq -1\}$ such that Y_t has continuous paths and for $n \leq -1$, $S'_n = Y_n$, $F'_n \subset G_n$.*

Proof. The lemma is the obvious reverse martingale analogue of Theorem A. We set

$$G_n = F'_n \vee \sigma\{B^j(s); 0 \leq s < \infty, -\infty < j \leq n\} \text{ if } n = -1, -2, \ldots$$

and

$$G_t = G_{[t]} \vee \sigma\left\{S_{[t]+1} + B^{[t]+1}(\phi(s)); 0 < s \leq t-[t]\right\} \text{ if } t \leq -1, \; t \text{ not an integer.}$$

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For $t \leq 0$, $[t]$ denotes the greatest negative integer not exceeding t . Also $\{B^k(t); k \leq -1\}$ is a countable family of Brownian motions independent of each other and of F_1 , whilst ϕ is any continuously differentiable function on $(0, 1]$ for which $\phi(0+) = +\infty$ and $\phi(1) = 0$ with $\phi' < 0$ as in Heath [4].

Now Heath's Theorem 1 depends only on reasoning on the interval $[0, 1]$ so we may now follow his proof, that is, let Y_t be a separable version of the process given by $Y_t = E\{S'_{[t]+1}|G_t\}$, $t \leq -1$, and the lemma follows as in [4].

To find our Brownian motion define $f = k \circ h \circ g : [-\infty, -1] \to [0, \infty]$ by letting $g : [-\infty, -1] \to [-1, -\frac{1}{2}]$ be given by $g(t) = t/(1+|t|)$, $h : [-1, -\frac{1}{2}] \to [0, 1]$ be given by $h(t) = 2(t+1)$ and $k : [0, 1] \to [0, \infty]$ be given by $k(t) = 1/(1-t)$. For $t \in [0, \infty)$ let $\hat{Y}_t = Y_{f^{-1}(t)}$ and $\hat{G}_t = G_{f^{-1}(t)}$. Then $\{\hat{Y}_t, \hat{G}_t; t \geq 0\}$ is a square integrable martingale and using our Theorem A we may find a Brownian motion $\{B(u), F^*_u; u \geq 0\}$ with stopping times $\{T_t; t \geq 0\}$ such that $\hat{Y}_t = B(T_t)$ almost surely and $\hat{G}_t \subset F^*_{T_t}$, $t \in [0, \infty)$. Now for $n \leq -1$, $S_{-n} = S'_n = Y_n = \hat{Y}_{f(n)}$ almost surely and $F_{-n} = F'_n \subset G_n = G_{f(n)}$ so that by setting, for $n \geq 1$, $\tau_n = T_{f(-n)}$ we have $B(\tau_n) = B(T_{f(-n)}) = \hat{Y}_{f(-n)} = S_n$ almost surely. Similarly set $G_n = \hat{G}_{f(-n)}$.

The argument used in Theorem 1 to obtain the conditional moment resutls is again applicable and the proof is complete.

As a consequence of the preceding two theorems it is a straightforward matter to show the following.

THEOREM 3. *If* $\{S_n, F_n; -\infty < n < \infty\}$ *is a square integrable doubly infinite martingale with* $S_{-\infty} = \lim_{n\to-\infty} S_n = 0$ *almost surely, then (on a possibly enlarged version of* (Ω, F, P) *) there exists a Brownian motion* $\{B(u), F^*_u; u \geq 0\}$ *and a non-decreasing sequence of stopping times*

$\{\tau_n;\ -\infty < n < \infty\}$ *such that* $B(\tau_n) = S_n$ *almost surely,* $F_n \subset F^*_{\tau_n}$, $-\infty < n < \infty$. *Furthermore there exists an increasing family of* σ*-fields* G_n *such that* τ_n *is* G_n *measurable and the following results hold:*

$$(2.8)\qquad E\{\tau_n - \tau_{n-1} \mid G_{n-1}\} = E\left\{(S_n - S_{n-1})^2 \mid F_{n-1}\right\} \text{ almost surely;}$$

and for $1 < p < \infty$ *there exist positive constants* M_p *and* N_p *depending only on* p *such that*

$$(2.9)\qquad M_p E\left\{(\tau_n - \tau_{n-1})^{p/2} \mid G_{n-1}\right\} \le E\left\{|S_n - S_{n-1}|^p \mid F_{n-1}\right\} \le N_p E\left\{(\tau_n - \tau_{n-1})^{p/2} \mid G_{n-1}\right\} \text{ almost surely.}$$

The proof of this theorem follows quite easily from the above if we change the function f of the proof of Theorem 2 to $f' = h \circ g$ and for $n \ge 2$ put $S^*_n = S_{n-2}$, that is, the part of the martingale indexed by the negative integers is contained in the interval $[0, 1]$, $S_0 = S^*_2$ and the rest of the martingale S^*_n is as above.

3. The law of the iterated logarithm for reverse martingales

As an application of the previous results we obtain a functional law of the iterated logarithm for reverse martingales. This result is new and unlike the central limit theorem cannot be obtained from the corresponding result for ordinary martingales.

Let $\{B(t);\ t \ge 0\}$ be a Brownian motion on $[0, \infty)$ starting from 0 and define

$$(3.1)\qquad \xi_u(t) = \left(2u \log_2 u^{-1}\right)^{-\frac{1}{2}} B(ut) = \phi(u)B(ut)$$

for $t \in [0, 1]$ and $u < e^{-1}$.

Denote by (C, ρ) the Banach space of continuous real valued functions on $[0, 1]$ with

$$\rho(x, y) = \sup_{0 \le t \le 1} |x(t) - y(t)| ,\quad x, y \in C .$$

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Let K be the set of absolutely continuous $x \in C$ such that $x(0) = 0$ and $\int_0^1 [\dot{x}(t)]^2 dt \le 1$ where $\dot{x}$ denotes the derivative of x with respect to Lebesque measure and determined almost everywhere.

THEOREM 4. *With probability one the set* $\left\{\xi_u ;\ u < e^{-1}\right\}$ *is relatively compact and the set of its limit points as* u *tends to* 0 *is* K .

The proof of this theorem depends on noting that for any constant c , $c^{-\frac{1}{2}}B(ct)$ and $c^{\frac{1}{2}}B(t/c)$ have the same distribution. The proof then follows that of Theorem 1 of Strassen [15] as Strassen's proof depends on estimates of probabilities which by the above have the same distribution as those that we require.

For the iterated logarithm law for reverse martingales we use a formulation similar to that of Hall and Heyde [3] which allows the use of random and deterministic norming sequences. Let $\{S_n,\ F_n;\ n \ge 1\}$ be a reverse martingale, with $ES_1^2 < \infty$, on a probability space $(\Omega,\ F,\ P)$. For $n \ge 1$ put $X_n = S_n - S_{n-1}$, $V_n^2 = \sum_{j=n}^{\infty} E\left\{X_j^2 \mid F_{j+1}\right\}$ and $s_n^2 = EV_n^2$. Note that $S_\infty = \lim_{n\to\infty} S_n$ almost surely always exists for a reverse martingale, $S_n - S_\infty = \sum_{j=n}^{\infty} X_j$ and $s_n^2 \to 0$ as $n \to \infty$.

For a non-increasing sequence of positive random variables $\{W_n;\ n \ge 1\}$ such that without loss of generality $W_1^2 < e^{-1}$ define

$$\begin{aligned}(3.2)\qquad \mu(u) &= S_{p+1} - S_\infty + X_p\left(u - W_{p+1}^2\right)\left(W_p^2 - W_{p+1}^2\right)^{-1}, \quad u \le W_1^2 ,\\ &= 0 , \quad u > W_1^2 ,\end{aligned}$$

where

$$p = p(u) = \max\left\{j : W_j^2 \ge u\right\}$$

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and

$$\mu_n(t) = \phi\left(W_n^2\right)\mu\left(W_n^2 t\right) , \quad t \in [0, 1] . \tag{3.3}$$

THEOREM 5. *Suppose that* $\{S_n ; n \geq 1\}$ *is a random sequence (not necessarily even a reverse martingale) such that* $S_n = B\left(T_n\right)$ *for some sequence of non-increasing, non-negative random variables* $\{T_n ; n \geq 1\}$, *the Brownian motion being defined on the same probability space. If*

$$T_n < e , \quad T_n \to 0 \text{ almost surely}, \quad T_{n+1}^{-1}T_n \to 1 \text{ almost surely}, \tag{3.4}$$

and

$$T_n^{-1}W_n \to 1 \text{ almost surely}, \tag{3.5}$$

then

$$|\mu(u)-B(u)| = o\left(\left(u \log_2 u^{-1}\right)^{\frac{1}{2}}\right) \text{ almost surely as } u \to 0$$

and with probability one the sequence $\{\mu_n ; n \geq 1\}$ *is relatively compact in* C , *the set of its limit points coinciding with* K .

Proof. The condition on $\{T_n\}$ and the definition of μ give

$$|\mu(u)-B(u)| \leq \max\left\{\left|B\left(T_{p(u)}\right)-B(u)\right| , \left|B\left(T_{p(u)+1}\right)-B(u)\right|\right\}$$

with

$$u^{-1}T_{p(u)} \to 1 \text{ almost surely and } u^{-1}T_{p(u)+1} \to 1 \text{ almost surely as } u \to 0 .$$

A suitable modification of p. 217 of Strassen [15] then yields the theorem. See also Hall and Heyde [3].

THEOREM 6. *Let* $\{Z_n ; n \geq 1\}$ *be a sequence of non-negative random variables and suppose* Z_n *and* W_n *are* F_{n+1} *measurable. If*

$$\lim_{n\to\infty} \phi\left(W_n^2\right) \sum_{k=n}^{\infty} \left[X_k I\left(|X_k| > Z_k\right)-E\left\{X_k I\left(|X_k| > Z_k\right) \mid F_{k+1}\right\}\right] = 0 \tag{3.6}$$

almost surely,

(3.7) $$\lim_{n\to\infty} W_n^{-2} \sum_{k=n}^{\infty} \left[E\left\{X_k^2 I(|X_k| \le Z_k) | F_{k+1}\right\} - \left(E\{X_k I(|X_k| \le Z_k) | F_{k+1}\}\right)^2\right] = 1$$ *almost surely,*

(3.8) $$\sum_{k=1}^{\infty} W_k^{-4} E\left\{X_k^4 I(|X_k| \le Z_k) | F_{k+1}\right\} < \infty \text{ almost surely,}$$

(3.9) $$\lim_{n\to\infty} W_{n+1}^{-1} W_n = 1 \text{ almost surely and } W_n \to 0 \text{ almost surely}$$

then with probability one the sequence $\{\mu_n ; n \ge 1\}$ *is relatively compact in* C *and the set of its limit points coincides with* K .

Proof. We follow the proof of Theorem 1 of Hall and Heyde [3] fairly closely with appropriate changes so in some places only a sketch of the proof is needed.

Set

(3.10) $$\tilde{X}_j = X_j I(|X_j| \le Z_j) ,$$

and

(3.11) $$X_j^* = \tilde{X}_j - E\{\tilde{X}_j | F_{j+1}\} .$$

Let $S_n^* = \sum_{j=n}^{\infty} X_j^*$, $V_n^{*2} = \sum_{j=n}^{\infty} E\left\{X_n^{*2} | F_{j+1}\right\}$ and for $u \in [0, 1]$ put

$$\mu_n^*(u) = \phi\left(W_n^2\right)\left[S_l^* + \left(uW_n^2 - W_{l+1}^2\right)\left(W_{l+1}^2 - W_l^2\right)^{-1} X_l^*\right]$$

where $l = l(n, u) = \max\left\{j \ge n : uW_n^2 \le W_j^2\right\}$.

Note that both S_n and S_n^* are reverse martingales with respect to the same σ-field $\{F_n ; n \ge 1\}$.

Now

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$$\sup_{0\le u\le 1} |\mu_n(u)-\mu_n^*(u)|$$

$$\le \phi\left(W_n^2\right) \sup_{n\le k<\infty} \left|\sum_{j=k}^{\infty} \left(X_j-X_j^*\right)\right|$$

$$\le \phi\left(W_n^2\right) \sup_{n\le k<\infty} \left|\sum_{j=k}^{\infty} [X_j-X_jI(|X_j| \le z_j)+E\{X_jI(|X_j| \le z_j)|F_{j+1}\}]\right|$$

$$= \phi\left(W_n^2\right) \sup_{n\le k<\infty} \left|\sum_{j=k}^{\infty} [X_jI(|X_j| > z_j)-E\{X_jI(|X_j| > z_j)|F_{j+1}\}]\right|$$

$$\to 0 \text{ almost surely,}$$

using (3.6).

We now introduce the Skorokhod embedding for the reverse martingale $\{S_n^*, F_n; n \ge 1\}$. From Theorem 2 there exists a Brownian motion and a sequence of stopping times T_n such that $S_n^* = B(T_n)$ almost surely. Also there exist σ-fields G_n such that T_n is G_n measurable and if we let $t_n = T_n - T_{n+1}$ then

$$E\{t_n|G_{n+1}\} = E\left\{X_n^{*2}|F_{n+1}\right\} \text{ almost surely,} \tag{3.12}$$

and

$$E\left\{t_n^{p/2}|G_{n+1}\right\} \le C_pE\left\{|X_n^*|^p|F_{n+1}\right\} \tag{3.13}$$

for some constant C_p depending only on p.

From Theorem 5 and the condition on W_n we need only show

$$W_n^{-2}T_n \to 1 \text{ almost surely} \tag{3.14}$$

and as a first step we obtain

$$W_n^{-2}\left(T_n-V_n^{*2}\right) \to 0 \text{ almost surely.} \tag{3.15}$$

We have

$$E\left\{X_j^{*4}|F_{j+1}\right\} \le 16E\left\{\tilde{X}_j^4|F_{j+1}\right\}$$

$$\le 16E\left\{X_j^4I(|X_j| \le z_j)|F_{j+1}\right\}$$

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and hence, from (3.8),

(3.16) $$\sum_{j=1}^{\infty} W_j^{-4} E\left\{X_j^{*4} \mid F_{j+1}\right\} < \infty \text{ almost surely.}$$

Clearly if (3.16) holds then

(3.17) $$\sum_{j=1}^{\infty} E\left\{X_j^{*4} \mid F_{j+1}\right\} < \infty \text{ almost surely.}$$

Also Lemma 2 below gives

(3.18) $$\sum_{j=1}^{\infty} W^{-2}\left[t_j - E\{t_j \mid G_{j+1}\}\right] < \infty \text{ almost surely}$$

and

(3.19) $$\sum_{j=1}^{\infty} \left[t_j - E\{t_j \mid G_{j+1}\}\right] < \infty \text{ almost surely.}$$

Finally an application of Lemma 1 of Heyde [5] gives

(3.20) $$W_n^{-2} \sum_{j=n}^{\infty} \left[t_j - E\{t_j \mid G_{j+1}\}\right] \to 0 \text{ almost surely}$$

which is equivalent to (3.15).

The following result which we used above is the reverse martingale analogue of Corollary 2.8.5 of Stout [14] and the martingale proof may be duplicated using the notion of starting times instead of stopping times. (For $\{F_j;\ j \geq 1\}$ a decreasing sequence of σ-fields a random variable T is a starting time if $\{T \leq j\} \in F_j$.)

LEMMA 2. *Suppose* $\{X_j,\ F_j;\ j \geq 1\}$ *is a reverse martingale difference sequence. If for some* p , $1 \leq p \leq 2$,

(3.21) $$\sum_{j=1}^{\infty} E\left\{|X_j|^p \mid F_{j+1}\right\} < \infty$$

then

(3.22) $$S_1 = \sum_{j=1}^{\infty} X_j \text{ converges almost surely.}$$

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The proof of the theorem will now be complete provided we show

(3.23) $$W_n^{-2} V_n^{*2} \to 1 \quad \text{almost surely}$$

which is essentially what condition (3.7) says.

To see this observe that

$$\begin{aligned} W_n^{-2} V_n^{*2} &= W_n^{-2} \sum_{j=n}^{\infty} E\left\{X_j^{*2} \mid F_{j+1}\right\} \\ &= W_n^{-2} \sum_{j=n}^{\infty} \left[E\left\{\tilde{X}_j^2 \mid F_{j+1}\right\} - \left(E\{\tilde{X}_j \mid F_{j+1}\}\right)^2\right] \\ &= W_n^{-2} \sum_{j=n}^{\infty} \left[E\left\{X_j^2 I(|X_j| \le z_j) \mid F_{j+1}\right\} - \left(E\{X_j I(|X_j| \le z_j) \mid F_{j+1}\}\right)^2\right] , \end{aligned}$$

so (3.23) follows. This completes the proof.

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