STANDARD AND ACCESSIBLE RINGS

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1. Introduction. A ring is defined to be standard **(1)** in case the following two identities hold:

$$(wx, y, z) + (xz, y, w) + (wz, y, x) = 0, \tag{1}$$

$$(x, y, z) + (z, x, y) - (x, z, y) = 0, \tag{2}$$

where the associator (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$. Albert has determined the structure of finite-dimensional, standard algebras **(1)**. The simple ones turn out to be either Jordan algebras or associative ones.

We focus attention here on a more general class of rings, which we shall call accessible. By permuting w and x in (1) and subtracting from (1) we obtain the identity

$$((w, x), y, z) = 0, \tag{3}$$

where the commutator (w, x) is defined by $(w, x) = wx - xw$. A ring is called accessible in case identities (2) and (3) hold. Thus a standard ring is automatically accessible. On the other hand, while (2) and (3) hold in any commutative ring, (1) need not.

The structure of accessible rings, without finiteness assumptions, can readily be determined. An accessible ring is defined to be simple in case it has no proper two-sided ideals. Simple, accessible rings are either associative or commutative. From this result it follows trivially that simple, standard rings of characteristic different from 3 are either Jordan or associative rings. A structure for semi-simple, accessible rings is given, utilizing the Jacobson-Brown radical and the fact that primitive, accessible rings are either associative or commutative.

The following result may also be of interest. If an accessible ring has no nilpotent ideals other than zero, then it is isomorphic to a subdirect sum of an associative and a commutative ring. Hence all identities common to the class of rings, which consist of all associative rings and all commutative rings, must hold in such a ring.

The methods of proof are quite elementary. Identities are obtained which enable the construction of certain significant ideals.

2. Preliminaries. Substituting $z = y$ in (2) one obtains the flexible law, $(y, x, y) = 0$. A linearization of this identity yields $(y, x, z) = - (z, x, y)$.

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As Albert observed, it can now be seen that (2) is equivalent to the flexible law and the identity

$$(x, y, z) + (y, z, x) + (z, x, y) = 0.$$

We note that similarly (1) is equivalent to (3) and the identity

$$(wx, y, z) + (xz, y, w) + (zw, y, x) = 0.$$

In an arbitrary ring the identity

$$(xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) - (x, z, y)$$

holds. Thus (2) is a consequence of the commutative law, as well as of the associative law. Moreover the identity

$$(4) \qquad (xy, z) = x(y, z) + (x, z)y,$$

holds in every accessible ring.

Another identity which holds in an arbitrary ring is

$$(5) \qquad (wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z.$$

The nucleus of an accessible ring is defined as the set of all elements n in R with the property $(n, R, R) = 0$. If n is an element of the nucleus N of R, then because of the flexible law $(R, R, n) = 0$. Finally, because of (2), it follows that also $(R, n, R) = 0$. If n is substituted for w in (5), it becomes obvious that

$$(6) \qquad (nx, y, z) = n(x, y, z), \qquad n \in N.$$

The center C of R is defined as the set of all elements c in N which have the additional property that $(c, R) = 0$.

We now proceed to develop further identities that hold in arbitrary accessible rings. The elements u, v, w, x, y, z will denote arbitrary elements of such rings.

Through repeated use of (4) one may break up $((w, x, y), z)$ as

$$\begin{aligned}((w, x, y), z) &= (wx \cdot y - w \cdot xy, z) = wx \cdot (y, z) + w(x, z) \cdot y \\ &\quad + (w, z)x \cdot y - (w, z) \cdot xy - w \cdot x(y, z) - w \cdot (x, z)y \\ &= (w, x, (y, z)) + (w, (x, z), y) + ((w, z), x, y).\end{aligned}$$

Since (3) implies that every commutator is in the nucleus, we obtain

$$(7) \qquad ((w, x, y), z) = 0.$$

Because of (6) and the fact that every commutator is in the nucleus we get $(v, x)(x, y, z) = ((v, x)x, y, z)$. It follows from (4) that $(v, x)x = (vx, x)$. Consequently

$$((v, x)x, y, z) = ((vx, x), y, z) = 0.$$

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Therefore $(v, x)(x, y, z) = 0$.[1] A linearization of this last identity becomes

$$(v, w)(x, y, z) = -(v, x)(w, y, z). \tag{8}$$

One can now prove that a product of a commutator with an associator always lies in the center. First one notes that

$$((v, w)(x, y, z), u) = ((v, w), u)(x, y, z),$$

because of (4) and (7). From the definition of the commutator it follows that

$$((v, w), u)(x, y, z) = -(u, (v, w))(x, y, z).$$

It is this last form to which we apply (8) to obtain

$$-(u, (v, w))(x, y, z) = (u, x)((v, w), y, z).$$

Finally (3) tells us that $((v, w), y, z) = 0$, so that $(u, x)((v, w), y, z) = 0$. Consequently $((v, w)(x, y, z), u) = 0$. It remains only to prove that $(v, w)(x, y, z)$ lies in the nucleus. It is easily seen that

$$((v, w)(x, y, z), t, u) = (v, w)((x, y, z), t, u),$$

using (6) and (3). At this point (8) is employed to yield

$$(v, w)((x, y, z), t, u) = -(v, (x, y, z))(w, t, u).$$

But $(v, (x, y, z)) = 0$ was proven with (7). Consequently $((v, w)(x, y, z), t, u) = 0$. We have established that

$$(v, w)(x, y, z) \in C. \tag{9}$$

Now let us consider the element $[(v, w)(x, y, z)]^2$. Clearly

$$[(v, w)(x, y, z)]^2 = (v, w)(x, y, z)(v, w)(x, y, z) = -(v, x)(w, y, z)(v, w)(x, y, z),$$

using (3) and (8). On the other hand $(w, y, z)(v, w) = (v, w)(w, y, z)$, because of (7). But we have already noted that $(v, w)(w, y, z) = 0$. Thus we have proved that

$$[(v, w)(x, y, z)]^2 = 0. \tag{10}$$

3. Structure theory. Decently behaved rings have no nilpotent elements in their center. For let R be any ring with nilpotent elements in its center. Then there must be an element $c \neq 0$ and in the center of R such that $c^2 = 0$. Consider the ideal D generated by c. It consists of all elements of the form $ic + cx$, where i is any integer and x an arbitrary element of R. It is now easy to verify that $D^2 = 0$, and so R has a non-zero, nilpotent ideal.

Henceforth we shall be considering accessible rings R without nilpotent

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[1]Independently R. L. San Soucie has announced in Abstract 672, Bull. A. M. S. *61* (1955) that rings satisfying (3), which have no divisors of zero, are either associative or commutative.

elements in their centers, unless otherwise noted. An immediate consequence of this assumption, taking into account (9) and (10), is that

$$(v, w)(x, y, z) = 0. \tag{11}$$

But then one can also obtain from (4) and (11) that

$$(v(x, y, z), w) = (v, w)(x, y, z) = 0.$$

Also

$$((v, w)x, y, z) = (v, w)(x, y, z) = 0,$$

because of (6) and (11). This last information allows us to construct ideals A and B in R, which have rather interesting properties. Let A consist of all finite sums of elements of the form (x, y, z) or of the form $w(x, y, z)$. This set A, as may be readily verified, is an ideal even in an arbitrary ring. It is the smallest ideal modulo which the ring is associative. With the present assumptions, namely that R is accessible and has no nilpotent elements in its center, we can assert that for any element a in A we have $(a, R) = 0$.

Let B consist of all finite sums of elements of the form (x, y) or of the form $(x, y)z$. In an arbitrary ring this set need not be an ideal, but by virtue of (3) and (4) it can be shown to be one. In addition it is also true that B is contained in the nucleus N. B is also the smallest ideal modulo which R is commutative.

From previous remarks, in conjunction with (7) and (11), it becomes clear that for any element a in A and any element b in B we must have $ab = 0$. Therefore $AB = 0$. Suppose that x is an element of $A \cap B$. Then since $AB=0$, $x^2 = 0$. But x lies in the center of R because of the previously mentioned properties of A and B. Hence $x = 0$.

At this point several theorems may be established.

THEOREM 1. *A simple, accessible ring R is either associative or commutative.*

Proof. If R has nilpotent elements in its center then the ideal D described previously is different from zero, so that $D = R$. Since $D^2 = 0$, R must be a trivial ring, which is both associative and commutative. The only remaining case is the one in which R has no nilpotent elements in its center. Then the ideal B constructed above is either zero or the whole ring. If $B = 0$ then R is commutative, while if $B = R$ then R is associative, since B is contained in the nucleus. This completes the proof.

By substituting $w = x$ and $z = x$ in (1), one obtains $3(x^2, y, x) = 0$. Consequently, in a ring in which $3a = 0$ implies $a = 0$ and which satisfies the identity (1), the Jordan identity $(x^2, y, x) = 0$ must hold. Therefore a commutative, standard ring of characteristic not 3 is automatically a Jordan ring. It follows as an immediate Corollary to Theorem 1 that a simple, standard ring of characteristic not 3 is either a Jordan ring or associative. This is a generalization to rings of the theorem of Albert's mentioned in the introduction.

THEOREM 2. *If an accessible ring R has no nilpotent ideal other than zero, then it is isomorphic to a subdirect sum of an associative and a commutative ring.*

Proof. By assumption R can have no nilpotent ideal other than zero, so that $D = 0$. Hence R has no nilpotent elements in its center. Consider the natural homomorphism from R into $R/A \oplus R/B$. The kernel of this homomorphism is $A \cap B = 0$. Hence R is a subdirect sum of R/A and R/B. We have already noted that R/A is associative and that R/B is commutative. This completes the proof of the theorem.

The following is a direct consequence of Theorem 2. If an accessible ring R has a maximal nilpotent ideal I then R/I satisfies the conclusion of Theorem 2. Any expression involving elements of R, which would be automatically zero if the elements came from either an associative or a commutative ring, therefore must generate a nilpotent ideal of R. Of course the definition of accessibility requires only that two expressions, namely those occurring in (2) and (3), be zero.

The last result is concerned with a conventional type of decomposition, the introduction of a radical. Since the class of accessible rings includes the associative ones, the maximal nilpotent ideal will in general prove an unsatisfactory radical. We turn to a larger radical, namely the generalization of the Jacobson radical suggested by Brown **(2)**. From this paper it follows that an accessible ring is semi-simple if and only if it is isomorphic to a subdirect sum of primitive accessible rings. A ring is defined as primitive in case it possesses a regular maximal right ideal F, which contains no two-sided ideal of the ring other than the zero ideal.

We assert

THEOREM 3. *A semi-simple, accessible ring is a subidrect sum of primitive, accessible rings. A primitive, accessible ring is either commutative or associative.*

Proof. Only the second statement remains to be proved. Let R be a primitive, accessible ring and F a regular maximal right ideal of R which contains no two-sided ideal of R other than the zero ideal. The first step will be proving that R is prime. That is to say, if G and H are ideals of R such that $GH = 0$ and $G \neq 0$, then $H = 0$. We note that $G \not\subset F$, so that $R = F + G$. Then

$$RH = (F + G)H = FH + GH = FH \subset F.$$

For arbitrary elements x and y in R and h in H we have

$$(x, y, h) = xy \cdot h - x \cdot yh = xy \cdot h - xh' \in RH.$$

But $(x, y, h) = -(h, y, x)$, because of the flexible law, so that $(h, y, x) \in RH$. Finally, by means of (2), it can be shown that $(y, h, x) \in RH$. At this point it is easy to see that RH is an ideal of R. Since $RH \subset F$, then in fact $RH = 0$.

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The regularity of F assures the existence of an element f in R with the property that for all x in R, $fx - x$ is always in F. Then in particular $fh - h = -h$ is an element of F. Consequently $H \subset F$. Since H is an ideal, $H = 0$. Since a prime ring has no nilpotent ideals other than the zero ideal it has no nilpotent elements in its center. As previously shown this implies that the ideals A, B of R have the property $AB = 0$. Hence either $A = 0$, in which case R is associative or $B = 0$, in which case R is commutative. This completes the proof.

References

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2. Bailey Brown, *An extension of the Jacobson radical*, Proc. Amer. Math. Soc., *2* (1951), 114–117.

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