

# ON SOME EXTENSIONS OF PANJER'S CLASS OF COUNTING DISTRIBUTIONS<sup>1</sup>

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## ABSTRACT

In this paper we discuss some properties of counting distributions whose discrete density  $\{p_n\}_{n=0}^{\infty}$  satisfies a recursion in the form

$$p_n = \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} \quad (n = 1, 2, \dots)$$

with  $p_n = 0$  for  $n < 0$  and present an algorithm for recursive evaluation of corresponding compound distributions.

## 1. INTRODUCTION

Following PANJER (1981) there has grown up an extensive literature on recursive evaluation of compound distributions. Panjer assumed that the discrete density  $\{p_n\}_{n=0}^{\infty}$  of the counting distribution satisfied the recursion

$$p_n = \left( a + \frac{b}{n} \right) p_{n-1} \quad (n = 1, 2, \dots)$$

for some constants  $a$  and  $b$ . SUNDT and JEWELL (1981) showed that the only non-degenerate members of this class are the Poisson, the binomial, and the negative binomial distributions.

SCHRÖTER (1990) generalised Panjer's recursive algorithm to the class of counting distributions satisfying the recursion

$$p_n = \left( a + \frac{b}{n} \right) p_{n-1} + \frac{c}{n} p_{n-2} \quad (n = 1, 2, \dots)$$

with  $p_{-1} = 0$  for some constants  $a$ ,  $b$ , and  $c$ , and discussed the properties of this class. In particular he showed that the convolution of a Poisson distribution and a distribution from Panjer's class belongs to this extended class.

In the present paper we study the even more general class satisfying the recursion

$$(1) \quad p_n = \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} \quad (n = 1, 2, \dots)$$

<sup>1</sup> This paper is dedicated to Professor W.S. JEWELL on the occasion of his 60th birthday July 2, 1992.

for some integer  $k$  and constants  $a_i$  and  $b_i$  ( $i = 1, \dots, k$ ) with  $p_n = 0$  for  $n < 0$ . We see that  $p_0 > 0$  for all distributions in this class.

In Section 2 we introduce some notation and definitions. In Section 3 we discuss some properties of counting distributions satisfying (1). In Section 4 we study convolutions of such distributions, and Section 5 is devoted to mixtures of distributions satisfying (1). In Section 6 we generalise Panjer’s recursive algorithm for compound distributions to counting distributions satisfying (1). Finally we discuss some possible generalisations in Section 7.

2. DEFINITIONS AND NOTATION

**2A.** We shall denote a counting distribution with discrete density satisfying (1) by  $R_k[\mathbf{a}, \mathbf{b}]$  with  $\mathbf{a} = (a_1, \dots, a_k)$  and  $\mathbf{b} = (b_1, \dots, b_k)$ . Let  $\mathcal{R}_k$  denote the class of such distributions for fixed  $k$ . We see that any distribution in  $\mathcal{R}_{k-1}$  can be considered as a distribution in  $\mathcal{R}_k$  with  $a_k = b_k = 0$ . Thus  $\mathcal{R}_{k-1} \subset \mathcal{R}_k$ . We introduce

$$\mathcal{R}_k^0 = \mathcal{R}_k \sim \mathcal{R}_{k-1}; \quad (k = 1, 2, \dots)$$

the class  $\mathcal{R}_0$  consists of the degenerate distribution concentrated at zero. We see that Panjer’s class is equal to  $\mathcal{R}_1$ , and Schröter’s class is contained in  $\mathcal{R}_2$ .

The definitions of  $R_k[\mathbf{a}, \mathbf{b}]$ ,  $\mathcal{R}_k$ , and  $\mathcal{R}_k^0$  easily extend to  $k = \infty$ . In that case (1) can be written as

$$(2) \quad p_n = \sum_{i=1}^n \left( a_i + \frac{b_i}{n} \right) p_{n-i}. \quad (n = 1, 2, \dots)$$

The class  $\mathcal{R}_\infty^0$  consists of all distributions in  $\mathcal{R}_\infty$  that cannot be expressed as a distribution in  $\mathcal{R}_j$  for any finite  $j$ .

For the rest of the paper we shall for simplicity silently assume that  $k > 0$ .

**2B.** The *stop loss transform*  $\bar{F}$  of a cumulative distribution  $F$  is defined by

$$\bar{F}(x) = \int_x^\infty (y-x) dF(y) = \int_x^\infty (1-F(y)) dy.$$

**2C.** We make the convention that  $\sum_{i=p}^q = 0$  if  $q < p$ .

3. SOME PROPERTIES OF  $\mathcal{R}_k$

**3A.** Let  $\{p_n\}_{n=0}^\infty$  denote the discrete density of  $R_k[\mathbf{a}, \mathbf{b}]$ , and let  $\psi$  be the probability generating function of this distribution, that is,

$$\psi(s) = \sum_{n=0}^{\infty} p_n s^n.$$

We have

$$\begin{aligned} \psi'(s) &= \sum_{n=1}^{\infty} p_n n s^{n-1} = \sum_{n=1}^{\infty} n s^{n-1} \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} \\ &= \sum_{i=1}^k \sum_{n=i}^{\infty} (n a_i + b_i) p_{n-i} s^{n-1} = \sum_{i=1}^k \sum_{n=0}^{\infty} (n a_i + i a_i + b_i) p_n s^{n+i-1}, \end{aligned}$$

which gives

$$(3) \quad \psi'(s) = \sum_{i=1}^k [a_i s^i \psi'(s) + (i a_i + b_i) s^{i-1} \psi(s)].$$

With

$$\rho(s) = \frac{d}{ds} \ln \psi(s) = \frac{\psi'(s)}{\psi(s)}$$

we obtain

$$(4) \quad \rho(s) = \frac{\sum_{i=1}^k (i a_i + b_i) s^{i-1}}{1 - \sum_{i=1}^k a_i s^i},$$

which together with the initial condition  $\psi(1) = 1$  determines the distribution  $R_k[a, b]$  uniquely. We therefore have the following theorem.

**Theorem 1.** A counting distribution belongs to  $\mathcal{P}_k$  if and only if the derivative of the natural logarithm of its probability generating function can be expressed as the ratio between a polynomial of degree at most  $k - 1$  and a polynomial of degree at most  $k$  with a non-zero constant term.

By multiplying numerator and denominator in (4) by  $1 + qs$  for an arbitrary number  $q$  and rearranging them, we obtain

$$\rho(s) = \frac{\sum_{i=1}^{k+1} (i c_i + d_i) s^{i-1}}{1 - \sum_{i=1}^{k+1} c_i s^i}$$

with

$$c_i = a_i + q a_{i-1} \quad d_i = b_i + q (b_{i-1} - a_{i-1}) \quad (i = 1, 2, \dots, k + 1)$$

$$a_0 = -1 \qquad b_0 = b_{k+1} = a_{k+1} = 0,$$

and thus  $R_k[\mathbf{a}, \mathbf{b}] = R_{k+1}[\mathbf{c}, \mathbf{d}]$  with  $\mathbf{c} = (c_1, \dots, c_{k+1})$  and  $\mathbf{d} = (d_1, \dots, d_{k+1})$ . From this we conclude that the  $k$ -tuples  $\mathbf{a}$  and  $\mathbf{b}$  are not uniquely determined by  $R_k[\mathbf{a}, \mathbf{b}]$  if  $R_k[\mathbf{a}, \mathbf{b}] \notin \mathcal{R}_k^0$ .

On the other hand, if  $R_k[\mathbf{a}, \mathbf{b}] \in \mathcal{R}_k^0$ , then there exists no  $k' < k$  such that  $\rho(s)$  can be written in the form

$$\rho(s) = \frac{\sum_{i=1}^{k'} (ia'_i + b'_i) s^{i-1}}{1 - \sum_{i=1}^{k'} a'_i s^i}.$$

This means that the numerator and the denominator in (4) do not have any common factors, and thus the coefficients of these polynomials must be uniquely determined by  $\rho$ , which implies that they are uniquely determined by  $R_k[\mathbf{a}, \mathbf{b}]$ .

We have now proved the following theorem.

**Theorem 2.** The  $k$ -tuples  $\mathbf{a}$  and  $\mathbf{b}$  are uniquely determined by  $R_k[\mathbf{a}, \mathbf{b}]$  if and only if  $R_k[\mathbf{a}, \mathbf{b}] \in \mathcal{R}_k^0$ .

**Example 1.** The Poisson distribution with discrete density

$$p_n = \frac{\lambda^n}{n!} e^{-\lambda} \quad (n = 0, 1, \dots)$$

satisfies the recursion

$$p_n = \frac{\lambda}{n} p_{n-1} \quad (n = 1, 2, \dots)$$

$$p_0 = e^{-\lambda},$$

that is, this distribution is equal to  $R_1[0, \lambda]$ , and we have  $\rho(s) = \lambda$ . However, we can also write

$$\rho(s) = \frac{\lambda + q\lambda s}{1 + qs},$$

and thus  $R_1[0, \lambda] = R_2[(-q, 0), (\lambda + q, q\lambda)]$ . Therefore this distribution satisfies the recursion

$$p_n = \left(-q + \frac{\lambda + q}{n}\right) p_{n-1} + \frac{q\lambda}{n} p_{n-2} \quad (n = 1, 2, \dots)$$

with  $p_{-1} = 0$ . This example has also been discussed by SCHRÖTER (1990).

**3B.** Let  $N$  be a random variable with distribution  $R_k[\mathbf{a}, \mathbf{b}]$ . As  $\rho(1) = EN$  and  $\rho'(1) = \text{Var } N - EN$ , we obtain from (4)

$$EN = \frac{\sum_{i=1}^k (ia_i + b_i)}{1 - \sum_{i=1}^k a_i}$$

$$\text{Var } N = \frac{\sum_{i=1}^k i(ia_i + b_i)}{1 - \sum_{i=1}^k a_i} + \frac{\sum_{i=1}^k (ia_i + b_i) \sum_{j=1}^k ja_j}{\left(1 - \sum_{i=1}^k a_i\right)^2} = \frac{\sum_{i=1}^k i[(i + EN)a_i + b_i]}{1 - \sum_{i=1}^k a_i}.$$

These formulae generalise Proposition 2 in SCHRÖTER (1990).

**3C.** The following theorem shows that any distribution on the range  $\{0, 1, \dots, k\}$  with positive probability at zero is contained in  $\mathcal{R}_k$ .

**Theorem 3.** A distribution on the range  $\{0, \dots, k\}$  with positive probability at zero and discrete density  $\{p_n\}_{n=0}^k$  can be expressed as  $R_k[\mathbf{a}, \mathbf{b}]$  with  $\mathbf{a} = (a_1, \dots, a_k)$  and  $\mathbf{b} = (b_1, \dots, b_k)$  given by

$$a_i = -\frac{p_i}{p_0} \quad b_i = 2i \frac{p_i}{p_0} \quad (i = 1, \dots, k)$$

**Proof.** With  $\psi$  denoting the probability generating function of the distribution  $\{p_n\}_{n=0}^k$ , we have

$$\frac{d}{ds} \ln \psi(s) = \frac{\psi'(s)}{\psi(s)} = \frac{\sum_{i=1}^k ip_i s^{i-1}}{\sum_{i=0}^k p_i s^i} = \frac{\sum_{i=1}^k i \frac{p_i}{p_0} s^{i-1}}{1 - \sum_{i=1}^k \left(-\frac{p_i}{p_0}\right) s^i},$$

and the theorem follows by comparison with (4). Q.E.D.

The distribution in Theorem 3 is not necessarily contained in  $\mathcal{R}_k^0$ . For instance, if it is binomial, then it is contained in  $\mathcal{R}_1^0$  regardless of  $k$ .

Theorem 3 holds in particular for  $k = \infty$ . Thus we see that all counting distributions with positive probability at zero belong to  $\mathcal{R}_\infty$ .

#### 4. CONVOLUTIONS

**4A.** Let  $\psi_1, \psi_2$ , and  $\psi$  be the probability generating functions of  $R_k[\mathbf{a}, \mathbf{b}]$ ,

$R_l[\mathbf{c}, \mathbf{d}]$ , and their convolution. From (4) we obtain

$$\begin{aligned} \frac{d}{ds} \ln \psi(s) &= \frac{d}{ds} \ln [\psi_1(s) \psi_2(s)] = \frac{d}{ds} \ln \psi_1(s) + \frac{d}{ds} \ln \psi_2(s) \\ &= \frac{\sum_{i=1}^k (ia_i + b_i) s^{i-1}}{1 - \sum_{i=1}^k a_i s^i} + \frac{\sum_{i=1}^l (ic_i + d_i) s^{i-1}}{1 - \sum_{i=1}^l c_i s^i} \\ &= \frac{\left( \sum_{i=1}^k (ia_i + b_i) s^{i-1} \right) \left( 1 - \sum_{i=1}^l c_i s^i \right) + \left( \sum_{i=1}^l (ic_i + d_i) s^{i-1} \right) \left( 1 - \sum_{i=1}^k a_i s^i \right)}{\left( 1 - \sum_{i=1}^k a_i s^i \right) \left( 1 - \sum_{i=1}^l c_i s^i \right)}. \end{aligned}$$

This is the ratio of a polynomial of degree at most  $k+l-1$  and a polynomial of degree at most  $k+l$  with a non-zero constant term, and from Theorem 1 we see that the convolution of  $R_k[\mathbf{a}, \mathbf{b}]$  and  $R_l[\mathbf{c}, \mathbf{d}]$  is contained in  $\mathcal{R}_{k+l}$ . Thus we have the following result.

**Theorem 4.** The convolution of a distribution in  $\mathcal{R}_k$  and a distribution in  $\mathcal{R}_l$  is a distribution in  $\mathcal{R}_{k+l}$ .

Even if  $R_k[\mathbf{a}, \mathbf{b}] \in \mathcal{R}_k^0$  and  $R_l[\mathbf{c}, \mathbf{d}] \in \mathcal{R}_l^0$ , we cannot conclude that their convolution is a distribution in  $\mathcal{R}_{k+l}^0$ ; from the way we constructed  $\frac{d}{ds} \ln \psi(s)$ ,

we see that if the polynomials  $1 - \sum_{i=1}^k a_i s^i$  and  $1 - \sum_{i=1}^l c_i s^i$  have a common factor of degree  $q$ , then the convolution is a distribution in  $\mathcal{R}_{k+l-q}$ . In particular, if  $l = k$  and  $\mathbf{c} = \mathbf{a}$ , we obtain

$$\begin{aligned} \frac{d}{ds} \ln \psi(s) &= \frac{\sum_{i=1}^k (ia_i + b_i) s^{i-1}}{1 - \sum_{i=1}^k a_i s^i} + \frac{\sum_{i=1}^k (ia_i + d_i) s^{i-1}}{1 - \sum_{i=1}^k a_i s^i} \\ &= \frac{\sum_{i=1}^k [ia_i + (ia_i + b_i + d_i)] s^{i-1}}{1 - \sum_{i=1}^k a_i s^i}, \end{aligned}$$

that is, the convolution of  $R_k[\mathbf{a}, \mathbf{b}]$  and  $R_k[\mathbf{a}, \mathbf{d}]$  is  $R_k[\mathbf{a}, \mathbf{e}]$  with  $\mathbf{e} = (e_1, \dots, e_k)$  given by

$$e_i = ia_i + b_i + d_i. \quad (i = 1, \dots, k)$$

The following theorem is an obvious generalisation of this result.

**Theorem 5.** The convolution of the distributions  $R_k[\mathbf{a}, \mathbf{b}^{(j)}]$  with  $\mathbf{a} = (a_1, \dots, a_k)$  and  $\mathbf{b}^{(j)} = (b_1^{(j)}, \dots, b_k^{(j)})$  ( $j = 1, \dots, m$ ) is  $R_k[\mathbf{a}, \boldsymbol{\beta}]$  with  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$  given by

$$\beta_i = (m-1)ia_i + \sum_{j=1}^m b_i^{(j)}. \quad (i = 1, \dots, k)$$

**Corollary 1.** The  $m$ -fold convolution of  $R_k[\mathbf{a}, \mathbf{b}]$  is  $R_k[\mathbf{a}, \boldsymbol{\beta}]$  with  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$  given by

$$\beta_i = (m-1)ia_i + mb_i. \quad (i = 1, \dots, k)$$

The following corollary is a simple consequence of Theorem 3 and Corollary 1.

**Corollary 2.** The  $m$ -fold convolution of a distribution on the range  $\{0, \dots, k\}$  with positive probability at zero and discrete density  $\{p_n\}_{n=0}^k$  is  $R_k[\mathbf{a}, \mathbf{b}]$  with  $\mathbf{a} = (a_1, \dots, a_k)$  and  $\mathbf{b} = (b_1, \dots, b_k)$  given by

$$a_i = -\frac{p_i}{p_0} \quad b_i = (m+1)i\frac{p_i}{p_0}. \quad (i = 1, \dots, k)$$

The recursive algorithm for evaluation of convolutions indicated by Corollary 2, was presented by DE PRIL (1985).

**4B.** Any counting distribution with positive probability at zero can be expressed in the form  $R_\infty[\mathbf{a}, \mathbf{b}]$  for any sequence  $\mathbf{a} = (a_1, a_2, \dots)$ ; if  $\{p_n\}_{n=0}^\infty$  is the discrete density of this distribution, then we can let  $\mathbf{b} = (b_1, b_2, \dots)$  with the  $b_n$ 's given by the recursive algorithm

$$b_n = \frac{1}{p_0} \left[ np_n - \sum_{i=1}^{n-1} (na_i + b_i)p_{n-i} \right] - na_n, \quad (n = 1, 2, \dots)$$

which is obtained by solving (2) with respect to  $b_n$ . By combining this result with Theorem 5, we obtain the following recursive algorithm for evaluating convolutions of counting distributions with positive probability at zero.

**Theorem 6.** The discrete density  $\{\pi_n\}_{n=0}^\infty$  of the convolution of  $m$  counting distributions with positive probability at zero and discrete densities respectively

$\{p_n^{(j)}\}_{n=0}^\infty$  ( $j = 1, \dots, m$ ) can be evaluated recursively by

$$(5) \quad \pi_n = \sum_{i=1}^n \left( a_i + \frac{\beta_i}{n} \right) \pi_{n-i} \quad (n = 1, 2, \dots)$$

$$(6) \quad \pi_0 = \prod_{j=1}^m p_0^{(j)}$$

with

$$\beta_n = (m-1)na_n + \sum_{j=1}^m b_n^{(j)} \quad (n = 1, 2, \dots)$$

$$b_n^{(j)} = \frac{1}{p_0^{(j)}} \left[ np_n^{(j)} - \sum_{i=1}^{n-1} (na_i + b_i^{(j)}) p_{n-i}^{(j)} \right] - na_n. \quad (n = 1, 2, \dots; j = 1, \dots, m)$$

This algorithm holds for any sequence  $(a_1, a_2, \dots)$  of real numbers.

The algorithm of Theorem 6 becomes much simpler in the special case  $\mathbf{a} = \mathbf{0}$ , and normally one would presumably apply this choice of  $\mathbf{a}$ . However, in some applications one might get computer overflow or underflow when performing the recursions, and we might be able to overcome this problem by using non-zero values of  $a_i$  for some values of  $i$ . For recursive evaluation of compound distributions when the counting distribution belongs to the Panjer class, the problem with overflow and underflow has been discussed by PANJER and WILLMOT (1986).

Let us now look at the special case when the  $m$  distributions are identical. For simplicity we drop the top-scripts in this case, and we put  $\mathbf{a} = \mathbf{0}$ . Under these assumptions, (5) and (6) reduce to

$$(7) \quad \pi_n = \frac{m}{n} \sum_{i=1}^n b_i \pi_{n-i} \quad (n = 1, 2, \dots)$$

$$(8) \quad \pi_0 = p_0^m.$$

It is remarkable that when we have calculated the  $b_i$ 's, then we can easily evaluate the  $m$ -fold convolution of  $\{p_n\}_{n=0}^\infty$  for any  $m$ . It is interesting to compare this algorithm with the algorithm implied by Corollary 2. It seems that if we want to evaluate the  $m$ -fold convolution for one particular value of  $m$ , then the algorithm of Corollary 2 would be preferable. However, if we want the  $m$ -fold convolutions for several values of  $m$ , then it might be more efficient to first calculate the  $b_i$ 's and then use (7) and (8).

The recursive algorithm of Theorem 6 was presented by DE PRIL (1989) with  $\mathbf{a} = \mathbf{0}$ . De Pril also deduced a closed-form expression for the  $b_n$ 's. As the algorithm is rather time-consuming, De Pril introduced a class of approximations, and he gave upper bounds for the inaccuracy of these approximations.

4C. For the rest of Section 4 we shall concentrate on convolutions of distributions in  $\mathcal{R}_1$ .

By putting  $k = 1$  in Theorem 5 we obtain the following corollary.

**Corollary 3.** The convolution of the  $m$  distributions  $R_1[a, b_1], \dots, R_1[a, b_m]$  is  $R_1[a, \beta]$  with

$$\beta = (m-1)a + \sum_{j=1}^m b_j.$$

The following theorem is proved by SUNDT and JEWELL (1981).

**Theorem 7.** A distribution  $R_1[a, b] \in \mathcal{R}_1^0$  is binomial if  $a < 0$ , Poisson if  $a = 0$ , and negative binomial if  $a > 0$ .

Let us apply Corollary 3 to each of the three cases described in Theorem 7.

i) *Binomial*

Let the  $j$ th distribution be binomial with parameters  $(t_j, q)$ , that is, it has discrete density

$$p_n^{(j)} = \binom{t_j}{n} q^n (1-q)^{t_j-n}. \quad (n = 0, 1, \dots, t_j)$$

Then

$$a = -\frac{q}{1-q} \quad b_j = \frac{q}{1-q} (t_j + 1),$$

and we obtain

$$b = (m-1) \left( -\frac{q}{1-q} \right) + \sum_{j=1}^m \frac{q}{1-q} (t_j + 1) = \frac{q}{1-q} \left( \sum_{j=1}^m t_j + 1 \right),$$

that is, the convolution is binomial with parameters  $\left( \sum_{j=1}^m t_j, q \right)$ .

ii) *Poisson*

Let the  $j$ th distribution be Poisson with parameter  $\lambda_j$ , that is, it has discrete density

$$p_n^{(j)} = \frac{\lambda_j^n}{n!} e^{-\lambda_j}. \quad (n = 0, 1, 2, \dots)$$

Then

$$a = 0 \quad b_j = \lambda_j,$$

and we obtain

$$b = (m-1)0 + \sum_{j=1}^m \lambda_j = \sum_{j=1}^m \lambda_j,$$

that is, the convolution is Poisson with parameter  $\sum_{j=1}^m \lambda_j$ .

iii) *Negative binomial*

Let the  $j$ th distribution be negative binomial with parameters  $(\alpha_j, q)$ , that is, it has discrete density

$$p_n^{(j)} = \binom{\alpha_j + n - 1}{n} (1-q)^{\alpha_j} q^n. \quad (n = 0, 1, 2, \dots)$$

Then

$$a = q \quad b_j = q(\alpha_j - 1),$$

and we obtain

$$b = (m-1)q + \sum_{j=1}^m q(\alpha_j - 1) = q \left( \sum_{j=1}^m \alpha_j - 1 \right),$$

that is, the convolution is negative binomial with parameters  $\left( \sum_{j=1}^m \alpha_j, q \right)$ .

The results shown above about convolutions in these three classes of distributions should be well known. However, it seems interesting to consider them in relation to Corollary 3.

**4D.** Let us now more generally consider the convolution of the  $m$  distributions  $R_1[a_1, b_1], \dots, R_1[a_m, b_m]$ .

We have the following result.

**Theorem 8.** The convolution of the  $m$  distributions  $R_1[a_1, b_1], \dots, R_1[a_m, b_m]$  is  $R_m[\mathbf{a}, \boldsymbol{\beta}]$ , with  $\mathbf{a} = (\alpha_1, \dots, \alpha_m)$  and  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$  given by

$$(9) \quad \alpha_i = (-1)^{i+1} \sum_{1 \leq j_1 < j_2 < \dots < j_i \leq m} \prod_{k=1}^i a_{j_k} \quad (i = 1, \dots, m)$$

$$(10) \quad \beta_i = (-1)^{i+1} \sum_{r=1}^m b_r \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_{i-1} \leq m \\ j_i \neq r (r=1, \dots, i-1)}} \prod_{k=1}^{i-1} a_{j_k} \quad (i = 2, \dots, m)$$

$$(11) \quad \beta_1 = \sum_{j=1}^m b_j.$$

**Proof.** The probability generating function  $\psi$  of the convolution is given by

$$(12) \quad \rho(s) = \frac{d}{ds} \ln \psi(s) = \sum_{j=1}^m \frac{a_j + b_j}{1 - a_j s}$$

with the initial condition  $\psi(1) = 1$ . On the other hand, by Theorem 4 we see that the convolution is a distribution in  $\mathcal{S}_m$ , which can be written in the form  $R_m[\mathbf{a}, \boldsymbol{\beta}]$ , and thus

$$(13) \quad \rho(s) = \frac{\sum_{i=1}^m (i\alpha_i + \beta_i) s^{i-1}}{1 - \sum_{i=1}^m \alpha_i s^i}.$$

It remains to show that  $\mathbf{a}$  and  $\boldsymbol{\beta}$  given by (9)-(11) satisfy (12) and (13).

We rewrite (12) as

$$(14) \quad \rho(s) = \frac{\sum_{j=1}^m (a_j + b_j) \prod_{k \neq j} (1 - a_k s)}{\prod_{k=1}^m (1 - a_k s)}.$$

We see that (13) and (14) are satisfied if

$$(15) \quad \sum_{i=1}^m (i\alpha_i + \beta_i) s^{i-1} = \sum_{j=1}^m (a_j + b_j) \prod_{k \neq j} (1 - a_k s)$$

$$(16) \quad 1 - \sum_{i=1}^m \alpha_i s^i = \prod_{k=1}^m (1 - a_k s).$$

From (16) we obtain (9), and (15) gives (11) and

$$(17) \quad \beta_i = (-1)^{i+1} \sum_{r=1}^m (a_r + b_r) \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_{i-1} \leq m \\ j_t \neq r \ (t=1, \dots, i-1)}} \prod_{k=1}^{i-1} a_{j_k} - i\alpha_i \quad (i = 2, \dots, m)$$

Insertion of (9) in (17) gives (10). This completes the proof of Theorem 8.

Q.E.D.

We see that the  $\alpha_i$ 's do not depend on the  $b_j$ 's.

In the special case when all the  $b_j$ 's have the same value  $b$ , then (10) simplifies to

$$(18) \quad \beta_i = -(m-i+1)b\alpha_{i-1}. \quad (i = 2, \dots, m)$$

In particular, if  $b = 0$ , then (11) and (18) give that all the  $\beta_i$ 's are equal to zero too. In this case the  $j$ th distribution is geometric with parameter  $a_j$ , that is, negative binomial with parameters  $(1, a_j)$ .

**4E.** Let us look at the case  $m = 2$ . In this case Theorem 8 reduces to the following corollary.

**Corollary 4.** The convolution of  $R_1[a_1, b_1]$  and  $R_1[a_2, b_2]$  is  $R_2[(a_1 + a_2, -a_1 a_2), (b_1 + b_2, -(a_1 b_2 + a_2 b_1))]$ .

Corollary 4 was proved by SCHRÖTER (1990) in the special case  $a_2 = 0$ .

Corollary 4 applies in particular when  $a_1 = a_2 = a$ ; in that case we obtain that the convolution is  $R_2[(2a, -a^2), (b_1 + b_2, -a(b_1 + b_2))]$ . However, by Corollary 3 this distribution does not belong to  $\mathcal{R}_2^0$ , and it is more convenient to express it as  $R_1[a, a + b_1 + b_2]$ .

**Example 2.** We consider the convolution of a binomial distribution with parameters  $(t, q)$  and a negative binomial distribution with parameters  $(\alpha, q)$ . Then

$$\begin{aligned} a_1 &= -\frac{q}{1-q} & b_1 &= \frac{q}{1-q}(t+1) \\ a_2 &= q & b_2 &= q(\alpha-1), \end{aligned}$$

and by Corollary 4 the convolution is equal to

$$R_2 \left[ \left( -\frac{q^2}{1-q}, \frac{q^2}{1-q} \right), \left( q \left( \frac{t+q}{1-q} + \alpha \right), \frac{q^2}{1-q}(\alpha-t-2) \right) \right].$$

### 5. MIXTURES

It is natural to ask whether a mixture of a distribution in  $\mathcal{R}_k$  and a distribution in  $\mathcal{R}_l$  belongs to  $\mathcal{R}_m$  for some finite  $m$  when  $k$  and  $l$  are finite. Unfortunately we cannot give a general yes or not to this question. There are cases where the property holds, but there are also cases where it does not. In this section we shall look at some examples.

We start with a trivial observation. As a finite mixture of counting distributions on a finite range with positive probability at zero is a counting distribution on a finite range with positive probability at zero, the property holds for distributions on finite ranges by Theorem 3.

Now let  $\{p_n^{(1)}\}_{n=0}^\infty$  and  $\{p_n^{(2)}\}_{n=0}^\infty$  be discrete densities in  $\mathcal{R}_k$  resp.  $\mathcal{R}_l$  with probability generating functions  $\psi_1$  resp.  $\psi_2$ . Let  $\{p_n\}_{n=0}^\infty$  be the mixture defined by

$$p_n = \nu p_n^{(1)} + (1 - \nu) p_n^{(2)}, \quad (n = 0, 1, 2, \dots; 0 < \nu < 1)$$

and let  $\psi$  denote its probability generating function. We have

$$\psi = \nu \psi_1 + (1 - \nu) \psi_2,$$

which implies

$$(19) \quad \rho(s) = \frac{d}{ds} \ln \psi(s) = \frac{\nu \psi_1'(s) + (1 - \nu) \psi_2'(s)}{\nu \psi_1(s) + (1 - \nu) \psi_2(s)}.$$

We can apply (19) and Theorem 1 to decide whether the mixture belongs to  $\mathcal{R}_m$  for some finite  $m$ .

**Example 3.** We look at a mixture between two Poisson distributions with parameters  $\lambda_1$  and  $\lambda_2$  with  $\lambda_1 \neq \lambda_2$ . Then

$$\psi_j(s) = e^{\lambda_j(s-1)} \quad \psi_j'(s) = \lambda_j e^{\lambda_j(s-1)}, \quad (j = 1, 2)$$

and insertion in (19) gives

$$\rho(s) = \frac{\nu \lambda_1 e^{\lambda_1(s-1)} + (1 - \nu) \lambda_2 e^{\lambda_2(s-1)}}{\nu e^{\lambda_1(s-1)} + (1 - \nu) e^{\lambda_2(s-1)}},$$

which obviously cannot be written as the ratio between two polynomials. Thus the mixture does not belong to  $\mathcal{R}_m$  for any finite  $m$ .

**Example 4.** Let us look at a mixture between two geometric distributions with parameters  $q_1$  and  $q_2$  with  $q_1 \neq q_2$ . Then

$$\psi_j(s) = \frac{1 - q_j}{1 - q_j s} \quad \psi_j'(s) = \frac{q_j(1 - q_j)}{(1 - q_j s)^2}, \quad (j = 1, 2)$$

Insertion in (19) and some rearranging gives

$$\rho(s) = \frac{\nu q_1(1 - q_1)(1 - q_2 s)^2 + (1 - \nu) q_2(1 - q_2)(1 - q_1 s)^2}{\nu(1 - q_1)(1 - q_1 s)(1 - q_2 s)^2 + (1 - \nu)(1 - q_2)(1 - q_2 s)(1 - q_1 s)^2}.$$

We see that the numerator in this fraction is a polynomial of degree two and the denominator a polynomial of degree three with a non-zero constant term. Thus the mixture is contained in  $\mathcal{R}_3$ .

**Example 5.** We consider a mixture between two negative binomial distributions with parameters  $(\alpha_1, q)$  and  $(\alpha_2, q)$  with  $\alpha_2 > \alpha_1$ . Then

$$\psi_j(s) = \left( \frac{1-q}{1-qs} \right)^{\alpha_j} \quad \psi'_j(s) = \frac{\alpha_j q (1-q)^{\alpha_j}}{(1-qs)^{\alpha_j+1}}. \quad (j = 1, 2)$$

Insertion in (19) and some rearranging gives

$$\rho(s) = \frac{vaq(1-qs)^\beta + (1-v)(a+\beta)q(1-q)^\beta}{v(1-qs)^{\beta+1} + (1-v)(1-q)^\beta(1-qs)}$$

with  $\alpha = \alpha_1$  and  $\beta = \alpha_2 - \alpha_1$ . If  $\beta$  is an integer, then the numerator is a polynomial of degree  $\beta$  and the denominator a polynomial of degree  $\beta + 1$  with a non-zero constant term, and thus the mixture belongs to  $\mathcal{R}_{\beta+1}$ . However, if  $\beta$  is not an integer, then the mixture does not belong to  $\mathcal{R}_m$  for any finite  $m$ .

### 6. COMPOUND DISTRIBUTIONS

**6A.** Let  $N$  be a non-negative integer-valued random variable with distribution  $R_k[\mathbf{a}, \mathbf{b}]$ , and let  $Y_1, Y_2, \dots$  be non-negative integer-valued random variables, mutually independent and identically distributed with common discrete density  $f$ , and independent of  $N$ . We denote by  $\{p_n\}_{n=0}^\infty$  the discrete density of  $N$ . Let  $g$  denote the discrete density of

$$X = \sum_{i=1}^N Y_i,$$

that is,

$$g = \sum_{n=0}^\infty p_n f^{n*}.$$

For convenience we introduce  $q = f(0)$ .

**Theorem 9.** We can evaluate  $g(x)$  recursively by the algorithm

$$(20) \quad g(x) = \frac{1}{1 - \sum_{i=1}^k a_i q^i} \sum_{y=1}^x g(x-y) \sum_{i=1}^k \left( a_i + \frac{b_i y}{i x} \right) f^{i*}(y) \quad (x = 1, 2, \dots)$$

$$(21) \quad g(0) = \sum_{n=0}^\infty p_n q^n.$$

**Proof.** Formula (21) obviously holds.

Let  $X_n = \sum_{i=1}^n Y_i$  ( $n = 1, 2, \dots$ ). For  $x > 0$  we have

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} p_n f^{n*}(x) = \sum_{n=1}^{\infty} \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} f^{n*}(x) \\ &= \sum_{i=1}^k \sum_{n=i}^{\infty} p_{n-i} \left( a_i + \frac{b_i}{n} \right) f^{n*}(x) = \sum_{i=1}^k \sum_{n=i}^{\infty} p_{n-i} \mathbf{E} \left[ a_i + \frac{b_i}{i} \frac{X_i}{x} \mid X_n = x \right] f^{n*}(x) \\ &= \sum_{i=1}^k \sum_{n=i}^{\infty} \sum_{y=0}^x p_{n-i} \left( a_i + \frac{b_i}{i} \frac{y}{x} \right) f^{i*}(y) f^{(n-i)*}(x+y) \\ &= \sum_{y=0}^x \sum_{i=1}^k \left( a_i + \frac{b_i}{i} \frac{y}{x} \right) f^{i*}(y) \sum_{n=i}^{\infty} p_{n-i} f^{(n-i)*}(x-y) \\ &= \sum_{y=0}^x \sum_{i=1}^k \left( a_i + \frac{b_i}{i} \frac{y}{x} \right) f^{i*}(y) g(x-y) = \sum_{y=0}^x g(x-y) \sum_{i=1}^k \left( a_i + \frac{b_i}{i} \frac{y}{x} \right) f^{i*}(y), \end{aligned}$$

which gives (20). This completes the proof of Theorem 9. Q.E.D.

As the severities are usually assumed to be strictly positive, we state the following corollary.

**Corollary 5.** If  $q = 0$ , then

$$g(x) = \sum_{y=1}^x g(x-y) \sum_{i=1}^k \left( a_i + \frac{b_i}{i} \frac{y}{x} \right) f^{i*}(y) \quad (x = 1, 2, \dots)$$

$$g(0) = p_0.$$

The recursive algorithm presented by PANJER (1981) is obtained as a special case of Corollary 5 by letting  $k = 1$ . With  $k = 2$  and  $a_2 = 0$  we obtain SCHRÖTER'S (1990) generalisation.

**6B.** Let

$$m = \max \{ y : f(y) > 0 \}.$$

As  $f^{i*}(y) = 0$  for all  $y > mi$ , (20) can be written

$$g(x) = \frac{1}{1 - \sum_{i=1}^k a_i q^i} \sum_{y=1}^{mk} g(x-y) \sum_{i=1}^k \left( a_i + \frac{b_i}{i} \frac{y}{x} \right) f^{i*}(y), \quad (x = 1, 2, \dots)$$

and we obtain the following result.

**Theorem 10.** The distribution of  $X$  is  $R_{mk}[c, d]$  with  $c = (c_1, \dots, c_{mk})$  and  $d = (d_1, \dots, d_{mk})$  given by

$$(22) \quad c_y = \frac{\sum_{i=1}^k a_i f^{i^*}(y)}{1 - \sum_{i=1}^k a_i q^i} \quad d_y = \frac{y \sum_{i=1}^k \frac{b_i}{i} f^{i^*}(y)}{1 - \sum_{i=1}^k a_i q^i}, \quad (y = 1, \dots, mk)$$

Let

$$M = \# \{Y_i > 0 : i \leq N\}.$$

If  $N$  is the number of claims occurred in an insurance portfolio during a given period, and  $Y_i$  is the amount of the  $i$ th of these claims, then  $M$  is the number of non-zero claims. The following corollary to Theorem 10 shows that the distribution of  $M$  belongs to the same class as the distribution of  $N$ . Analogous results have been discussed by PANJER and WILLMOT (1984) and SUNDT (1991b) for the case  $k = 1$ .

**Corollary 6.** The distribution of  $M$  is  $R_k[c, d]$  with  $c = (c_1, \dots, c_k)$  and  $d = (d_1, \dots, d_k)$  given by

$$c_y = (1 - q)^y \frac{\sum_{i=y}^k a_i \binom{i}{y} q^{i-y}}{1 - \sum_{i=1}^k a_i q^i} \quad d_y = (1 - q)^y \frac{\sum_{i=y}^k b_i \binom{i-1}{y-1} q^{i-y}}{1 - \sum_{i=1}^k a_i q^i}. \quad (y = 1, \dots, k)$$

**Proof.** We obtain the corollary from Theorem 10 by letting  $f$  be the discrete density  $f_0$  defined by

$$f_0(0) = q = 1 - f_0(1)$$

and using that

$$f_0^{i^*}(y) = \binom{i}{y} (1 - q)^y q^{i-y}. \quad (y = 0, \dots, i) \quad \text{Q.E.D.}$$

**Corollary 7.** If  $N$  has the distribution  $R_k[a, b]$  and  $m$  is a positive integer, then  $mN$  has the distribution  $R_{mk}[c, d]$  with  $c = (c_1, \dots, c_{mk})$  and  $d = (d_1, \dots, d_{mk})$  given by

$$c_y = a_{y/m} \quad d_y = m b_{y/m} \quad (y = m, 2m, \dots, km)$$

and  $c_y = d_y = 0$  for all other values of  $y$ .

**Proof.** We can apply Theorem 10 with the  $Y_i$ 's identically equal to  $m$ , that is,  $f(y) = \delta_{my}$  (Kronecker delta). Then  $f^{i*}(y) = \delta_{(im), y}$ , and the corollary follows by insertion in (22). Q.E.D.

**6C.** For the present subsection we assume that  $q = 0$ . Furthermore we assume that all the  $b_i$ 's are equal to zero, like in the case with convolutions of geometric distributions mentioned at the end of subsection 4D. Then (20) simplifies to

$$g(x) = \sum_{y=1}^x g(x-y) \sum_{i=1}^k a_i f^{i*}(y). \quad (x = 1, 2, \dots)$$

In this case we have similar recursions for the corresponding cumulative distribution and its stop loss transform. Let  $F$  and  $G$  be the cumulative distributions corresponding to  $f$  and  $g$  respectively. Analogous to the deduction of the corresponding formulae in the special case  $k = 1$  in subsection 10.4D in SUNDT (1991a) we obtain

$$(23) \quad G(x) = p_0 + \sum_{y=1}^x G(x-y) \sum_{i=1}^k a_i f^{i*}(y) \quad (x = 0, 1, \dots)$$

$$\bar{G}(x) = \sum_{y=1}^x \bar{G}(x-y) \sum_{i=1}^k a_i f^{i*}(y) \quad (x = 1, 2, \dots)$$

$$\bar{G}(0) = EX = EN EY.$$

If all the  $a_i$ 's are non-negative with  $\sum_{i=1}^k a_i < 1$ , then (23) is a renewal equation with defective distribution  $\sum_{i=1}^k a_i F^{i*}$  (cf. FELLER (1971, Section XI.6)),

and we can obtain asymptotic expressions for  $g$ ,  $G$ , and  $\bar{G}$  analogous to the ones deduced for the case  $k = 1$  in SUNDT (1982). By Theorem 10, in our case the distribution of  $N$  is a compound distribution; its counting distribution is geometric with parameter

$$r = \sum_{i=1}^k a_i,$$

and its severity distribution has discrete density  $\{c_i\}_{i=1}^k$  given by

$$c_i = \frac{a_i}{r}. \quad (i = 1, \dots, k)$$

Thus  $G$  is a compound distribution with geometric counting distribution

with parameter  $r$  and compound severity distribution  $\sum_{i=1}^k c_i F^{i*}$ , and by using this representation of the  $G$ , we can apply the asymptotic results in SUNDT (1982).

**6D.** Generalisation of Theorem 9 to cases where the  $Y_i$ 's can also take negative values, is in most cases rather complicated. However, if there exist finite numbers  $y_0$  and  $n_0$  such that  $Y_i \geq y_0$  and  $N \leq n_0$  with probability one, then we can proceed like in Section 6 of SUNDT and JEWELL (1981).

7. GENERALISATIONS

**7A.** SUNDT and JEWELL (1981) generalised PANJER'S (1981) recursive algorithm to the class of counting distributions with discrete density  $\{p_n\}_{n=0}^\infty$  satisfying the recursion

$$p_n = \left( a + \frac{b}{n} \right) p_{n-1}. \quad (n = m+1, m+2, \dots)$$

Panjer's class is obtained with  $m = 0$ . We make a similar extension of  $\mathcal{R}_k$  and consider the class of counting distributions satisfying the recursion

$$(24) \quad p_n = \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} \quad (n = m+1, m+2, \dots)$$

with  $p_n = 0$  for  $n < 0$ . We obtain the following generalisation of Theorem 9.

**Theorem 11.** If  $\{p_n\}_{n=0}^\infty$  satisfies the recursion (24), then  $g(x)$  can be evaluated by the recursive algorithm

$$(25) \quad g(x) = \frac{1}{1 - \sum_{i=1}^k a_i q^i} \left( \sum_{n=1}^m \left[ p_n - \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} \right] f^{n*}(x) + \sum_{y=1}^x g(x-y) \sum_{i=1}^k \left( a_i + \frac{b_i y}{i x} \right) f^{i*}(y) \right) \quad (x = 1, 2, \dots)$$

$$(26) \quad g(0) = \sum_{n=0}^\infty p_n q^n.$$

**Proof.** Formula (26) obviously holds. For  $x > 0$  we have

$$g(x) = \sum_{n=1}^{\infty} p_n f^{n*}(x) = \sum_{n=1}^m p_n f^{n*}(x) + \sum_{n=m+1}^{\infty} \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} f^{n*}(x),$$

that is,

$$(27) \quad g(x) = \sum_{n=1}^m \left[ p_n - \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} \right] f^{n*}(x) + \sum_{n=1}^{\infty} \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} f^{n*}(x).$$

Like in the proof of Theorem 9 we obtain

$$\sum_{n=1}^{\infty} \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} f^{n*}(x) = \sum_{y=0}^x g(x-y) \sum_{i=1}^k \left( a_i + \frac{b_i}{i} \frac{y}{x} \right) f^{i*}(y),$$

and insertion in (27) and solving for  $g(x)$  gives (25). This completes the proof of Theorem 11. Q.E.D.

**7B.** A natural question is, could we extend the results shown in Sections 3 and 4 to the classes of counting distributions satisfying (24)? Unfortunately, possible extensions are not trivial. The deductions in Sections 3 and 4 depended very much on the simple form of  $\rho$  given by (4); in extended classes we do not get such a simple form.

To indicate the difficulties, we look at a simple case. Let

$$p_n = \begin{cases} \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} & (n > m) \\ 0, & (n < m) \end{cases}$$

and let  $\psi$  denote the probability generating function of this distribution. Analogous to the deduction of (3) we obtain

$$\begin{aligned} \psi'(s) &= \sum_{n=m}^{\infty} p_n n s^{n-1} = m p_m s^{m-1} + \sum_{n=m+1}^{\infty} n s^{n-1} \sum_{i=1}^k \left( a_i + \frac{b_i}{n} \right) p_{n-i} \\ &= m p_m s^{m-1} + \sum_{i=1}^k [a_i s^i \psi'(s) + (i a_i + b_i) s^{i-1} \psi(s)], \end{aligned}$$

which gives

$$\psi(s) \sum_{i=1}^k (ia_i + b_i) s^{i-1} - \psi'(s) \left( 1 - \sum_{i=1}^k a_i s^i \right) + mp_m s^{m-1} = 0.$$

We see that the presence of the term  $mp_m s^{m-1}$  makes the situation much more complicated for  $m > 0$ .

#### REREFENCES

- DE PRIL, N. (1985) Recursion for convolutions of arithmetic distributions. *ASTIN Bulletin* **15**, 135–139.
- DE PRIL, N. (1989) The aggregate claims distribution with arbitrary positive claims. *ASTIN Bulletin* **19**, 9–24.
- FELLER, W. (1971) *An introduction to probability theory and its applications*. Vol. II. (2 ed.) Wiley, New York.
- PANJER, H.H. (1981) Recursive evaluation of compound distributions. *ASTIN Bulletin* **12**, 22–26.
- PANJER, H.H. and WILLMOT, G.E. (1984) Computational techniques in reinsurance models. *Transactions of the 22nd International Congress of Actuaries* **4**, 111–120.
- PANJER, H.H. and WILLMOT, G.E. (1986) Computational aspects of recursive evaluation of compound distributions. *Insurance: Mathematics and Economics* **5**, 113–116.
- SCHRÖTER, K.J. (1990) On a class of counting distributions and recursions for related compound distributions. *Scandinavian Actuarial Journal*, 161–175.
- SUNDT, B. (1982) Asymptotic behaviour of compound distributions and stop-loss premiums. *ASTIN Bulletin* **13**, 89–98. Corrigendum. *ASTIN Bulletin* **15**, 44.
- SUNDT, B. (1991a) *An introduction to non-life insurance mathematics*. (2 ed.) Verlag Versicherungswirtschaft e.V., Karlsruhe.
- SUNDT, B. (1991b) On asymptotic rates on lines in excess of loss reinsurance. *Insurance: Mathematics and Economics* **10**, 61–67.
- SUNDT, B. and JEWELL, W.S. (1981) Further results on recursive evaluation of compound distributions. *ASTIN Bulletin* **12**, 27–39.

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