THE EVALUATION FUNCTIONALS ASSOCIATED WITH AN ALGEBRA OF BOUNDED OPERATORS

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1. Introduction. In this note we shall employ the notation of [1] without further mention. Thus X denotes a normed space and P the subset of $X \times X'$ given by

$$P = \{(x,f): \|x\| = 1, f(x) = 1 = \|f\|\}.$$

Given a subalgebra $\mathfrak{A}$ of $B(X)$, the set $\{\Phi_{(x,f)}: (x,f) \in P\}$ of evaluation functionals on $\mathfrak{A}$ is denoted by Π. We shall prove that if X is a Banach space and if $\mathfrak{A}$ contains all the bounded operators of finite rank, then Π is norm closed in $\mathfrak{A}'$. We give an example to show that Π need not be weak* closed in $\mathfrak{A}'$. We show also that Π need not be norm closed in $\mathfrak{A}'$ if X is not complete.

2. The main result. Given $x \in X, f \in X'$, we write as usual

$$x \otimes f(y) = f(y)x \qquad (y \in X),$$

so that $x \otimes f$ is a bounded operator on X with rank at most one. It is well known that any bounded operator on X with rank one may be written in this form with $x \neq 0$ and $f \neq 0$.

LEMMA 1. *Let X be a normed space and $\{T_n\}$ a sequence of bounded operators on X with rank one, such that $\lim_{n\to\infty} | T_n - S | = 0$ for some $S \in B(X)$. Then S has rank at most one.*

Proof. Suppose that $y_1 = Sx_1$, $y_2 = Sx_2$ with y_1, y_2 linearly independent. By the Hahn–Banach theorem we may choose $g \in X'$ with $g(y_1) = 1$, $g(y_2) = 0$. We may write $T_n = x_n \otimes f_n$, where $\{x_n\}$, $\{f_n\}$ are bounded sequences in X, X' respectively. Then

$$\lim_{n\to\infty} f_n(x_1)g(x_n) = \lim_{n\to\infty} g(T_n x_1) = g(Sx_1) = 1,$$

$$\lim_{n\to\infty} f_n(x_2)g(x_n) = \lim_{n\to\infty} g(T_n x_2) = g(Sx_2) = 0.$$

Since $\{f_n(x_1)\}$, $\{g(x_n)\}$ are bounded sequences, it follows that $\lim_{n\to\infty} f_n(x_2) = 0$ and then $\lim_{n\to\infty} T_n x_2 = 0$. This gives the contradiction $y_2 = 0$, and so S has rank at most one.

THEOREM. *Let X be a Banach space and let $\mathfrak{A}$ be a subalgebra of $B(X)$ containing all the bounded operators of finite rank. Then Π is norm closed in $\mathfrak{A}'$.*

Proof. Given F in the norm closure of Π, there is a sequence $\{(x_n, f_n)\}$ in P such that

$$\lim_{n\to\infty} \| \Phi_{(x_n,f_n)} - F \| = 0.$$

We write $T_n = x_n \otimes f_n$, so that $T_n \in B(X)$, $T_n^2 = T_n$, and $| T_n | = 1$. If $T = x \otimes f$, where $x \in X$,

$f \in X'$, $\|x\| \leqq 1$, $\|f\| \leqq 1$, then $T \in \mathfrak{A}$ and $|T| \leqq 1$. Since

$$\Phi_{(x_n,f_n)}(T) = f_n(Tx_n) = f_n(x)f(x_n) = f(T_n x),$$

it follows that $\{f(T_n x)\}$ converges uniformly for $\|x\| \leqq 1$, $\|f\| \leqq 1$. Since X is a Banach space, there exists $S \in B(X)$ with $\lim_{n\to\infty} |T_n - S| = 0$. From $T_n^2 = T_n$, $|T_n| = 1$ we deduce that $S^2 = S$, $|S| = 1$. It now follows from Lemma 1 that S has rank one and so we may write S in the form $S = x_0 \otimes f_0$ for some $(x_0, f_0) \in P$. Since $\lim_{n\to\infty} \|T_n x_0 - Sx_0\| = 0$, we have $\lim_{n\to\infty} \|f_n(x_0)x_n - x_0\| = 0$ and so $\lim_{n\to\infty} |f_n(x_0)| = 1$. By compactness there is a subsequence $\{f_{n_j}\}$ and a scalar λ with $|\lambda| = 1$ such that $\lim_{j\to\infty} f_{n_j}(x_0) = \lambda$. Since

$$\lim_{j\to\infty} \|\Phi_{(x_{n_j}, f_{n_j})} - F\| = 0, \qquad \Phi_{(\lambda x_0, \bar{\lambda} f_0)} = \Phi_{(x_0, f_0)},$$

we may clearly suppose that $\{f_{n_j}\} = \{f_n\}$ and $\lambda = 1$. We thus have

$$\lim_{n\to\infty} f_n(x_0) = 1, \qquad \lim_{n\to\infty} \|x_n - x_0\| = 0.$$

From $\lim_{n\to\infty} \|T_n^* f_0 - S^* f_0\| = 0$ we deduce that $\lim_{n\to\infty} \|f_0(x_n)f_n - f_0\| = 0$. Since $\lim_{n\to\infty} f_0(x_n) = f_0(x_0) = 1$, we also have $\lim_{n\to\infty} \|f_n - f_0\| = 0$. Finally

$$|\Phi_{(x_n,f_n)}(T) - \Phi_{(x_0,f_0)}(T)| \leqq |f_n(Tx_n - Tx_0)| + |(f_n - f_0)(Tx_0)|,$$
$$\leqq |T|(\|x_n - x_0\| + \|f_n - f_0\|),$$

so that

$$\lim_{n\to\infty} \|\Phi_{(x_n,f_n)} - \Phi_{(x_0,f_0)}\| = 0.$$

Therefore $F = \Phi_{(x_0,f_0)}$ and Π is norm closed.

Remarks. (i) The above argument uses only the norm of $\mathfrak{A}'$ and so we may replace $\mathfrak{A}$ by any dense subalgebra of it.

(ii) Let B be an arbitrary Banach algebra and let $a \to T_a$ be a representation of B on a Banach space X whose image contains all the bounded operators of finite rank. If

$$\Psi_{(x,f)}(a) = f(T_a x) \qquad (a \in B),$$

we readily see that $\{\Psi_{(x,f)} : (x,f) \in P\}$ is norm closed in B'.

(iii) Given (x_1, f_1), $(x_2, f_2) \in P$, write $(x_1, f_1) \sim (x_2, f_2)$ if there is a scalar λ with $|\lambda| = 1$ and $x_2 = \lambda x_1$. Then $f_2 = \bar{\lambda} f_1$ and clearly $\sim$ is an equivalence relation on P. Using the argument of the above proof and Lemma 2 below, we may verify that $P/\sim$ with the quotient topology induced from $(P, \|.\| \times \|.\|')$ is homeomorphic with $(\Pi, \|.\|)$.

3. Some examples. We begin with a simple lemma.

LEMMA 2. *Let X be a normed space and let $\mathfrak{A}$ be a subalgebra of $B(X)$ containing all the bounded operators of finite rank. If (x_1, f_1), $(x_2, f_2) \in P$ and $\Phi_{(x_1,f_1)} = \Phi_{(x_2,f_2)}$, there is a scalar λ such that $|\lambda| = 1$, $x_2 = \lambda x_1$, $f_2 = \bar{\lambda} f_1$.*

Proof. Let $x \in X, f \in X'$, and let $T = x \otimes f$ so that $T \in \mathfrak{A}$. Then

$$f(x_1)f_1(x) = f(x_2)f_2(x) \qquad (x \in X, f \in X').$$

If x_1, x_2 are linearly independent, we may choose $f \in X'$ such that $f(x_1) = 1, f(x_2) = 0$. Then $f_1(x) = 0$ $(x \in X)$, which is impossible since $\|f_1\| = 1$. Hence there is a scalar λ with $|\lambda| = 1$, $x_2 = \lambda x_1$. It follows that $f_2 = \bar{\lambda} f_1$ as required.

Let c_0, l_1, l_∞ denote respectively the Banach spaces of all complex sequences that converge to zero, that have absolutely convergent series, and that are bounded. We make the usual identifications $c_0' = l_1$ and $l_1' = l_\infty$.

EXAMPLE 1. *If $X = c_0$ and $\mathfrak{A} = B(X)$, then Π is not weak* closed.*

Proof. We define elements of c_0, l_1, l_∞ respectively by

$$x_n(r) = \begin{cases} 1 & (1 \leqq r \leqq n), \\ 0 & (r > n), \end{cases}$$

$$f(r) = \begin{cases} 1 & (r = 1), \\ 0 & (r > 1), \end{cases}$$

$$z(r) = 1 \quad (r \geqq 1).$$

For each n we have $(x_n, f) \in P$. For each $T \in \mathfrak{A}$ we have

$$\lim_{n \to \infty} \Phi_{(x_n, f)}(T) = \lim_{n \to \infty} \hat{x}_n(T^* f) = z(T^* f).$$

If $F(T) = z(T^* f)$ $(T \in \mathfrak{A})$, then $F \in \mathfrak{A}'$. It follows from the method of the proof of Lemma 2 that $F \notin \Pi$ and so Π is not weak* closed.

Let c_{00} denote the normed space of all complex sequences with finite support, with the supremum norm.

EXAMPLE 2. *If $X = c_{00}$ and $\mathfrak{A} = B(X)$, then Π is not norm closed.*

Proof. We define elements of c_{00}, c_0 respectively by

$$y_n(r) = \begin{cases} 1/r & (1 \leqq r \leqq n), \\ 0 & (r > n), \end{cases}$$

$$y(r) = 1/r \quad (r \geqq 1).$$

If f is as in Example 1, we easily verify that

$$\lim_{n \to \infty} \| \Phi_{(y_n, f)} - F \| = 0,$$

where

$$F(T) = f(\tilde{T} y) \qquad (T \in \mathfrak{A}),$$

$\tilde{T}$ being the unique extension of T to a bounded operator on c_0. It follows readily from Lemma 2 that $F \notin \Pi$, and so Π is not norm closed.

If the subalgebra $\mathfrak{A}$ of $B(X)$ contains the identity operator, then the weak* closure of Π is a subset of $D_{\mathfrak{A}}(I) \subset S(\mathfrak{A}')$. On the other hand if $\mathfrak{A}$ does not contain the identity operator, then the zero functional may belong to the weak* closure of Π, even if X is a Hilbert space.

Let l_2 be the Hilbert space of all complex square-summable sequences.

EXAMPLE 3. *If $X = l_2$ and $\mathfrak{A}$ is the algebra of compact operators on X, then the zero functional belongs to the weak* closure of* Π.

Proof. Let $\{e_n\}$ be the usual basis for l_2, so that $(e_n, e_n) \in P$ for each n. If T is a bounded operator on X of rank one, say $T = x \otimes y$, then

$$\lim_{n\to\infty} |\Phi_{(e_n,e_n)}(T)| = \lim_{n\to\infty} |x(n)|\,|y(n)| = 0.$$

It follows that $\lim_{n\to\infty} \Phi_{(e_n,e_n)}(T) = 0$ for each T of finite rank and thence for uniform limits of such operators, i.e. for each compact operator T. The proof is complete.

REFERENCE

1. F. F. Bonsall, The numerical range of an element of a normed algebra, *Glasgow Math. J.* **10** (1969), 68–72.

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