Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 376–383

POINTS OF LOCAL NONCONVEXITY AND FINITE UNIONS OF CONVEX SETS

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1. Introduction. Let S be a subset of $\mathbf{R}^d$. A point x in S is a *point of local convexity of* S if and only if there is some neighborhood U of x such that, if $y, z \in S \cap U$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Several interesting properties are known about sets whose lnc points Q may be decomposed into n convex sets. For S closed, connected, $S \sim Q$ connected, and Q having cardinality n, Guay and Kay [**2**] have proved that S is expressible as a union of $n + 1$ or fewer closed convex sets (and their result is valid in a locally convex topological vector space). For S closed, connected, and Q decomposable into n convex sets, Valentine [**7**] has shown that S is an L_{2n+1} set, and Stavrakas [**4**] has obtained conditions which insure that S be an L_{n+1} set. In this paper, we show that with suitable hypothesis, S may be decomposed into 2^n or fewer closed convex sets.

Throughout the paper, S is a closed, connected subset of $\mathbf{R}^d$, where $d =$ dim aff S. Q denotes the set of lnc points of S, and $S \sim Q$ is connected. We assume that $Q = \bigcup_{i=1}^n C_i$ where each C_i is convex. Since Q is a closed set, without loss of generality we consider each C_i to be closed. Further, we assume that n is minimal in the following sense:

For every i, there are points of C_i which do not belong to any C_j for $j \neq i, 1 \leqq i, j \leqq n$. That is, $C_i \not\subseteq \bigcup\{C_j : 1 \leqq j \leqq n, j \neq i\}$.

2. The dimension of the C_i sets. Guay and Kay [**2**] have proved that when Q is finite and nonempty, then S is planar. A similar result is obtained in our setting, for if $Q = \bigcup_{i=1}^n C_i$ where C_i is convex and essential, then dim $C_i = d - 2$. The following lemma will be important in our proof.

LEMMA 1. $S =$ cl (int S).

Proof. By appropriately adapting techniques employed in [**1**], it may be shown that $S \sim Q$ is dense in S. Thus $S \subseteq$ cl $(S \sim Q)$. Stavrakas [**3**] has proved that if S (not necessarily closed) is a nonplanar subset of $\mathbf{R}^3$ with $S \subseteq$ cl $(S \sim Q)$ and $S \sim Q$ connected, then $S \subseteq$ cl (int S). His proof generalizes easily to $\mathbf{R}^d$ where $d =$ dim aff S, and so $S \subseteq$ cl (int S) in our setting. Furthermore, since S is closed, cl (int S) $\subseteq S$, and the lemma is proved.

Received September 17, 1973 and in revised form, September 26, 1974.

Definition 1. If $Q = \bigcup_{i=1}^{n} C_i$ where C_i is convex, we say C_i is *essential* if and only if for every x in C_i, there is some neighborhood $\mathcal{N}'$ of x such that for $\mathcal{N}$ convex and $\mathcal{N} \subseteq \mathcal{N}'$, $(S \cap \mathcal{N}) \sim C_i$ is connected.

Note that for x in $C_i \equiv C$, C essential, and $\mathcal{N}$ a neighborhood of x satisfying Definition 1, x is an lnc point for the connected set cl T, where $T \equiv (S \cap \mathcal{N}) \sim C$. Moreover, every point of $C \cap \text{cl}\,\mathcal{N}$ is an lnc point for cl T: For y in $C \cap \mathcal{N}$, y is an lnc point for S and hence for $S \cap \mathcal{N}$. Since $S = \text{cl (int } S)$, cl $(S \cap \mathcal{N}) = \text{cl } [(S \cap \mathcal{N}) \sim C] = \text{cl } T$, and y is an lnc point for cl T. Since the lnc points of cl T form a closed set, each point of $C \cap \text{cl}\,\mathcal{N}$ is an lnc point for cl T. Trivially, $C \cap \text{cl } \mathcal{N}$ is essential for cl T when $\mathcal{N}$ is convex.

The following version of a result by Valentine [**7**, Corollary 2] will be needed.

LEMMA 2. *If* $[x, y] \cup [y, z] \subseteq S$ *and no point of* Q *lies in* conv $\{x, y, z\} \sim [x, z]$, *then* conv $\{x, y, z\} \subseteq S$.

THEOREM 1. *If* C_i *is essential, then* dim $C_i \leqq d - 2$ (where d = dim aff S).

Proof. Clearly dim $C_i \leqq d - 1$. Assume that for $C_i \equiv C$, dim $C = d - 1$ to obtain a contradiction. Since by an early remark, $Q = \bigcup_{i=1}^{n} C_i$ where n is minimal, we may select x in rel int C and in no other C_j set. Let $\mathcal{N}'$ be a neighborhood of x satisfying Definition 1. There is some convex neighborhood $\mathcal{N}$ of x, $\mathcal{N} \subseteq \mathcal{N}'$, with cl $\mathcal{N}$ disjoint from the remaining C_j sets. Moreover, $\mathcal{N}$ may be selected so that cl $\mathcal{N} \cap$ aff $C \subseteq C$. Letting $T = (S \cap \mathcal{N}) \sim C$, clearly all the lnc points for cl T lie in C.

Since $\mathcal{N} \subseteq \mathcal{N}'$, $T = T \sim C$ is connected. Also $T \sim C \subseteq \text{cl } T \sim C \subseteq \text{cl } T$, so cl $T \sim C$ is connected, and cl $T \sim$ aff C is connected.

Let H be the hyperplane determined by C, H_1, H_2 the corresponding open halfspaces. Certainly one of these sets, say H_1, contains points in cl T. The set $H_1 \cap$ cl T is locally convex, and since cl $T \sim$ aff C = cl $T \sim H$ is connected, $H_1 \cap$ cl T is connected. Thus $H_1 \cap$ cl T is polygonally connected, and for x, y in $H_1 \cap$ cl T, there is a polygonal path λ in $H_1 \cap$ cl T from x to y. By repeated use of Lemma 2, $[x, y] \subseteq$ cl T, and $H_1 \cap$ cl T is convex. Since cl T is not convex, there must be points of cl T in H_2, and cl $T \sim C$ = cl $T \sim H$ cannot be connected, a contradiction. Thus, our assumption is false and dim $C \leqq d - 2$.

The proof that dim $C_i \geqq d - 2$ will require two easy lemmas. We adopt the following standard terminology: For x, y in S, we say *x sees y via S* if and only if $[x, y] \subseteq S$. For $\mathcal{N}$ a subset of S, we say *x sees $\mathcal{N}$ via S* if and only if x sees every point of $\mathcal{N}$ via S.

LEMMA 3. *If* $[x, z] \subseteq S \sim (\bigcup_{i=1}^{n} \text{aff } C_i)$, *then there is a neighborhood* $\mathcal{N}$ *of* x *such that* z *sees* $\mathcal{N} \cap S$ *via* $S \sim (\bigcup_{i=1}^{n} \text{aff } C_i)$.

Proof. Since $S \sim (\bigcup_{i=1}^{n} \text{aff } C_i) \neq \emptyset$ is open in S, for every point q on $[x, z]$ there exists a convex neighborhood of q disjoint from $\bigcup_{i=1}^{n}$ aff C_i. Since $[x, z]$ is compact, clearly there is an open cylinder about $[x, z]$ disjoint from

$\bigcup_{i=1}^{n}$ aff C_i. Choose $\mathscr{N}$ to be a neighborhood of x interior to the cylinder with $\mathscr{N} \cap S$ convex. For p in $\mathscr{N} \cap S$, $[p, x] \cup [x, z] \subseteq S$, no point of Q lies in conv $\{p, x, z\}$, so by Lemma 2, $[p, z] \subseteq S$. Furthermore, $[p, z] \subseteq S \sim (\bigcup_{i=1}^{n}$ aff $C_i)$.

LEMMA 4. *If C_i is essential, $1 \leqq i \leqq n$, then $S \sim (\bigcup_{i=1}^{n}$ aff $C_i)$ is connected.*

Proof. Since $S \sim (\bigcup_{i=1}^{n} C_i)$ is connected and locally convex, it is polygonally connected, and by standard arguments, since $S =$ cl (int S), int $S \sim (\bigcup_{i=1}^{n} C_i)$ is polygonally connected and hence connected. By Theorem 1, dim $C_i \leqq d - 2$, so int $S \sim (\bigcup_{i=1}^{n}$ aff $C_i)$ is connected. Again using the fact that $S =$ cl(int S), $S \sim (\bigcup_{i=1}^{n}$ aff $C_i)$ is connected.

THEOREM 2. *If $C_i \neq \emptyset$ is essential, then* dim $C_i \geqq d - 2$.

Proof. For the moment, let $Q = C \neq \emptyset$ and assume dim $C \leqq d - 3$ to obtain a contradiction. Select points x, y in $S \sim$ aff C for which $[x, y] \not\subseteq S$. (Clearly such points exist for otherwise S would be convex.) Then since $S \sim$ aff C is connected (by Lemma 4) and locally convex, there is a polygonal path λ in $S \sim$ aff C from x to y. Without loss of generality, assume there is some z in $S \sim$ aff C for which $[x, z]$, $[z, y] \subseteq S \sim$ aff C.

Use Lemma 3 to select a neighborhood $\mathscr{N}$ of x such that z sees $\mathscr{N} \cap S$ via $S \sim$ aff C. Moreover, since $[x, y] \not\subseteq S$, $\mathscr{N}$ may be chosen so that y sees no point of $\mathscr{N} \cap S$ via S. Since dim $C \leqq d - 3$ and $S =$ cl (int S), there is some x_0 in $\mathscr{N} \cap S$ with $x_0 \notin$ aff $(C \cup \{z\})$. Clearly $[z, x_0] \subseteq S$, $[y, x_0] \not\subseteq S$.

Similarly, select a neighborhood $\mathscr{M}$ of y such that x_0 sees no point of $\mathscr{M} \cap S$ via S and z sees $\mathscr{M} \cap S$ via $S \sim$ aff C. Choose y_0 in $\mathscr{M}$ with $y_0 \notin$ aff $(C \cup \{x_0, z\})$.

Then no point p of aff C may lie relatively interior to conv $\{z, x_0, y_0\}$, for otherwise $y_0 \in$ aff $\{x_0, p, z\}$ and $y_0 \in$ aff $(C \cup \{x_0, z\})$, a contradiction.

Hence $[x_0, z]$, $[z, y_0]$ are in $S \sim$ aff C, no point of aff C is in conv $\{x_0, y_0, z\} \sim [x_0, y_0]$, so by Lemma 2, $[x_0, y_0] \subseteq S$. We have a contradiction, our assumption is false, and dim $C \geqq d - 2$.

To complete the proof, let $Q = \bigcup_{i=1}^{n} C_i$ where n is minimal. For C_i essential, let $C = C_i$. As in the proof of Theorem 1, select x in rel int C and in no other C_j set, and let $\mathscr{N}'$ satisfy Definition 1. Select a convex neighborhood $\mathscr{N}$ of x, $\mathscr{N} \subseteq \mathscr{N}'$, with cl $\mathscr{N}$ disjoint from the remaining C_j sets and with $\mathscr{N} \cap$ aff $C \subseteq C$. Letting $T = (S \cap \mathscr{N}) \sim C$, by previous remarks, $C \cap$ cl $\mathscr{N} \equiv Q_T$ is the set of lnc points for cl T. Also, cl $T \sim C =$ cl $T \sim$ aff C is connected.

Now since $S =$ cl (int S), dim $T = d$. By applying the first part of this proof to cl T, dim $Q_T \geqq d - 2$ and hence dim $C \geqq d - 2$, finishing the proof of the theorem.

COROLLARY. *If $C_i \neq \emptyset$ is essential, then* dim $C_i = d - 2$.

3. Expressing S as a finite union of convex sets. The representation theorem requires the following lemma. (We note that a form of Lemma 5 appears in [**4**, Theorem 4].)

LEMMA 5. *If $Q = C$ and C is essential, then every point of C sees S via S.*

Proof. By Lemma 4, $S \sim \text{aff } C$ is connected. Let $q \in C$ and examine the set A of points in $S \sim \text{aff } C$ which q sees via S. We assert that A is open and closed in $S \sim \text{aff } C$:

Clearly if (x_j) is a sequence in A converging to $x \in S \sim \text{aff } C$, then q sees x via S, and A is closed in $S \sim \text{aff } C$. To show A open in $S \sim \text{aff } C$, let $p \in A$. Then $[p, q) \subseteq S \sim \text{aff } C$ and we may select a sequence (q_i) on $[p, q)$ converging to q. Choose a neighborhood $\mathscr{M}'$ of p with $\mathscr{M}' \cap S \equiv \mathscr{M}$ convex and disjoint from aff C. For i arbitrary and r in $\mathscr{M}$, $[r, p] \cup [p, q_i] \subseteq S$, no point of C lies in conv $\{r, p, q_i\}$, so by Lemma 2, $[r, q_i] \subseteq S$. Thus for every i, q_i sees $\mathscr{M}$ via S. Since S is closed, q sees $\mathscr{M}$ via S, $\mathscr{M} \subseteq A$, and A is open in $S \sim \text{aff } C$.

Thus A is open and closed in the connected set $S \sim \text{aff } C$. If $A \neq \emptyset$, then $A = S \sim \text{aff } C$, and since $S = \text{cl}(\text{int } S)$, q sees S via S.

Now define

$$D \equiv \{q;\ q \text{ in } C \text{ and } q \text{ sees some point of } S \sim \text{aff } C \text{ via } S\}.$$

By a theorem of Valentine [**7**, Lemma 1], every point of S sees some point of C via S, so $D \neq \emptyset$. Also, by the preceding paragraph, each point of D sees S via S, so D is closed. Clearly conv $D \subseteq S$.

Since C is essential, we assert that $C = D$: Let $q \in C$ and let $\mathscr{N}'$ be any neighborhood of q satisfying Definition 1. For $\mathscr{N}$ a convex neighborhood of q with $\text{cl}\,\mathscr{N} \subseteq \mathscr{N}'$, let $T = (S \cap \mathscr{N}) \sim C$. Then cl T is a closed connected set having $C \cap \text{cl}\,\mathscr{N}$ as its set of lnc points. Again by Valentine's theorem, for every t in $\text{cl } T \sim \text{aff } C$, t sees via cl T some lnc point p of cl T. Hence t sees p via S, $p \in C$, and $p \in D$. We conclude that every neighborhood of q contains some point of D, so $q \in \text{cl } D$. Since D is closed, $q \in D$, and $C \subseteq D$. The reverse inclusion is obvious, and $C = D$, completing the proof.

Guay and Kay [**2**] have proved that for Q a singleton point, S is expressible as a union of two closed convex sets. The following theorem generalizes this result to the case in which Q is convex.

THEOREM 3. *If $Q = C \neq \emptyset$ and C is essential, then S may be represented as a union of two closed convex sets.*

Proof. By the corollary to Theorem 2, dim $C = d - 2$. Select points x, y in int $(S \sim \text{aff } C)$ with $[x, y] \not\subseteq S$ and $x \notin \text{aff}(C \cup \{y\})$. Then $[x, y] \cap \text{aff } C = \emptyset$. (Clearly this is possible by techniques used in Theorem 2, since $S = \text{cl}(\text{int } S)$.) Let H, J denote the hyperplanes determined by $C \cup \{x\}$, $C \cup \{y\}$, respectively. And let R denote the closed convex region determined by H, J which contains $[x, y]$. Note that int $R \cap \text{aff } C = \emptyset$ since $C \subseteq H \cap J$.

For z in $(R \cap S) \sim \text{aff } C$, z cannot see both x and y via S, for otherwise $[x, z] \cup [z, y] \subseteq S \sim C$, no point of C would be in conv $\{x, y, z\}$, and hence $[x, y] \subseteq S$, a contradiction.

Now let S_x denote the closed subset of S which x sees via S. Then we assert that $(R \cap S_x) \sim \text{aff } C$ is a convex set: For p, q in $(R \cap S_x) \sim \text{aff } C$, $[p, x] \cup [x, q] \subseteq S$, no point of aff C lies in conv $\{p, x, q\}$, and so conv $\{p, x, q\} \subseteq S \sim \text{aff } C$. Thus x sees $[p, q]$ via S, so $[p, q] \subseteq (R \cap S_x) \sim \text{aff } C$. Similarly, if S_y is the closed subset of S which y sees via S, $(R \cap S_y) \sim \text{aff } C$ is convex.

We will show that

$$(S \cap R) \sim \text{aff } C = [(S_x \cap R) \sim \text{aff } C] \cup [(S_y \cap R) \sim \text{aff } C]:$$

Let $z \in (S \cap R) \sim \text{aff } C$. Lemma 4 implies that $S \sim \text{aff } C$ is polygonally connected, so there is a polygonal path λ in $S \sim \text{aff } C$ from z to x. Let w denote the first point of λ in bdry R. By repeated use of Lemma 2, z sees w via S and $[z, w] \subseteq S \sim \text{aff } C$. To finish the argument, we consider two cases.

Case 1. Suppose $w \in H$. By Lemma 5, C sees S via S, so the $d - 1$ dimensional sets conv $(C \cup \{w\})$, conv $(C \cup \{x\})$ lie in S. Moreover, since H contains x, w and C, each of the above convex sets lies in $H \cap R$. Since dim $C = d - 2$, the sets conv $(C \cup \{x\})$, conv $(C \cup \{w\})$ necessarily intersect in some $p \in S \sim \text{aff } C$. Then $[w, p] \cup [p, x] \subseteq S$, no point of C lies in conv $\{w, p, x\}$, and by Lemma 2, $[w, x] \subseteq S \sim \text{aff } C$. Recall that by the preceding paragraph, $[z, w] \subseteq S \sim \text{aff } C$, and again by Lemma 2, $[x, z] \subseteq S$. Hence $z \in S_x \cap R$, the desired result.

Case 2. If $w \in J$, then by a similar argument, $[w, y] \subseteq S \sim \text{aff } C$, $[y, z] \subseteq S$, and $z \in S_y \cap R$.

We conclude that $(S \cap R) \sim \text{aff } C \subseteq [(S_x \cap R) \sim \text{aff } C] \cup [(S_y \cap R) \sim \text{aff } C]$. Since the reverse inclusion is obvious, the sets are equal.

The sets $(S_x \cap R) \sim \text{aff } C$, $(S_y \cap R) \sim \text{aff } C$ are disjoint, non-empty convex sets and can be separated by a hyperplane M. Since C sees $R \cap S$ via S, C is in the boundary of each of the above convex sets, and aff $C \subseteq M$. Furthermore, M can contain no point s of $H \sim \text{aff } C$ (or $J \sim \text{aff } C$), for otherwise aff $(C \cup \{s\}) = H \subseteq M$, clearly impossible. Since dim (aff C) $= d - 2 =$ dim $(H \cap M)$, $H \cap M = \text{aff } C$. Similarly $J \cap M = \text{aff } C$ and (bdry R) $\cap M \subseteq \text{aff } C$.

We assert that there is some point w in $(S \cap M) \sim R$: Otherwise $S \sim \text{aff } C$ would be the union of the disjoint sets $(S \cap M_1) \cup [(S_x \cap R) \sim \text{aff } C]$, $(S \cap M_2) \cup [(S_y \cap R) \sim \text{aff } C]$, where M_1, M_2 are the open halfspaces determined by M, with $S_x \cap R \subseteq \text{cl } M_1$ and $S_y \cap R \subseteq \text{cl } M_2$. But each of these sets is closed in $S \sim \text{aff } C$. The proof follows. Let

$$K \equiv (S \cap M_1) \cup [(S_x \cap R) \sim \text{aff } C]$$

and let p be any limit point of K. Then either p is in K or p is in M and hence in $R \cap M$. If $p \in (R \cap M) \sim \text{bdry } R$, then $p \in (S_x \cap R) \sim \text{aff } C \subseteq K$. If $p \in (\text{bdry } R) \cap M$, then $p \in \text{aff } C$. We conclude that K contains its limit points in $S \sim \text{aff } C$, and K is closed in $S \sim \text{aff } C$. Similarly $(S \cap M_2) \cup [(S_y \cap R) \sim \text{aff } C]$ is closed in $S \sim \text{aff } C$. However, this contradicts the connectedness of $S \sim \text{aff } C$, and we conclude that $(S \cap M) \sim R \neq \emptyset$.

Now let w be any point in $(S \cap M) \sim R$. We show that w sees S via S:

https://doi.org/10.4153/CJM-1975-046-6 Published online by Cambridge University Press

For p in $S \sim \text{aff } C$, there is a polygonal path λ in $S \sim \text{aff } C$ from w to p. For $[w, q] \cup [q, r]$ in λ, the only way that a point of aff C may lie in conv $\{w, q, r\}$ is for $[q, r]$ to contain some point of $(\text{int } R) \cap S \cap M$. Then $q \notin M$ for otherwise either $[w, q]$ or $[q, r]$ would cut aff C. Without loss of generality, assume $q \in M_1$. If $r \in M_2$, then some point of $[q, r]$ would lie in $(S_x \cap \text{int } R) \cap (S_y \cap \text{int } R)$, a contradiction since these sets are disjoint. Furthermore, $r \notin M_1$, for otherwise $[q, r]$ could not cut M. Hence r must lie in M. For s with $[r, s] \subseteq \lambda$, we may use Lemma 3 to select a neighborhood $\mathcal{N}$ of r such that both q and s see $\mathcal{N} \cap S$ via $S \sim \text{aff } C$. Select r_0 in $[q, r) \cap \mathcal{N} \cap S$ and replace r by r_0 in λ. Since $r_0 \in M_1$, no point of aff C lies in conv $\{w, q, r_0\}$, and $[w, r_0] \subseteq S \sim \text{aff } C$, by Lemma 2. We may repeat the argument for $[w, r_0] \cup [r_0, s]$. Inductively, if $[w, t_0] \cup [t_0, p] \subseteq S \sim \text{aff } C$, then $[w, p_0] \subseteq S \sim \text{aff } C$, where $p_0 \in [t_0, p)$ is selected arbitrarily close to p. Hence $[w, p] \subseteq S$ and w sees $S \sim \text{aff } C$ via S. Since $S = \text{cl }(\text{int } S)$, w sees S via S, the desired result.

Finally, if u, v belong to $S \cap M_1$, then $[u, w] \cup [w, v] \subseteq S$, no point of C is in conv $\{u, w, v\}$, and $[u, v] \subseteq S$. Thus $S \cap M_1$ is convex. Similarly, $S \cap M_2$ is convex, and the sets cl $(S \cap M_1)$, cl $(S \cap M_2)$ are convex sets whose union is S, completing the proof of the theorem.

The set C must be essential for Theorem 3 to hold, as the following example illustrates.

Example 1. Let D denote the unit disk centered at the origin in the complex plane, P the infinite sided convex polygon having sides $s_n = [t_{n-1}, t_n]$, where $t_n = \exp(\pi i/2^n)$ for $n \geqq 0$. Further, let R_n represent the closed region bounded by s_n and bdry D which does not contain P, let $P_n = n/(n+1)P$, and let $D_1 = D \times [0, -1]$ in $\mathbf{R}^3$.

Inductively, for each $n \geqq 1$, attach a copy P_{2n}' of P_{2n} to D_1 along s_{2n} at an appropriate angle so that for $S_{2n} \equiv D_1 \cup \text{conv } (R_{2n} \cup P_{2n}')$, the lnc points of S_{2n} are exactly s_{2n}, $S_{2n} \sim s_{2n}$ is connected, $S_{2n} \cap S_{2i} = D_1$ for $i < n$, and the P_{2n}' sets converge to P. If $S = \bigcup_{n=1}^{\infty} S_{2n}$, then S has P as its set of lnc points, $S \sim P$ is connected, yet S is not a finite union of convex sets.

Using Theorem 3, it is possible to obtain the following representation theorem.

THEOREM 4. *If* $Q = \bigcup_{i=1}^{n} C_i$, C_i *is essential for* $1 \leqq i \leqq n$, *and*

$$(\text{rel int } C_i) \cap C_j = \emptyset \textit{ for } i \neq j,$$

then S may be represented as a union of 2^n or fewer closed convex sets.

Proof. By the corollary to Theorem 2, dim $C_i = d - 2$ for each i. For C_1, as in the proof of Theorem 1, select a point x relatively interior to C_1 and in no $C_i, i \neq 1$. Let $\mathcal{N}'$ be a neighborhood of x satisfying Definition 1, and let $\mathcal{N}$ be a convex neighborhood of x disjoint from the remaining C_i sets, with cl $\mathcal{N} \cap \text{aff } C_1 \subseteq C_1$, $\mathcal{N} \subseteq \mathcal{N}'$. Letting $T = (S \cap \mathcal{N}) \sim C_1$, by previous arguments, cl $T \sim C_1$ is connected, dim $T = d$, and $C_1 \cap \text{cl } \mathcal{N} \equiv Q_T$ is the

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set of lnc points of cl T. Clearly Q_T is convex and essential and has dimension $d - 2$. By repeating the argument used in the proof of Theorem 3, we may select a hyperplane M so that cl $(T \cap M_1)$, cl $(T \cap M_2)$ are convex sets whose union is cl T.

We assert that all lnc points for cl$(S \cap M_1)$, cl$(S \cap M_2)$ are in $\bigcup_{i=1}^{n} C_i$: For y in $C_1 \sim [\bigcup_{i=2}^{n} C_i]$, since $x \in$ rel int C_1, $[x, y]$ is disjoint from $\bigcup_{i=2}^{n} C_i$. For every point p on $[x, y]$, select a neighborhood $\mathcal{N}_p$ of p disjoint from $\bigcup_{i=2}^{n} C_i$ and with $(S \cap \mathcal{N}_p) \sim C_1$ connected. Reduce to a finite subcollection $\mathcal{N}_1, \ldots, \mathcal{N}_j$ of the $\mathcal{N}_p$ sets which covers $[x, y]$. Choose a convex neighborhood U' of $[x, y]$ with cl $U' \subseteq \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_j$, and let $U \equiv (U' \cap S) \sim C_1$. Clearly the lnc points for cl U are exactly $C_1 \cap$ cl U, cl U is closed, connected, and cl $U \sim C_1$ is connected. Using the fact that cl $U \cap$ cl T is d dimensional, the previous argument for cl T may be adapted to cl U to show that M separates cl U into two convex sets, cl$(U \cap M_1)$ and cl$(U \cap M_2)$. Moreover, y cannot be an lnc point for cl$(S \cap M_i)$, for U' is a neighborhood of y whose intersection with cl$(S \cap M_i)$ is convex, $i = 1, 2$. Thus the lnc points for cl$(S \cap M_1)$, cl$(S \cap M_2)$ lie in $\bigcup_{i=2}^{n} C_i$, the desired result.

Let A_1, A_2 denote the components of $S \sim M$ containing $T \cap M_1$, $T \cap M_2$ respectively. Let B_1 denote the union of those components of $S \sim$ cl$(A_1 \cup A_2)$ whose closure contains points of A_1, B_2 the union of the remaining components of $S \sim$ cl$(A_1 \cup A_2)$. Define $P_1 \equiv$ cl$(A_1 \cup B_1)$, $P_2 \equiv$ cl$(A_2 \cup B_2)$. Since $S =$ cl(int S) and $S \sim Q$ is connected, the closure of every component of $S \sim$ cl$(A_1 \cup A_2)$ necessarily contains points of at least one of A_1, A_2. Hence P_1, P_2 are connected, and $S = P_1 \cup P_2$.

By our choice of B_1, B_2, it is clear that $P_1 \neq P_2$. Moreover, our previous argument for cl$(S \cap M_1)$, cl$(S \cap M_2)$ shows that the lnc points for P_1, P_2 lie in $\bigcup_{i=2}^{n} C_i$. We assert that $P_1 \cap C_i$, $P_2 \cap C_i$ are convex for $2 \leqq i \leqq n$: If $P_1 \cap C_i = \emptyset$ or $P_1 \cap C_i = C_i$, the result is trivial. Otherwise, each of A_1, A_2 contains points of C_i, and $P_1 \cap C_i =$ (cl M_1) $\cap C_i$, $P_2 \cap C_i =$ (cl M_2) $\cap C_i$, each a convex set.

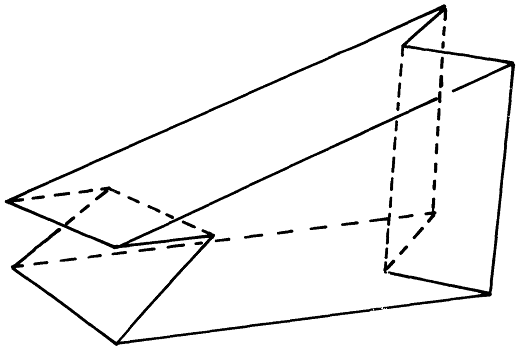
We have P_1 closed, connected, $P_1 \sim (\bigcup_{i=2}^{n} C_i)$ connected, and the lnc points for P_1 a union of $\leqq n - 1$ essential convex sets (and similarly for P_2). Hence the argument may be repeated for each of P_1, P_2, and for C_2, to obtain $\leqq 2^2$ sets, each having the above properties and with lnc points a union of $\leqq n - 2$ convex sets. Inductively, repeating the argument n times, we obtain $\leqq 2^n$ closed connected sets having no lnc points (and thus convex by a theorem of Tietze [**5**]). Therefore, S is expressible as a union of $\leqq 2^n$ closed convex sets, completing the proof.

The following example shows that the number 2^n in Theorem 4 is best possible.

Example 2. Let $S \subseteq \mathbf{R}^3$ be the set in Figure 1. Then Q is expressible as a union of $n = 2$ essential convex sets, yet S may not be decomposed into fewer than 4 convex sets. The example may be extended to higher values of n by

https://doi.org/10.4153/CJM-1975-046-6 Published online by Cambridge University Press

considering a prism whose basis is a $2n$ — gon and removing wedges appropriately from non-basis facets.



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