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ASYMPTOTIC RESULTS FOR CLASS NUMBER DIVISIBILITY IN CYCLOTOMIC FIELDS

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ABSTRACT. Let $n \geq 3$ and $m \geq 3$ be integers. Let K_n be the cyclotomic field obtained by adjoining a primitive nth root of unity to the field of rational numbers. Let K_n^+ denote the maximal real subfield of K_n. Let h_n (resp., h_n^+) denote the class number of K_n (resp., K_n^+). For fixed m we show that m divides h_n and h_n^+ for asymptotically almost all n. Also for those K_n and K_n^+ with a given number of ramified primes, we obtain lower bounds for certain types of densities for m dividing h_n and h_n^+.

1. **Introduction.** Let $n \geq 3$ be an integer, and let ζ_n be a primitive nth root of unity. Let h_n denote the class number of the cyclotomic field $K_n = \mathbf{Q}(\zeta_n)$, and let h_n^+ denote the class number of the maximal real subfield K_n^+ of K_n. It is well known that $h_n^+ \mid h_n$. Recently Cornell and Rosen [1] have announced that if n is divisible by at least five distinct odd primes, then $2 \mid h_n^+$ (and of course $2 \mid h_n$). Hence asymptotically $2 \mid h_n^+$ and $2 \mid h_n$ for almost all n. Now let $m \geq 3$ be an integer. Cornell and Washington [2] have recently given sufficient conditions for $m \mid h_n^+$ and $m \mid h_n$. In section 2 we describe the results of Cornell and Washington. In section 3 we obtain an asymptotic result which indicates that for fixed m almost all K_n^+ and K_n have class numbers divisible by m. In section 4 we consider those K_n^+ and K_n with a fixed number of ramified primes and obtain lower bounds for certain types of densities for $m \mid h_n^+$ and $m \mid h_n$.

2. **Results of Cornell and Washington.** Let notations be as in section 1. First we consider $m \mid h_n^+$. Let

$$M = \begin{cases} 4m & \text{if } m \text{ is odd} \\ 2m & \text{if } m \text{ is even.} \end{cases} \tag{2.1}$$

Let r be the number of distinct primes congruent to 1 (mod M) that divide n, and let $p_{i_1}, \ldots, p_{i_r}$ denote these distinct primes. Then the proofs in [2] show that $m \mid h_n^+$ if

(i) $r \geq 4$, or

(ii) $r = 3$ and at least two of $(p_{i_j}/p_{i_l}) = 1$ or $1 \leq l < j \leq 3$, where (p_{i_j}/p_{i_l}) is the Legendre symbol. Alternately we may describe the situation as follows. If

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$m \nmid h_n^+$, it is necessary that

(a) $r \le 2$, or

(b) $r = 3$ and at most one of $(p_{i_j}/p_{i_l}) = 1$ for $1 \le l < j \le 3$.

Since $h_n^+ \mid h_n$, we see that condition (i) or (ii) is sufficient for $m \mid h_n$. Equivalently condition (a) or (b) is necessary for $m \nmid h_n$. However when m is odd, the proofs in [2] indicate more restrictive necessary conditions for $m \nmid h_n$. Let s be the number of distinct primes congruent to 1 (mod m) that divide n, and let $p_{i_1}, \ldots, p_{i_s}$ denote these distinct primes. Then if m is odd and $m \nmid h_n$, it is necessary that one of the following conditions be satisfied:

(c) $s \le 1$

(d_1) $s = 2$, and both $p_{i_1} \equiv 1 \pmod 4$ and $p_{i_2} \equiv 1 \pmod 4$

(d_2) $s = 2$, exactly one of p_{i_1} and p_{i_2} is congruent to 1 (mod 4), and $(p_{i_2}/p_{i_1}) = -1$

(e_1) $s = 3$, each $p_{i_j} \equiv 1 \pmod 4$ for $j = 1, 2, 3$, and at most one of $(p_{i_j}/p_{i_l}) = 1$ for $1 \le l < j \le 3$

(e_2) exactly one of p_{i_1}, p_{i_2}, p_{i_3} is congruent to 3 (mod 4), say p_{i_1}, $(p_{i_j}/p_{i_1}) = -1$ for $j = 2, 3$, and $(p_{i_3}/p_{i_2}) = 1$.

3. $m \mid h_n^+$ **and** $m \mid h_n$ **for almost all** n. Let notations be as in previous sections. In section 1 we remarked that $2 \mid h_n^+$ (and $2 \mid h_n$) if n is divisible by at least five odd primes. So if $2 \nmid h_n^+$, it is necessary that n be divisible by at most four odd primes. If x is a positive real number and $B_x = \{n \le x\colon n$ is divisible by at most four distinct odd primes$\}$, then it can be proved by standard techniques that

$$|B_x| \ll \frac{x(\log\log x)^3}{\log x}.$$

In particular $|B_x| = o(x)$, from which it follows that $2 \mid h_n^+$ (and $2 \mid h_n$) for asymptotically almost all n.

Now suppose $m \ge 3$ is an integer. From section 2 we know that if $m \nmid h_n^+$, it is necessary that n be divisible by at most three distinct primes congruent to 1 (mod M), where M is given by (2.1). Our goal in this section is to prove the following theorem.

THEOREM 1. *Let $m \ge 3$ be an integer, and define M by* (2.1). *Let x be a positive real number, and let $C_x = \{n \le x\colon m \nmid h_n^+\}$. Then*

$$|C_x| \ll \frac{x(\log\log x)^3}{(\log x)^{1/\varphi(M)}}$$

where φ is the Euler φ-function, and where the constant implied by the symbol $\ll$ depends only on M. In particular, for fixed m, $|C_x| = o(x)$. Hence for fixed m, $m \mid h_n^+$ and $m \mid h_n$ for asymptotically almost all n.

The proof of Theorem 1 depends on certain sieve estimates. Let z be a positive real number and b a positive integer. Let $A(x)=\{n\le x\}$, $A_b(x)=\{n\in A(x):\quad n\equiv 0 \pmod b\}$, $P=\{\text{primes } p\equiv 1 \pmod M\}$, $P(z)=\prod_{p\le z}^{p\in P} p$, $S(A(x), P, z)=|\{n\in A(x):\quad (n, P(z))=1\}|$. We shall need the following lemma, which follows immediately from [5], Corollary 2.3.1.

LEMMA 1. *Let G be a set of primes with*

$$\sum_{\substack{p\in G\\ p\le x}} \frac{1}{p} \ge \frac{1}{\varphi(M)} \log\log x - a \tag{3.1}$$

for some constant a. Then

$$|\{n\in A(x): (n, p)=1 \text{ for all } p\in G\}| \ll \frac{x}{(\log x)^{1/\varphi(M)}} \tag{3.2}$$

where the constant implied by the $\ll$ symbol depends only on a.

Now we let

$D=|\{n\in A(x)$: n is divisible by at most three distinct primes in $P\}|$. We claim that

$$D\le F_0+F_1+F_2+F_3 \tag{3.3}$$

where

$$F_0=S(A(x), P, x^{1/4})$$

$$F_1=\sum_{\substack{p\in P\\ p\le x^{1/4}}} S\left(A_p(x), P-\{p\}, \left(\frac{x}{p}\right)^{1/4}\right)$$

$$F_2=\sum_{\substack{p_1,p_2\in P\\ p_1,p_2\le x^{1/4}}} S\left(A_{p_1p_2}(x), P-\{p_1, p_2\}, \left(\frac{x}{p_1p_2}\right)^{1/4}\right)$$

$$F_3=\sum_{\substack{p_1,p_2,p_3\in P\\ p_1,p_2,p_3\le x^{1/4}}} S\left(A_{p_1p_2p_3}(x), P-\{p_1, p_2, p_3\}, \frac{x}{p_1p_2p_3}\right).$$

If n is divisible by no prime in P, then n is counted in F_0. If n is divisible by exactly one prime in P, then we see that n is counted in F_0 or F_1. If n is divisible by exactly two distinct primes in P, then we see that n is counted in F_0, F_1, or F_2. Finally if n is divisible by exactly three distinct primes in P, we see that n is counted in F_0, F_1, F_2, or F_3. Thus Inequality 3.3 is valid. We also note that

$$S\left(A_p(x), P-\{p\}, \left(\frac{x}{p}\right)^{1/4}\right)=S\left(A\left(\frac{x}{p}\right), P-\{p\}, \left(\frac{x}{p}\right)^{1/4}\right)$$

$$S\left(A_{p_1p_2}(x), P-\{p_1, p_2\}, \left(\frac{x}{p_1p_2}\right)^{1/4}\right)=S\left(A\left(\frac{x}{p_1p_2}\right), P-\{p_1, p_2\}, \left(\frac{x}{p_1p_2}\right)^{1/4}\right)$$

$$S\left(A_{p_1p_2p_3}(x), P-\{p_1, p_2, p_3\}, \frac{x}{p_1p_2p_3}\right)=S\left(A\left(\frac{x}{p_1p_2p_3}\right), P-\{p_1, p_2, p_3\}, \frac{x}{p_1p_2p_3}\right).$$

We first use Lemma 1 with $G=\{p \in P : p \leq x^{1/4}\}$. The well-known formula

$$\sum_{\substack{p \in P \\ p \leq x}} \frac{1}{p} = \frac{1}{\varphi(M)} \log \log x + 0(1)$$

implies that

$$\sum_{\substack{p \in G \\ p \leq x}} \frac{1}{p} = \sum_{\substack{p \in P \\ p \leq x^{1/4}}} \frac{1}{p} = \frac{1}{\varphi(M)} \log \log (x^{1/4}) + 0(1)$$

$$= \frac{1}{\varphi(M)} \log \log x + 0(1).$$

So Inequality 3.1 is satisfied with a constant a that depends on M. Thus from Inequality 3.2,

$$F_0 = S(A(x), P, x^{1/4}) \ll \frac{x}{(\log x)^{1/\varphi(M)}}.$$

Next we use Lemma 1 with $G=\{q \in P-\{p\} : q \leq (x/p)^{1/4}\}$. Since

$$\sum_{\substack{q \in G \\ q \leq x/p}} \frac{1}{q} = \frac{1}{\varphi(M)} \log \log \left(\frac{x}{p}\right) + 0(1),$$

then

$$S\left(A\left(\frac{x}{p}\right), P-\{p\}, \left(\frac{x}{p}\right)^{1/4}\right) \ll \frac{x}{p(\log(x/p))^{1/\varphi(M)}}.$$

For $p \leq x^{1/4}$, we have

$$\frac{1}{\left(\log\left(\frac{x}{p}\right)\right)^{1/\varphi(M)}} \ll \frac{1}{(\log x)^{1/\varphi(M)}} \quad \text{and} \quad \sum_{\substack{p \in P \\ p \leq x^{1/4}}} \frac{1}{p} \ll \log \log x.$$

So

$$F_1 = \sum_{\substack{p \in P \\ p \leq x^{1/4}}} S\left(A\left(\frac{x}{p}\right), P-\{p\}, \left(\frac{x}{p}\right)^{1/4}\right) \ll \frac{x \log \log x}{(\log x)^{1/\varphi(M)}}.$$

Next with

$$G = \left\{q \in P-\{p_1, p_2\} : q \leq \left(\frac{x}{p_1 p_2}\right)^{1/4}\right\},$$

Lemma 1 gives

$$S\left(A\left(\frac{x}{p_1 p_2}\right), P-\{p_1, p_2\}, \left(\frac{x}{p_1 p_2}\right)^{1/4}\right) \ll \frac{x}{p_1 p_2 (\log (x/p_1 p_2))^{1/\varphi(M)}}.$$

For $p_1p_2 \le x^{1/2}$, we have

$$\frac{1}{\left(\log\left(\frac{x}{p_1p_2}\right)\right)^{1/\varphi(M)}} \ll \frac{1}{(\log x)^{1/\varphi(M)}} \quad \text{and} \quad \sum_{\substack{p_1,p_2\in P\\ p_1,p_2\le x^{1/4}}} \frac{1}{p_1p_2} \ll (\log\log x)^2.$$

So

$$F_2 = \sum_{\substack{p_1,p_2\in P\\ p_1,p_2\le x^{1/4}}} S\left(A\left(\frac{x}{p_1p_2}\right), P-\{p_1, p_2\}, \left(\frac{x}{p_1p_2}\right)^{1/4}\right) \ll \frac{x(\log\log x)^2}{(\log x)^{1/\varphi(M)}}.$$

Finally with

$$G = \left\{q \in P-\{p_1, p_2, p_3\}: q \le \frac{x}{p_1p_2p_3}\right\},$$

Lemma 1 gives

$$S\left(A\left(\frac{x}{p_1p_2p_3}\right), P-\{p_1, p_2, p_3\}, \frac{x}{p_1p_2p_3}\right) \ll \frac{x}{p_1p_2p_3(\log(x/p_1p_2p_3))^{1/\varphi(M)}}.$$

For $p_1p_2p_3 \le x^{3/4}$, we have

$$\frac{1}{\left(\log\left(\frac{x}{p_1p_2p_3}\right)\right)^{1/\varphi(M)}} \ll \frac{1}{(\log x)^{1/\varphi(M)}} \quad \text{and} \quad \sum_{\substack{p_1,p_2,p_3\in P\\ p_1,p_2,p_3\le x^{1/4}}} \frac{1}{p_1p_2p_3} \ll (\log\log x)^3.$$

So

$$F_3 = \sum_{\substack{p_1,p_2,p_3\in P\\ p_1,p_2,p_3\le x^{1/4}}} S\left(A\left(\frac{x}{p_1p_2p_3}\right), P-\{p_1, p_2, p_3\}, \frac{x}{p_1p_2p_3}\right) \ll \frac{x(\log\log x)^3}{(\log x)^{1/\varphi(M)}}.$$

Then from Inequality 3.3, we have

$$D \ll \frac{x(\log\log x)^3}{(\log x)^{1/\varphi(M)}}.$$

Since $|C_x| \le D$, we have proved Theorem 1.

4. **Density results.** We let notations be the same as in previous sections. In this section we shall suppose that n is divisible by exactly t distinct primes. So $n = p_1^{e_1} \cdots p_t^{e_t}$, where $p_1, \ldots, p_t$ are distinct primes and each $e_i \ge 1$. We note that the conditions in section 2 do not distinguish between $m \mid h_n^+$ for $n = p_1 \cdots p_t$ and $m \mid h_n^+$ for $n = p_1^{e_1} \cdots p_t^{e_t}$ if each $e_i \ge 1$. Similarly the conditions in section 2 do not distinguish between $m \mid h_n$ for $n = p_1 \cdots p_t$ and $m \mid h_n$ for $n = p_1^{e_1} \cdots p_t^{e_t}$ if each $e_i \ge 1$. Hence we shall consider only square-free integers n with t prime factors. Let x be a positive real number, and let $R_{t,x} = \{n \le x: n = p_1 \cdots p_t$ with primes $p_1 < p_2 < \cdots < p_t\}$. If T_x is a subset of $R_{t,x}$ for each x,

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we define a density

$$d(T_x)=\lim_{x\to\infty}\frac{|T_x|}{|R_{t,x}|}$$

provided the limit exists. We define a lower density

$$\mathbf{d}(T_x)=\liminf_{x\to\infty}\frac{|T_x|}{|R_{t,x}|}.$$

If $d(T_x)$ exists, then of course $\mathbf{d}(T_x)=d(T_x)$. We also note the well-known asymptotic formula

$$|R_{t,x}|\sim\frac{x(\log\log x)^{t-1}}{(t-1)!\log x} \tag{4.1}$$

as $x\to\infty$ with t fixed. Our main result in this section is the following theorem.

THEOREM 2. *Let $m\geq 3$ and $t\geq 3$ be integers. Define M by* (2.1). *Let $V_{m,t,x}=\{n\in R_{t,x}: m \mid h_n\}$ and $V^+_{m,t,x}=\{n\in R_{t,x}: m \mid h^+_n\}$. Then*
(i) $\mathbf{d}(V_{m,t,x})\geq\mathbf{d}(V^+_{m,t,x})\geq 1-y_{m,t}$, where

$$y_{m,t}=\frac{(\varphi(M)-1)^{t-3}}{(\varphi(M))^t}\left[(\varphi(M)-1)^3+t(\varphi(M)-1)^2+\frac{t(t-1)}{2}(\varphi(M)-1)+\frac{t(t-1)(t-2)}{12}\right].$$

If m is odd, we have the stronger result (ii) $\mathbf{d}(V_{m,t,x})\geq 1-z_{m,t}$, *where*

$$z_{m,t}=\frac{(\varphi(m)-1)^{t-3}}{(\varphi(m))^t}\left[(\varphi(m)-1)^3+t(\varphi(m)-1)^2+\frac{t(t-1)}{4}(\varphi(m)-1)+\frac{7t(t-1)(t-2)}{384}\right].$$

Now let $t=2$ and let m be odd. Then (iii) $\mathbf{d}(V_{m,t,x})\geq\dfrac{1}{2(\varphi(m))^2}$.

Proof. For $t\geq 3$, let $Y_{m,t,x}=\{n\in R_{t,x}$: n satisfies condition (a) or (b) of section 2$\}$ and $Z_{m,t,x}=\{n\in R_{t,x}$: n satisfies one of the conditions (c), (d_1), (d_2), (e_1), (e_2) of section 2$\}$. To prove (i) and (ii) of Theorem 2, it suffices to show that $d(Y_{m,t,x})=y_{m,t}$ and $d(Z_{m,t,x})=z_{m,t}$. Now

$$|Y_{m,t,x}|=\sum_{p_1\cdots p_t\leq x}\delta_0+\sum_{p_1\cdots p_t\leq x}\delta_1+\sum_{p_1\cdots p_t\leq x}\delta_2+\sum_{p_1\cdots p_t\leq x}\delta_3 \tag{4.2}$$

where for $0\leq i\leq 2$,

$$\delta_i=\begin{cases}1 \text{ if exactly } i \text{ of } p_1,\dots,p_t \text{ are congruent to 1 (mod } M)\\ 0 \text{ otherwise}\end{cases}$$

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and

$$\delta_3 = \begin{cases} 1 & \text{if condition (b) of section 2 is satisfied} \\ 0 & \text{otherwise.} \end{cases}$$

Now for fixed M and t, standard calculations show

$$\sum_{p_1\cdots p_t\le x} \delta_0 = \sum_{\substack{p_1\cdots p_t\le x \\ \text{each } p_i\not\equiv 1 \pmod M}} 1 \sim \left(\frac{\varphi(M)-1}{\varphi(M)}\right)^t \frac{x(\log\log x)^{t-1}}{(t-1)!\log x}. \tag{4.3}$$

Similarly

$$\sum_{p_1\cdots p_t\le x} \delta_1 \sim \binom{t}{1}\frac{1}{\varphi(M)}\left(\frac{\varphi(M)-1}{\varphi(M)}\right)^{t-1}\frac{x(\log\log x)^{t-1}}{(t-1)!\log x} \tag{4.4}$$

and

$$\sum_{p_1\cdots p_t\le x} \delta_2 \sim \binom{t}{2}\left(\frac{1}{\varphi(M)}\right)^2\left(\frac{\varphi(M)-1}{\varphi(M)}\right)^{t-2}\frac{x(\log\log x)^{t-1}}{(t-1)!\log x} \tag{4.5}$$

where $\binom{t}{1}$ and $\binom{t}{2}$ are binomial coefficients.

The calculation of $\sum_{p_1\cdots p_t\le x}\delta_3$ is slightly more complicated. We will obtain the factor

$$\binom{t}{3}\left(\frac{1}{\varphi(M)}\right)^3\left(\frac{\varphi(M)-1}{\varphi(M)}\right)^{t-3}$$

from the congruence conditions mod M. Next since

$$\left(\frac{p_{i_2}}{p_{i_1}}\right)=\pm1,\ \left(\frac{p_{i_3}}{p_{i_1}}\right)=\pm1,\ \left(\frac{p_{i_3}}{p_{i_2}}\right)=\pm1,$$

there are eight possible combinations for the Legendre symbols. Four of these combinations satisfy the requirements of condition (b), and hence we expect an additional factor of $\frac{4}{8}=\frac{1}{2}$ in the calculation. Thus we expect

$$\sum_{p_1\cdots p_t\le x} \delta_3 \sim \frac{1}{2}\binom{t}{3}\left(\frac{1}{\varphi(M)}\right)^3\left(\frac{\varphi(M)-1}{\varphi(M)}\right)^{t-3}\frac{x(\log\log x)^{t-1}}{(t-1)!\log x}. \tag{4.6}$$

Concerning the proof of this type of result, we make the following observations. For simplicity, we consider the case $t=3$. We let χ_p be the quadratic character defined by

$$\chi_p(a)=\left(\frac{a}{p}\right) \quad \text{for} \quad (a,p)=1.$$

Then suppose, for example, that we want to count the integers $n\le x$ of the form $n=p_1p_2p_3$ with primes $p_1<p_2<p_3$, each $p_i\equiv 1 \pmod M$, and with

$$\left(\frac{p_2}{p_1}\right)=-1,\quad \left(\frac{p_3}{p_1}\right)=1,\quad \left(\frac{p_3}{p_2}\right)=-1.$$

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Then we would compute the sum

$$(4.7)\quad \sum_{\substack{p_1 \le x^{1/3} \\ p_1 \equiv 1 (\mathrm{mod}\, M)}} \sum_{\substack{p_1 < p_2 \le (x/p_1)^{1/2} \\ p_2 \equiv 1 (\mathrm{mod}\, M)}} \sum_{\substack{p_2 < p_3 \le x/p_1 p_2 \\ p_3 \equiv 1 (\mathrm{mod}\, M)}} \tfrac{1}{2}(1-\chi_{p_1}(p_2))\tfrac{1}{2}(1+\chi_{p_1}(p_3))\tfrac{1}{2}(1-\chi_{p_2}(p_3)).$$

We have performed calculations similar to this in [3] and [4]. The coefficient in the main term in the asymptotic formula for (4.7) comes from the factor $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$ and the congruence conditions mod M. The remaining terms are character sums which are part of the error term. Thus we see that the sum (4.7) is asymptotic to

$$\frac{1}{8}\left(\frac{1}{\varphi(M)}\right)^3 \frac{x(\log\log x)^2}{2!\log x}.$$

Since there are four possible combinations of Legendre symbols that give $\delta_3 = 1$, we get

$$\sum_{p_1p_2p_3 \le x} \delta_3 \sim \frac{1}{2}\left(\frac{1}{\varphi(M)}\right)^3 \frac{x(\log\log x)^2}{2!\log x} \quad \text{when} \quad t=3.$$

It is then easy to see that (4.6) is true for arbitrary $t \ge 3$. Now using (4.1)–(4.6), we see that $d(Y_{m,t,x}) = y_{m,t}$, where $y_{m,t}$ is specified in the statement of Theorem 2.

Next we consider $|Z_{m,t,x}|$. We note that

$$(4.8)\qquad |Z_{m,t,x}| = \sum_{p_1\cdots p_t \le x} \varepsilon_0 + \sum_{p_1\cdots p_t \le x} \varepsilon_1 + \sum_{p_1\cdots p_t \le x} \varepsilon_2 + \sum_{p_1\cdots p_t \le x} \varepsilon_3$$

where for $0 \le i \le 1$,

$$\varepsilon_i = \begin{cases} 1 & \text{if exactly } i \text{ of } p_1, \ldots, p_t \text{ are congruent to 1 (mod } m) \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_2 = \begin{cases} 1 & \text{if condition } (\mathrm{d}_1) \text{ or } (\mathrm{d}_2) \text{ of section 2 is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_3 = \begin{cases} 1 & \text{if condition } (\mathrm{e}_1) \text{ or } (\mathrm{e}_2) \text{ of section 2 is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

Now analogous to (4.3) and (4.4), we have

$$(4.9)\qquad \sum_{p_1\cdots p_t \le x} \varepsilon_0 \sim \left(\frac{\varphi(m)-1}{\varphi(m)}\right)^t \frac{x(\log\log x)^{t-1}}{(t-1)!\log x}$$

and

$$(4.10)\qquad \sum_{p_1\cdots p_t \le x} \varepsilon_1 \sim \binom{t}{1}\frac{1}{\varphi(m)}\left(\frac{\varphi(m)-1}{\varphi(m)}\right)^{t-1} \frac{x(\log\log x)^{t-1}}{(t-1)!\log x}.$$

We now examine the effect of the conditions (d_1), (d_2), (e_1), (e_2). In (d_1), the congruence conditions (mod 4) introduce a factor $\frac{1}{4}$. In (d_2), the congruence conditions (mod 4) introduce a factor $\frac{2}{4}$, and the condition $(p_{i_2}/p_{i_1})=-1$ introduces a factor $\frac{1}{2}$. Thus from (d_1) and (d_2) we get a factor $\frac{1}{4}+\frac{2}{4}\cdot\frac{1}{2}=\frac{1}{2}$ to multiply the coefficient

$$\binom{t}{2}\left(\frac{1}{\varphi(m)}\right)^2\left(\frac{\varphi(m)-1}{\varphi(m)}\right)^{t-2}.$$

So

$$\sum_{p_1\cdots p_t\le x}\varepsilon_2\sim\frac{1}{2}\binom{t}{2}\left(\frac{1}{\varphi(m)}\right)^2\left(\frac{\varphi(m)-1}{\varphi(m)}\right)^{t-2}\frac{x(\log\log x)^{t-1}}{(t-1)!\log x}. \tag{4.11}$$

In (e_1), the congruence conditions (mod 4) introduce a factor $\frac{1}{8}$, and the condition on Legendre symbols introduces a factor $\frac{4}{8}$. In (e_2), the congruence conditions (mod 4) introduce a factor $\frac{3}{8}$, and the condition on Legendre symbols introduces a factor $\frac{1}{8}$. So from (e_1) and (e_2) we get a factor $\frac{1}{8}\cdot\frac{4}{8}+\frac{3}{8}\cdot\frac{1}{8}=\frac{7}{64}$ to multiply the factor

$$\binom{t}{3}\left(\frac{1}{\varphi(m)}\right)^3\left(\frac{\varphi(m)-1}{\varphi(m)}\right)^{t-3}.$$

Thus

$$\sum_{p_1\cdots p_t\le x}\varepsilon_3\sim\frac{7}{64}\binom{t}{3}\left(\frac{1}{\varphi(m)}\right)^3\left(\frac{\varphi(m)-1}{\varphi(m)}\right)^{t-3}\frac{x(\log\log x)^{t-1}}{(t-1)!\log x}. \tag{4.12}$$

Then from (4.1), (4.8), (4.9), (4.10), (4.11), and (4.12), we get $d(Z_{m,t,x})=z_{m,t}$, where $z_{m,t}$ is specified in the statement of Theorem 2.

Statement (iii) in Theorem 2 can be obtained by using conditions (c), (d_1), (d_2) and by observing that

$$1-\left(\frac{1}{\varphi(m)}\right)^2[(\varphi(m)-1)^2+2(\varphi(m)-1)+\tfrac{1}{2}]=\frac{1}{2(\varphi(m))^2}.$$

REMARK. For fixed m in Theorem 2, $\lim_{t\to\infty} y_{m,t}=0$ and $\lim_{t\to\infty} z_{m,t}=0$.

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