TOTALLY REAL PSEUDO-UMBILICAL SUBMANIFOLDS OF A QUATERNION SPACE FORM

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1. Introduction. Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature c, and let $P(H)$ denote the $4n$-dimensional quaternion projective space of constant quaternion sectional curvature 4. Let N be an n-dimensional Riemannian manifold isometrically immersed in $M(c)$. We call N a totally real submanifold of $M(c)$ if each tangent 2-plane of N is mapped into a totally real plane in $M(c)$. B. Y. Chen and C. S. Houh proved in [**1**].

THEOREM A. *Let M be an n-dimensional compact totally real minimal submanifold of the quaternion projective space $P(H)$. If*

$$S \leq \frac{3n(n+1)}{(6n-1)},$$

then N is totally geodesic. Here S is the square of the length of the second fundamental form of N.

Let h be the second fundamental form of the immersion, and ξ the mean curvature vector. Let $\langle \cdot, \cdot \rangle$ denote the scalar product of $M(c)$. If there exists a function λ on N such that

$$\langle h(X, Y), \xi \rangle = \lambda \langle X, Y \rangle \tag{$*$}$$

for any tangent vector X, Y on N, then N is called a *pseudo-umbilical submanifold* of $M(c)$. It is clear that $\lambda \geq 0$. If the mean curvature vector $\xi = 0$ identically, then N is called a minimal submanifold of $M(c)$. Every minimal submanifold of $M(c)$ is itself a pseudo-umbilical submanifold of $M(c)$. In this paper, we consider the case when N is pseudo-umbilical and extend Theorem A. Our main results are

THEOREM 1. *Let N be an n-dimensional compact totally real pseudo-umbilical submanifold of $M(c)$. Then*

$$\int_N \{6S^2 - [(n+1)c + 16nH^2]S + 4n^2H^2c + 10n^2H^4\}\, dN \geq 0,$$

where H and dN denote the mean curvature of N and the volume element of N respectively.

THEOREM 2. *Let N be an n-dimensional compact totally real submanifold of $M(c)$. If*

$$6S^2 - ((n+1)c + 16nH^2)S + 4n^2H^2c + 10n^2H^4 - 4nH\Delta H \leq 0, \tag{1.1}$$

then the second fundamental form of N is parallel. In particular, if the equality of (1.1) *holds, then either N is totally geodesic or N is flat.*

When $H \equiv 0$, i.e. N is minimal, from Theorem 1 we may get (cf. [**4**]).

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COROLLARY. *Let N be an n-dimensional compact totally real minimal submanifold of P(H). If*

$$S \leq \tfrac{2}{3}(n+1),$$

then N is totally geodesic or $S = \tfrac{2}{3}(n+1)$.

2. Local formulas. We use the same notation and terminologies as in [1] unless otherwise stated. Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature c, and let N be an n-dimensional totally real submanifold of $M(c)$. We choose a local field of orthonormal frames,

$$e_1, \ldots, e_n, \qquad e_{I(1)} = Ie_1, \ldots, e_{I(n)} = Ie_n,$$
$$e_{J(1)} = Je_1, \ldots, e_{J(n)} = Je_n, \qquad e_{k(1)} = Ke_1, \ldots, e_{k(n)} = Ke_n,$$

in such a way that when restricted to $N, e_1, \ldots, e_n$ are tangent to N. Here I, J, K are the almost Hermitan structures and satisfy

$$IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J,\ I^2 = J^2 = K^2 = -1.$$

We shall use the following convention on the range of indices:

$$A, B, \ldots = 1, \ldots, n, I(1), \ldots, I(n), J(1), \ldots, J(n), K(1), \ldots, K(n),$$
$$\alpha, \beta, \ldots = I(1), \ldots, I(n), J(1), \ldots, J(n), K(1), \ldots, K(n),$$
$$i, j, \ldots = 1, \ldots, n, \qquad \phi = I, J, K.$$

Let $\{\omega_A\}$ be the dual frame field. Then the structure equations of $M(c)$ are given by

$$\begin{aligned} d\omega_A &= -\sum_B \omega_{AB} \wedge \omega_B,\ \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= -\sum_C \omega_{AC} \wedge \omega_{CB} + \tfrac{1}{2}\sum_{CD} K_{ABCD}\omega_C \wedge \omega_D, \\ K_{ABCD} &= \frac{c}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + I_{AC}I_{BD} - I_{AD}I_{BC} + 2I_{AB}I_{CD} + J_{AC}J_{BD} \\ &\quad - J_{AD}J_{BC} + 2J_{AB}J_{CD} + K_{AC}K_{BD} - K_{AD}K_{BC} + 2K_{AB}K_{CD}). \end{aligned} \tag{2.1}$$

Restricting these forms to N, we get the following structure equations of the immersion:

$$\omega_\alpha = 0, \qquad \omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \qquad h^\alpha_{ij} = h^\alpha_{ji}, \qquad h^{\phi(i)}_{jk} = h^{\phi(j)}_{ik} = h^{\phi(k)}_{ij},$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \tfrac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l,$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.2}$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \tfrac{1}{2}\sum_{ij} R_{\alpha\beta ij}\omega_i \wedge \omega_j,$$

$$R_{\alpha\beta ij} = K_{\alpha\beta ij} + \sum_k (h^\alpha_{ik}h^\beta_{kj} - h^\beta_{ik}h^\alpha_{kj}). \tag{2.3}$$

We call $h = \sum_{ij\alpha} h^\alpha_{ij}\omega_i\omega_j e_\alpha$ the second fundamental form of the immersed manifold N. We denote by $S = \sum_{ij\alpha} (h^\alpha_{ij})^2$ the square of the length of h. $\xi = \frac{1}{n}\sum_\alpha \operatorname{tr} H_\alpha e_\alpha$ and $H = \frac{1}{n}\sqrt{\sum_\alpha (\operatorname{tr} H_\alpha)^2}$ denote the mean curvature vector and the mean curvature of N, respectively. Here tr is the trace of the matrix $H_\alpha = (h^\alpha_{ij})$. Now, let $e_{k(n)}$ be parallel to ξ. Then we have

$$\operatorname{tr} H_{k(n)} = nH, \qquad \operatorname{tr} H_\alpha = 0,\ \alpha \neq k(n). \tag{2.4}$$

We define h^α_{ijk} and h^α_{ijkl} by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_l h^\alpha_{il}\omega_{lj} - \sum_l h^\alpha_{lj}\omega_{li} + \sum_\beta h^\beta_{ij}\omega_{\alpha\beta}, \tag{2.5}$$

and

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ijl}\omega_{lk} + \sum_\beta h^\beta_{ijk}\omega_{\alpha\beta},$$

respectively. Where

$$h^\alpha_{ijk} = h^\alpha_{ikj}$$

and

$$h^\alpha_{ijkl} - h^\alpha_{ijkl} = \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{mj}R_{mikl} - \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}.$$

The Laplacian h^α_{ij} of the second fundamental form h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. By a direct calculation we have (cf. [**1**, **2**, **3**])

$$\begin{aligned}\tfrac{1}{2}\Delta S &= \sum_{ijk\alpha} (h^\alpha_{ijk})^2 + \sum_{ij\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} \\ &= \sum_{ijk\alpha} (h^\alpha_{ijk})^2 + \sum_{ijk\alpha} h^\alpha_{ij}h^\alpha_{kkij} + \sum_{ijkl\alpha} h^\alpha_{ij}h^\alpha_{lk}R_{lijk} + \sum_{ijkl\alpha} h^\alpha_{ij}h^\alpha_{li}R_{lkjk} + \sum_{ijk\alpha\beta} h^\alpha_{ij}h^\beta_{ik}R_{\beta\alpha jk}.\end{aligned} \tag{2.6}$$

3. Proofs of Theorems. From (*) and (2.4) we get $\sum_\alpha \operatorname{tr} H_\alpha h^\alpha_{ij} = n\lambda\delta_{ij}$, $H^2 = \lambda$ and

$$h^{k(n)}_{ij} = H\delta_{ij}. \tag{3.1}$$

Using (3.1) we have

$$\sum_{ijk\alpha} h^\alpha_{ij}h^\alpha_{kkij} = nH\Delta H. \tag{3.2}$$

Using (2.1)–(2.4) and (3.1), we derive (cf. [**1**, **2**, **3**])

$$\begin{aligned}
&\sum_{ijkl\alpha} h^{\alpha}_{ij}h^{\alpha}_{kl}R_{lijk} + \sum_{ijkl\alpha} h^{\alpha}_{ji}h^{\alpha}_{li}R_{lkjk} + \sum_{ijk\alpha\beta} h^{\alpha}_{ij}h^{\beta}_{ik}R_{\beta\alpha jk} \\
&= \frac{c}{4}(n+1)S - n^2H^2c + \sum_{ijkl\alpha\beta} h^{\alpha}_{kk}h^{\alpha}_{ij}h^{\beta}_{jl}h^{\beta}_{li} + \sum_{\alpha\beta}\operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta}(\operatorname{tr} H_\alpha H_\beta)^2 \\
&= \frac{c}{4}(n+1)S - n^2H^2c + nH^2S + \sum_{\alpha\beta}\operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta}(\operatorname{tr} H_\alpha H_\beta)^2. \qquad (3.3)
\end{aligned}$$

Substituting (3.2) and (3.3) into (2.6), we obtain

$$\tfrac{1}{2}\Delta S = \sum_{ijk\alpha}(h^{\alpha}_{ijk})^2 + nH\Delta H + \frac{c}{4}(n+1)S - n^2H^2c + \sum_{\alpha\beta}\operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta}(\operatorname{tr} H_\alpha H_\beta)^2. \qquad (3.4)$$

In order to prove our Theorems, we need the following Lemmas.

LEMMA 1 [**4**]. *Let H_i, $i \geq 2$ be symmetric $n \times n$-matrices, $S_i = \operatorname{tr} H_i^2$, $S = \sum_i S_i$. Then*

$$\sum_{ij}\operatorname{tr}(H_iH_j - H_jH_i)^2 - \sum_{ij}(\operatorname{tr} H_iH_j)^2 \geq -\frac{3}{2}S^2,$$

and the equality holds if and only if all $H_i = 0$ or there exist two of H_i different from zero. Moreover, if $H_1 \neq 0$, $H_2 \neq 0$, $H_i = 0$, $i \neq 1, 2$, then $S_1 = S_2$ and there exists an orthogonal $(n \times n)$-matrix T such that

$$TH_1{}'T = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & \\ 0 & & 0 \end{pmatrix}, \qquad TH_2{}'T = \begin{pmatrix} 0 & a & 0 \\ a & 0 & \\ 0 & & 0 \end{pmatrix},$$

where $a = \sqrt{\dfrac{S_1}{2}}$.

LEMMA 2.

$$\sum_{\alpha\beta}\operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta}(\operatorname{tr} H_\alpha H_\beta)^2 \geq \tfrac{3}{2}S^2 + 3nH^2S - \tfrac{5}{2}n^2H^4.$$

In fact, using (2.4), (3.1) and noting that α runs up to $3n > 2$, we have

$$\begin{aligned}
\sum_{\alpha\beta}\operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta}(\operatorname{tr} H_\alpha H_\beta)^2 &= \sum_{\alpha\beta\neq k(n)}\operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\
&\quad - \sum_{\alpha\beta\neq k(n)}(\operatorname{tr} H_\alpha H_\beta)^2 - (\operatorname{tr} H_{k(n)}^2)^2. \qquad (3.5)
\end{aligned}$$

Applying Lemma 1 to (3.5), we get

$$\sum_{\alpha\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta} (\operatorname{tr} H_\alpha H_\beta)^2 \geq -\frac{3}{2}\Big(\sum_{\alpha\neq k(n)} \operatorname{tr} H_\alpha^2\Big)^2 - (\operatorname{tr} H_{k(n)}^2)^2$$

$$= -\tfrac{3}{2}(S - \operatorname{tr} H_{k(n)}^2)^2 - (\operatorname{tr} H_{k(n)}^2)^2 = -\tfrac{3}{2}(S - nH^2)^2 - n^2H^4 = -\tfrac{3}{2}S^2 + 3nH^2S - \tfrac{5}{2}n^2H^4.$$

On the other hand, by (3.1) we have

$$\sum_{ijk\alpha} (h_{ijk}^\alpha)^2 \geq \sum_{ik} (h_{iik}^{k(n)})^2 = n \sum_i (\nabla_i H)^2 = n\,|\nabla H|^2. \tag{3.6}$$

It is obvious that

$$\tfrac{1}{2}\Delta H^2 = H\Delta H + |\nabla H|^2. \tag{3.7}$$

Therefore, using Lemma 2, (2.6) and (2.7) by (3.4) we get

$$\begin{aligned}
\tfrac{1}{2}\Delta S &= \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + \sum_{ij\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha \\
&= \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + nH\Delta H + \frac{c}{4}(n+1)S + nH^2S - n^2H^2c \\
&\quad + \sum_{\alpha\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta} (\operatorname{tr} H_\alpha H_\beta)^2 \\
&\geq \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + nH\Delta H + \frac{c}{4}(n+1)S + 4nH^2S - n^2H^2c - \tfrac{3}{2}S^2 - \tfrac{5}{2}n^2H^4 \\
&\geq n\,|\nabla H|^2 + nH\Delta H + \frac{c}{4}(n+1)S + 4nH^2S - n^2H^2c - \tfrac{3}{2}S^2 - \tfrac{5}{2}n^2H^4 \\
&= \tfrac{1}{2}n\Delta H^2 + \frac{c}{4}(n+1)S + 4nH^2S - \tfrac{3}{2}S^2 - n^2H^2c - \tfrac{5}{2}n^2H^4.
\end{aligned} \tag{3.8}$$

Since N is compact, we obtain from (3.8)

$$\int_N \{6S^2 - [(n+1)c + 16nH^2]S + 4n^2H^2c + 10n^2H^4\}\, dN \geq 0.$$

From the first inequality of (3.8) we know that if N is compact and

$$6S^2 - [(n+1)c + 16nH^2]S + 4n^2H^2c + 10n^2H^4 - 4nH\Delta H \leq 0, \tag{3.9}$$

then $\sum_{ijk\alpha} (h_{ijk}^\alpha)^2 = 0$, that is, the second fundamental form h_{ij}^α is parallel. In particular, when the equality of (3.9) holds, we see from (3.8) that the equality

$$\sum_{\alpha\beta\neq k(n)} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta\neq k(n)} (\operatorname{tr} H_\alpha H_\beta)^2 = -\frac{3}{2}\Big(\sum_{\alpha\neq k(n)} \operatorname{tr} H_\alpha^2\Big)^2$$

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holds. Thus, by Lemma 1 we see that (i) $H_\alpha = 0$ ($\alpha \neq k(n)$) or (ii) there exist two non-zero H_α. In the case (i), we get $S = nH^2$. Hence noting H = constant and substituting it into the equality of (3.9), we obtain

$$(3n-1)cnH^2 = 0.$$

This implies $H = 0$, so that N is totally geodesic or $c = 0$ so that N is flat. Now, we will prove that the case (ii) can not occur. Otherwise, using the same method as in [**3**]), we may see $n = 2$. Thus we may assume

$$H_{I(1)} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \qquad H_{I(2)} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \qquad H_{K(2)} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \qquad H_\alpha = 0. \tag{3.10}$$

Here $a \neq 0$, $\alpha \neq I(1), I(2), K(2)$.

Let the codimension of N be $p(=3n)$. Put

$$S_\alpha = \sum_{ij} (h_{ij}^\alpha)^2,$$

$$p\sigma_1 = \sum_\alpha S_\alpha = S,$$

$$p(p-1)\sigma_2 = 2\sum_{\alpha<\beta} S_\alpha S_\beta.$$

It can be easily seen (cf. [**3**])

$$p^2(p-1)(\sigma_1^2 - \sigma_2) = \sum_{\alpha<\beta} (S_\alpha - S_\beta)^2. \tag{3.11}$$

By a direct calculation using (3.10), we get

$$p^2(p-1)\sigma_1^2 = (p-1)(4a^2 + 2H^2)^2, \tag{3.12}$$

$$p^2(p-1)\sigma_2 = p(8a^4 + 16a^2H^2), \tag{3.13}$$

and

$$\sum_{\alpha<\beta} (S_\alpha - S_\beta)^2 = 8(a^2 - H^2)^2. \tag{3.14}$$

Substituting (3.12)–(3.14) into (3.11), we obtain

$$(p-1)(4a^2 + 2H^2)^2 - p(8a^4 + 16a^2H^2) = 8(a^2 - H^2)^2. \tag{3.15}$$

From (3.15) we get

$$(p-3)(2a^4 + H^4) = 0,$$

namely

$$(3n-3)(2a^4 + H^4) = 0,$$

implying $n = 1$, because $2a^4 + H^4 \neq 0$. This is a contradiction, since $n = 2$.

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