THE UNEARNED NO CLAIM BONUS

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1. The claims experience of a motorcar insurance is assumed to give some indication about the risk (basic claim frequency) of that insurance. The experience rating systems in motor insurance are based on this supposition. In these systems the premium to be paid in some year is a function of the individual claims experience of preceding years. That premium can be considered as the difference between:

— a basic premium, being the expected value of the premium to be paid for insurances with a basic claim frequency equal to the mean value of the basic claim frequency of the portfolio or the tariff class.
— a positive or negative bonus; although in the negative case the word bonus is misleading this word will nevertheless be used for both cases in this paper.

The bonus usually consists of at least the following components:

a. a component concerning the individual claim frequency.
b. an individual random factor.
c. a collective random factor.

Other factors, like the effect of the trend in the claim frequencies and the effect of the dependence of the claim frequency of age and experience of the driver are not taken into account in this paper.

2. The meaning of the three components mentioned in section 1 can be demonstrated by the following example.

The structure function $U(x)$ of the portfolio, being the distribution function of the individual basic claim frequencies x is assumed to be defined by

$$d\,U(x) = \frac{\left(\frac{1}{b}\right)^{\frac{q}{b}} x^{\frac{q}{b}-1} e^{-\frac{x}{b}}}{\Gamma\frac{q}{b}}$$

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representing a gamma distribution with expected value q and variance qb. The probability P_i for an insurance to have i claims in a certain year is supposed to be

$$P_i = \int_0^\infty \int_0^\infty \frac{(xy)^i e^{-xy}}{i!} d\,U(x)\,d\,G(y)$$

In this formula the factor y has the same value for all insurances in the portfolio in a certain year but varies from year to year according to the distribution function $G(y)$ with expected value 1 and variance σ^2_y. The amount of a claim is supposed to be 1. For $\sigma^2_y = 0$ several authors have shown that if an insurance has had n claims in the preceding t years the risk premium $p_{t+1\,|\,nt}$ for the year $t + 1$ should be:

$$p_{t+1\,|\,nt} = \frac{q + b.n}{1 + b.t}$$

This premium can be explained as the difference between the basic premium q and the bonus:

$$B_{t+1\,|\,nt} = \frac{b}{1 + b.t}(qt - n)$$

If the claim frequency is x, and if the value attained by y in the year s is y_s ($s = 1, 2 \ldots t$), with $\sum_{s=1}^{t} y_s = Y_t$, the three components mentioned in section 1 can be formulated as follows:

a. component $B(a)$ concerning the individual claim frequency:

$$B(a) = \frac{b}{1 + bt}(qt - xt)$$

b. component $B(b)$ concerning the individual random factor:

$$B(b) = \frac{b}{1 + bt}(x\,Y_t - n)$$

c. component $B(c)$ concerning the collective random factor:

$$B(c) = \frac{b}{1 + bt}(xt - x\,Y_t)$$

The term "unearned bonus" in the title of this paper is used for the sum of $B(b)$ and $B(c)$, these components not being due to the individual claim frequency.

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3. $B(a)$ tends asymptotically to $q - x$ for increasing t; $B(b)$ and $B(c)$ both tend to zero. From the point of view of the insured a bonus system like this one can therefore be said to be asymptotically correct.

4. From the point of view of the insurance company, however, the situation is more complicated. The profit of the company in a certain year is due partly to the difference between the expected number and the actual number of claims and partly to the difference between the expected and the actual mean amount of the claims. According to the supposition that all claims are equal to 1, the second part of the profit is assumed to be zero; where in the following the word profit is used this word stands for "part of the profit due to the number of claims".

If in some year y is attaining a low value or if the individual random effects are leading to a low total number of claims, or if both influences are working, two consequences can be expected:

a. a positive profit in that year.

b. a tendency to positive unearned bonuses in future years, probably leading to negative profits in those years.

For this reason it seems necessary to pay a part of the profit to a "bonus reserve" to cover the future unearned bonuses concerning that profit. In this paper we will try to estimate the part of the profit to be reserved for a special case and to show that this part may be too large to neglect in the practice of motorcar insurance.

5. An insurance portfolio consists of M_t policies, all being t years in existence, out of which M_{nit} $(n = 0, 1, \dots \infty; i = 0, 1, \dots \infty)$ with n claims in that period of t years and with i claims in the year $t + 1$; $\sum_{i=0}^{\infty} M_{nit} = M_{nt}$; $\sum_{n=0}^{\infty} M_{nt} = M_t$; $\sum_{i=0}^{\infty} i\, M_{nit} = S_n$; $\sum_{n=0}^{\infty} S_n = S$.

The structure function and the distribution function $G(y)$ are assumed to be as mentioned in section 2.

6. If the factor y attained the value y_s in year s $(s = 1, 2 \dots t)$, with $Y_t = \sum_{s=1}^{t} y_s$, the risk premium $p_{t+1} \mid_{nt}$ for an insurance in group M_{nt} for the year $t + 1$ should be, as can be demonstrated easily:

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$$p_{t+1\,|\,nt} = \frac{q + bn}{1 + b\,Y_t}$$

The expected value of $p_{t+1\,|\,nt}$ with respect to Y_t is:

$$p^{t+1}{}_{|\,nt} = \int_0^\infty \ldots \int_0^\infty \int_0^\infty \frac{q + bn}{1 + b \sum\limits_{s=1}^{t} y_s} \, d\,G(y_1)\, d\,G\,(y_1) \ldots d\,G\,(y_t)$$

A useful approximation is given by:

$$p_{t+1\,|\,nt} = \frac{q + bn}{1 + bt} \left\{ 1 + \frac{b^2\, t\, \sigma^2 y}{(1 + bt)^2} \right\}$$

In practice the correction term

$$\frac{b^2\, t\, \sigma^2 y}{(1 + bt)^2}$$

proves to be less than 0,001 and can therefore be neglected; then the wellknown expression mentioned in section 2 results:

$$p_{t+1\,|\,nt} = \frac{q + bn}{1 + bt}$$

This formula will be used in the following.

7. Assuming that the value attained by y in the year $t + 1$ is y_{t+1} we can write for the probability $P_{i|nt}$ that an insurance with n claims in the preceding t years will have i claims in the year $t + 1$:

$$P_{i|nt} = \frac{\int\limits_0^\infty \frac{(x\; y_{t+1})^i\, e^{-xy_{t+1}}}{i!} \; \frac{(x\; Y_t)^n\, e^{-xY_t}}{n!} \, d\,U\,(x)}{\int\limits_0^\infty \frac{(x\; Y_t)^n\, e^{-xY_t}}{n!} \, d\,U\,(x)}$$

which formula may be reduced to

$$P_{i|nt} = \frac{\Gamma\left(\frac{q}{b} + n + i\right) b^i\, y_{t+1}{}^i\, (1 + b\,Y_t)^{\frac{q}{b}+n}}{i!\,\Gamma\left(\frac{q}{b} + n\right) \left\{1 + b\,(Y_t + y_{t+1})\right\}^{\frac{q}{b}+n+i}}$$

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On account of Bayes' theorem the a posteriori conditional distribution function $G(y_{t+1}|M\ldots)$ of y_{t+1}, given M_{nit} $(n, i = 0, 1, 2 \ldots \infty)$ is defined by:

$$d\,G(y_{t+1}\,|\,M\ldots) = \frac{\prod\limits_{n=0}^{\infty}\left\{M_{nt}\,!\prod\limits_{i=0}^{\infty}\frac{(P_{i|nt})\,M_{nit}}{M_{nit}\,!}\right\}d\,G(y)}{\int\limits_{0}^{\infty}\prod\limits_{n=0}^{\infty}\left\{M_{nt}\,!\prod\limits_{i=0}^{\infty}\frac{(P_{i}|_{nt})\,M_{nit}}{M_{nit}\,!}\right\}d\,G(y)}$$

This formula can be reduced to

$$\mathrm{d}\,G(y_{t+1}\,|\,M\ldots) = \frac{\dfrac{y_{t+1}^{S}}{\{1 + b\,(Y_t + y_{t+1})\}^{\frac{q}{b}M_t + \sum\limits_{n=0}^{\infty} nM_{nt} + s}}\,d\,G(y)}{\int\limits_{0}^{\infty}\dfrac{y_{t+1}^{S}}{\{1 + b\,(Y_t + y_{t+1})\}^{\frac{q}{b}M_t + \sum\limits_{n=0}^{\infty} nM_{nt} + S}}\,d\,G(y)}$$

8. The total risk premium p_{t+1} to be received in year $t + 1$ is

$$p_{t+1} = \sum_{n=0}^{\infty}\frac{M_{nt}(q + bn)}{1 + bt} = \frac{q\,M_t + b\sum\limits_{n=0}^{\infty} n\,M_{nt}}{1 + bt}$$

The profit of the year $t + 1$ (in the sense defined in section 4) W is:

$$W = p_{t+1} - S$$

The expected value $p_{t+u+1}\,|\,_{nit}$ of the premium that will be paid in year $t + u + 1$ for an insurance belonging to M_{nit} is:

$$p_{t+u+1}|_{nit} =$$

$$= \int\limits_{0}^{\infty}\frac{q + b(n+i) + b\,(u-1)\,\dfrac{q + b(n+i)}{1 + b(t + Y_{t+1})}}{1 + b\,(t + u)}\,d\,G(Y_{t+1}\,|\,M\ldots)$$

$$= \frac{q + b\,(n+i)}{1 + b\,(t+u)}\int\limits_{0}^{\infty}\frac{1 + b\,(t + u - 1 + y_{t+1})}{1 + b\,(t + y_{t+1})}\,d\,G(y_{t+1}\,|\,M\ldots)$$

The expected value $\bar{p}_{t+u+1}\,|\,_{nit}$ of the premium that should be received in year $t + u + 1$ for an insurance belonging to M_{nit} is:

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$$\bar{p}_{t+u+1} \mid {}_{nit} = \int_0^\infty \int_0^\infty \frac{\int_0^\infty x\, y_{t+u+1}\, (x\, y_{t+1})^i\, (x\, Y_t)^n\, e^{-xy_{t+1}}\, e^{-xY_t}\, d\, U(x)}{\int_0^\infty (x\, y_{t+1})^i\, (x\, Y_t)^n\, e^{-xy_{t+1}}\, e^{-xY_t}\, d\, U(x)} \, d\, G(y_{t+u+1})\, d\, G(y_{t+1} \mid M \ldots)$$

This may be reduced to:

$$\bar{p}_{t+u+1} \mid {}_{nit} = \{q + b\,(n+i)\} \int_0^\infty \frac{d\, G(y_{t+1} \mid M \ldots)}{1 + b\,(Y_t + y_{t+1})}$$

$$\sim \{q + b\,(n+i)\} \int_0^\infty \frac{d\, G\,(y_{t+1} \mid M \ldots)}{1 + b\,(t + y_{t+1})}$$

The unearned part of the bonus in the year $t + u + 1$ is:

$$\bar{p}_{t+u+1} \mid {}_{nit} - p_{t+u+1} \mid {}_{nit} = \frac{q + b\,(n+i)}{1 + b\,(t+n)} \int_0^\infty \frac{b(1 - y_{t+1})\, d\, G(y_{t+1} \mid M \ldots)}{1 + b\,(t + y_{t+1})}$$

We assume that out of the M_t insurances existing at the end of the year $t + 1$ a number $r_u \,.\, M_t$ will still be existing during the year $t + u + 1$ $(u = 1, 2, \ldots)$. Then the total amount D_u of unearned bonuses in the year $t + u + 1$ concerning the profit of the year $t + 1$ is:

$$D_u = r_u \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} M_{nit} \frac{q + b\,(n+1)}{1 + b\,(t+u)} \int_0^\infty \frac{b\,(1 - y_{tn1})\, d\, G(y_{t+1} \mid M \ldots)}{1 + b\,(t + y_{t+1})}$$

After addition over u and further reduction this leads to:

$$D = \sum_{u=1}^{\infty} D_u = [(1 + bt)\, W + \{1 + b\,(t+1)\}\, S].$$

$$\int_0^\infty \frac{b\,(1 - y_{t+1})\, d\, G(y_{t+1} \mid M \ldots)}{1 + b\,(t + y_{t+1})} \,.\, \sum_{u=1}^{\infty} \frac{r_u}{1 + b\,(t+u)}$$

9. If M_t is large and if some other conditions are fulfilled the a posteriori distribution function $G\,(y_{t+1} \mid M \ldots)$ tends to:

$$G(y_{t+1} \mid M \ldots) = 0 \text{ if } y_{t+1} < \frac{S}{p_{t+1}} = \frac{S}{W+S}$$

$$G(y_{t+1} \mid M \ldots) = 1 \text{ if } y_{t+1} \geqslant \frac{S}{W+S}$$

Then the formula for D may be reduced to:

$$D = b.W. \sum_{u=1}^{\infty} \frac{r_u}{1 + b(t+u)}$$

So an approximation of the relative part of the profit of any year that has to be paid to a bonus reserve to cover future unearned bonuses concerning that profit is given by:

$$\frac{D}{W} = b \sum_{u=1}^{\infty} \frac{r_u}{1 + b(t+u)}$$

10. If we assume:

$b = 0{,}1$

$r_u = 1 - 0{,}1u$ if $u \leqslant 10$; $r_u = 0$ if $u > 10$

t = mean "age" of portfolio = 5

then $\frac{D}{W} = 0{,}24$

This result seems too high to neglect in the balance sheet of the company.

11. The aim of this paper is to demonstrate that the bonus reserve in motorcar insurance may be important in practice, and to estimate this reserve in a special case. The methods described may, however, be used in other cases and also for other bonus systems. In case M_t or σ_y is so small that $G(y_{t+1} \mid M \ldots)$ has to be calculated the calculation is relatively simple. To illustrate the effect of a more precise calculation in such a case we mention that if in the example of section 10, $M_t = 25.000$; $\sigma_y^2 = 0{,}001$; $p_{t+1} = 3.800$; $S = 3.600$ (so that $W = 200$) the exact value of $\frac{D}{W}$ is 0,19 instead of 0,24.

12. The above mentioned methods are concerned with the profit of one year only. In fact the bonus reserve has to be formed as the

sum of additions and reductions regarding positive and negative profits of successive years and regarding positive and negative payments of unearned bonuses. The problems concerning this point and concerning the procedure for a negative reserve are not discussed here.

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