COMPLETELY CONTINUOUS MOVEMENTS IN TOPOLOGICAL VECTOR SPACES

by J. H. MICHAEL

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1. Introduction. Let A be a closed subset of a topological space X and f a continuous mapping of A into X with the following two properties :

1.1. $f\{\text{Fr}(A)\}$ and $f\{\text{Int}(A)\}$ are disjoint.

1.2. The mapping $f_* = f \mid \text{Fr}(A)$ is 1–1.

It is proved in [**5**], that if X is the euclidean n-sphere $S^n = \{x\,;\ x \in R^{n+1} \text{ and } \|x\| = 1\}$, then

1.3. $\quad f\{\text{Fr}(A)\} = \text{Fr}\{f(A)\}$.

[Hence $f\{\text{Int}(A)\} = \text{Int}\{f(A)\}$].

The purpose of the present paper is to prove (Theorem 4.4.) that 1.3 is true when X is a real convex topological vector space with a Hausdorff topology and f satisfies the additional requirement of being a completely continuous movement. The proof of this theorem makes use of the degree of completely continuous movements.

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2. Notation. We shall assume henceforth that E denotes a real convex topological vector space with a Hausdorff topology. $\mathscr{U}$ is the collection of all convex symmetrical open neighbourhoods of the origin. In the usual way, a mapping f of a subset A of E into E is defined to be *completely continuous* on A if it is continuous and there exists a compact subset K of E with $f(A) \subseteq K$. Following Nagumo [**6**], we define a *completely continuous movement* of a subset A of E to be a mapping f of A into E, such that the function

$$\phi(x) \equiv f(x) - x$$

is completely continuous on A. Set complementation is denoted by $\sim$; i.e., if $B \subseteq A$, then $A \sim B$ denotes the complement of B in A.

3. The degree of a completely continuous movement.

3.1. We first of all observe that completely continuous movements have the following properties. These are proved in [**6**].

3.1.1. If A is a closed subset of E and f is a completely continuous movement of A, then $f(A)$ is closed.

3.1.2. If f, g are completely continuous movements of A, $f(A)$, respectively, then gf is a completely continuous movement of A.

3.1.3. If f is a 1–1 completely continuous movement of A and A is closed, then f^{-1} is a completely continuous movement of $f(A)$.

3.2. Consider the triple (f, A, b), where A is a subset of E, f is a completely continuous movement of $\text{Fr}(A)$ and b is a point of $E \sim f\{\text{Fr}(A)\}$. With each such triple there is associated an integer

$$d(f, A, b)$$

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called its degree.† This degree has the following properties.

3.2.1. If f is the identity mapping of Fr (A), then

$$d(f, A, x) = 1 \quad \text{when } x \in \operatorname{Int}(A),$$
$$= 0 \quad \text{when } x \in E \sim \bar{A}.$$

3.2.2. If f is a completely continuous movement of $\bar{A}$ and $b \notin f(\bar{A})$, then

$$d(f, A, b) = 0.$$

3.2.3. Let $\mathscr{G}$ be a collection of mutually disjoint open subsets of Int (A). Put

$$U = \bigcup_{G \in \mathscr{G}} G.$$

If f is a completely continuous movement of $\bar{A} \sim U$ and $b \in E \sim f(\bar{A} \sim U)$, then $d(f, G, b) = 0$ for all but a finite number of $G \in \mathscr{G}$ and

$$d(f, A, b) = \sum_{G \in \mathscr{G}} d(f, G, b).$$

3.2.4. If f is a completely continuous movement of Fr (A) and b_1, b_2 are points of the same component of $E \sim f\{\operatorname{Fr}(A)\}$, then

$$d(f, A, b_1) = d(f, A, b_2).$$

3.2.5. If f is a completely continuous movement of Fr (A), $b \in E \sim f\{\operatorname{Fr}(A)\}$, $U \in \mathscr{U}$ and U is such that $b + U$ does not intersect $f\{\operatorname{Fr}(A)\}$ and if f_1 is a completely continuous movement of Fr (A) such that

$$f(x) - f_1(x) \in U$$

for all $x \in \operatorname{Fr}(A)$, then

$$d(f, A, b) = d(f_1, A, b).$$

3.2.6. Let B be a second subset of E, f and g be completely continuous movements of $\bar{A}$ and $\bar{B}$ such that $f(\bar{A}) \subseteq \bar{B}$, and b be a point of $E \sim g[\operatorname{Fr}(B) \cup f\{\operatorname{Fr}(A)\}]$ such that $d(f, A, y)$ is constant for $y \in g^{-1}(b)$. Then

$$d(gf, A, b) = d(g, B, b) \,.\, d\{f, A, g^{-1}(b)\} \quad \text{if } g^{-1}(b) \neq \varnothing,$$
$$= 0 \quad \text{if } g^{-1}(b) = \varnothing.$$

[Here $d\{f, A, g^{-1}(b)\}$ denotes the constant value of $d(f, A, y)$ for $y \in g^{-1}(b)$].

Further properties of the degree are given by Theorems 3.2.7 and 3.2.9, which appear below. Theorem 3.2.9 is not new (it is used by Leray in [**3**], for example), but since the proof does not appear to be readily available in the literature, the theorem is proved here.

3.2.7. Theorem. *Let A be a subset of E, f be a completely continuous movement of* Fr (A) *and $b \in E \sim f\{\operatorname{Fr}(A)\}$. Let K be a compact subset of E such that*

$$f(x) - x \in K$$

for all $x \in$ Fr (A). *If F is a linear manifold of E which contains b and K, then*

$$d(f, A, b) = d(f, F \cap A, b) \text{ in } F.$$

Proof. Choose $U \in \mathscr{U}$ such that $b + U$ does not intersect $f\{\operatorname{Fr}(A)\}$. K is evidently compact in F; hence, by Theorem 2 of [**6**], there exist a finite dimensional linear manifold G of F containing b, and a continuous mapping ϕ of K into G such that

$$\phi(x) - x \in U,$$

† For the definition and properties of the degree, see [**3**] and [**6**]. Actually, in [**3**] and [**6**] the degree is defined for open A. However, it can easily be defined for arbitrary A by putting

$$d(f, A, b) = d\{f, \operatorname{Int}(A), b\}.$$

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for all $x \in K$. Put

$$f_1(x) = \phi\{f(x) - x\} + x, \quad \text{...(1)}$$

for all $x \in \text{Fr}(A)$. Since

$$f_1(x) - x = \phi\{f(x) - x\} \in \phi(K),$$

and $\phi(K)$ is compact, $f_1(x)$ is a completely continuous movement of $\text{Fr}(A)$. Also

$$f(x) - f_1(x) = -[\phi\{f(x) - x\} - \{f(x) - x\}];$$

hence

$$f(x) - f_1(x) \in U, \quad \text{...(2)}$$

for all $x \in \text{Fr}(A)$. Furthermore, it follows from (1) that

$$f_1(x) - x \in G, \quad \text{..(3)}$$

for all $x \in \text{Fr}(A)$. Now, by (2) and 3.2.5,

$$d(f, A, b) = d(f_1, A, b),$$

and, by 3.2.3, this is

$$d(f_1, \text{Interior of } A \text{ in } E, b),$$

which, by (3) and [**3**], equals

$$\{df_1, G \cap (\text{Interior of } A \text{ in } E), b\} = d\{f_1, F \cap (\text{Interior of } A \text{ in } E), b\};$$

hence, by 3.2.3,

$$d(f, A, b) = d(f_1, F \cap A, b),$$

and, since (Frontier of $(F \cap A)$ in F) $\subseteq$ (Frontier of A in E) $\cap F$, we have, by (2) and 3.2.5,

$$d(f, A, b) = d(f, F \cap A, b).$$

3.2.8. Lemma. *If F is a linear manifold of E and K is a compact subset of F which spans F, then the relative topology of F is normal.*

Proof. The relative topology of F is regular and Lemma 1 on p. 113 of [**1**] shows that a regular Lindelöf space is normal. Hence we have only to prove that F is a Lindelöf space, i.e. that each covering of F by open sets of F has a countable subcovering.

To this end, let $\mathscr{V}$ be an open covering of F. For each positive integer n, let K_n be the set of all points

$$\lambda_1 x_1 + \ldots + \lambda_n x_n$$

of F, where $x_1, \ldots, x_n \in K$ and $\lambda_1, \ldots, \lambda_n$ are real numbers such that $|\lambda_1| \leqslant n, \ldots, |\lambda_n| \leqslant n$. If J_n denotes the closed interval $[-n, n]$, then K_n is a continuous image of the compact space $J_n \times \ldots \times J_n \times K \times \ldots \times K$ ($2n$ factors); hence K_n is compact. Evidently

$$F = \bigcup_{n=1}^{\infty} K_n.$$

For each n we can choose a finite subcollection $\mathscr{V}'_n$ of $\mathscr{V}$ which covers K_n. Let

$$\mathscr{V}' = \bigcup_{n=1}^{\infty} \mathscr{V}'_n.$$

$\mathscr{V}'$ is countable, $\mathscr{V}' \subseteq \mathscr{V}$ and $\mathscr{V}'$ covers F. This completes the proof.

3.2.9. Theorem. *If A and B are subsets of E with* $\text{Int}(B) \neq \varnothing$, *$f$ is a completely continuous movement of $\bar{A}$ into $\bar{B}$ such that* $f\{\text{Fr}(A)\} \subseteq \text{Fr}(B)$, *$g$ is a completely continuous movement of* $\text{Fr}(B)$, $b \in E \sim g\{\text{Fr}(B)\}$ *and $d(f, A, y)$ is constant for* $y \in \text{Int}(B)$, *then*

$$d(gf, A, b) = d(g. B, b) \,.\, d\{f, A, \text{Int}(B)\}.$$

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Proof. Let K and L be compact subsets of E such that

$$f(x)-x \in K$$

for all $x \in \bar{A}$ and

$$g(x)-x \in L$$

for all $x \in \text{Fr}(B)$. Let $b_1 \in \text{Int}(B)$ and F be the linear manifold spanned by the compact set $K \cup L \cup \{b, b_1\}$. By 3.2.7,

$$d\{f, A, \text{Int}(B)\} = d\{f, F\cap A, F\cap \text{Int}(B)\} \quad \text{.............................(4)}$$

and

$$d(g, B, b) = d(g, F\cap B, b). \quad \text{.......................................(5)}$$

But, since $K+L \subseteq F$, $K+L$ is compact and, for all $x \in \text{Fr}(A)$, we have

$$gf(x)-x = \{f(x)-x\} + [g\{f(x)\} - f(x)] \in K+L,$$

it also follows from 3.2.7 that

$$d(gf, A, b) = d(gf, F\cap A, b). \quad \text{.......................................(6)}$$

Thus, it will be sufficient to prove that

$$d(gf, F\cap A, b) = d(g, F\cap B, b) \,.\, d\{f, F\cap A, F\cap \text{Int}(B)\}. \quad \text{.....................(7)}$$

By 3.1.1, $g\{F\cap \text{Fr}(B)\}$ is closed in F; hence there exists an open, convex, symmetrical neighbourhood U of the origin in F such that

$$(b+U)\cap g\{F\cap \text{Fr}(B)\} = \varnothing. \quad \text{.......................................(8)}$$

By Theorem 2 of [6], we can find a finite dimensional linear manifold F^m of F and a continuous mapping θ of L into F^m such that

$$\theta(x)-x \in U$$

for all $x \in L$. Put

$$\psi_1(x) = \theta\{g(x)-x\}$$

for all $x \in F\cap \text{Fr}(B)$. Now $\theta(L)$ is a compact subset of F^m, $F\cap \text{Fr}(B)$ is closed in F, ψ_1 is a continuous mapping of $F\cap \text{Fr}(B)$ into $\theta(L)$ and, by Lemma 3.2.8, F is normal. Hence one can apply Tietze's Extension Theorem ([2], p. 28) to extend ψ_1 to a continuous mapping of $F\cap \bar{B}$ into a compact subset L_1 of F^m. Put

$$g_1(x) = \psi_1(x) + x$$

for all $x \in F\cap \bar{B}$. Then g_1 is a completely continuous movement of $F\cap \bar{B}$ into F and for all $x \in F\cap \text{Fr}(B)$ we have $g(x)-g_1(x) = \{g(x)-x\} - \theta\{g(x)-x\}$, and hence

$$g(x)-g_1(x) \in U \quad \text{..(9)}$$

for all $x \in F\cap \text{Fr}(B)$. Since the frontier of $F\cap B$ in F is contained in $F\cap \text{Fr}(B)$, it follows from (8), (9) and 3.2.5 that

$$d(g, F\cap B, b) = d(g_1, F\cap B, b). \quad \text{...................................(10)}$$

Since $f\{F\cap \text{Fr}(A)\} \subseteq F\cap \text{Fr}(B)$, we obtain from (9)

$$gf(x) - g_1 f(x) \in U$$

for all $x \in F\cap \text{Fr}(A)$; hence

$$d(gf, F\cap A, b) = d(g_1 f, F\cap A, b). \quad \text{................................(11)}$$

Now it follows from (8) and (9) that $g_1^{-1}(b) \subseteq F\cap \text{Int}(B)$; hence, by (4), $d(f, F\cap A, y)$ is constant for $y \in g_1^{-1}(b)$. Therefore, by 3.2.6,

$$d(g_1 f, F\cap A, b) = d(g_1, F\cap B, b) \,.\, d\{f, F\cap A, F\cap \text{Int}(B)\}. \quad \text{..................(12)}$$

Equation (7) now follows immediately from (10), (11) and (12), and the proof is complete.

4. The main result. In this section we prove the theorem that was discussed in § 1. Throughout the section A denotes a closed subset of E and f a completely continuous movement of A with the properties 1.1 and 1.2. As in 1.2, $f_* = f \mid \mathrm{Fr}\,(A)$.

4.1. LEMMA. *If Q is a component of $E \sim f\{\mathrm{Fr}\,(A)\}$ which intersects $f(A)$ and if $P = f^{-1}(Q)$, then*

(i) *Q does not intersect $f\{\mathrm{Fr}\,(P)\}$, and*

$$d(f, P, y) \neq 0$$

for all $y \in Q$;

(ii) $\mathrm{Fr}\,(Q) \subseteq f\{\mathrm{Fr}\,(A)\}$, *$f_*^{-1}\{\mathrm{Fr}\,(Q)\}$ does not intersect P or $E \sim \overline{P}$ and*

$$\begin{aligned} d(f_*^{-1}, Q, x) &\neq 0 \quad \textit{for } x \in P, \\ &= 0 \quad \textit{for } x \in E \sim \overline{P}. \end{aligned}$$

Proof. P and Q are evidently open sets of E and

$$\mathrm{Fr}\,(Q) \subseteq f\{\mathrm{Fr}\,(A)\}. \qquad \text{(13)}$$

If a is an arbitrary point of $\mathrm{Fr}\,(P)$, then $f(a) \in \overline{Q}$ and $f(a) \notin Q$; for $f(a) \in Q$ would imply $a \in P$. Hence $f(a) \in \mathrm{Fr}\,(Q)$. Thus

$$f\{\mathrm{Fr}\,(P)\} \subseteq \mathrm{Fr}\,(Q). \qquad \text{(14)}$$

Hence

$$Q \cap f\{\mathrm{Fr}\,(P)\} = \emptyset.$$

Also

$$P \cap f_*^{-1}\{\mathrm{Fr}\,(Q)\} = \emptyset;$$

for, if $a' \in P$, then $f(a') \in Q$. Hence $f(a') \notin \mathrm{Fr}\,(Q)$; i.e., $a' \notin f_*^{-1}\{\mathrm{Fr}\,(Q)\}$. Now

$$\mathrm{Fr}\,(P) \subseteq \mathrm{Fr}\,(A). \qquad \text{(15)}$$

For otherwise there would exist a point $p \in \mathrm{Fr}\,(P)$ with $p \in \mathrm{Int}\,(A)$; then, by 1.1,

$$f(p) \notin f\{\mathrm{Fr}\,(A)\},$$

which contradicts (13) and (14). By (14), 3.2.4 and Theorem 3.2.9,

$$d(f_*^{-1}f, P, x) = d(f_*^{-1}, Q, x)\,.\,d(f, P, y) \qquad \text{(16)}$$

for all $x \in E \sim f_*^{-1}\{\mathrm{Fr}\,(Q)\}$ and all $y \in Q$. Therefore by (15), 1.1 and 3.2.1,

$$\begin{aligned} d(f_*^{-1}, Q, x)\,.\,d(f, P, y) &= 1, \quad \text{for } x \in P, \\ &= 0, \quad \text{for } x \in E \sim [\overline{P} \cup f_*^{-1}\{\mathrm{Fr}\,(Q)\}], \end{aligned} \qquad \text{(17)}$$

for all $y \in Q$; consequently, since P is not empty,

$$d(f, P, y) \neq 0 \qquad \text{(18)}$$

for all $y \in Q$. It now follows from (18) and 3.2.2 that $Q \subseteq f(\overline{P})$; hence, since $f(\overline{P})$ is closed, $\overline{Q} \subseteq f(\overline{P})$; therefore

$$(E \sim \overline{P}) \cap f_*^{-1}\{\mathrm{Fr}\,(Q)\} = \emptyset. \qquad \text{(19)}$$

For, if $c \in f_*^{-1}\{\mathrm{Fr}\,(Q)\}$, then $f(c) \in \overline{Q}$, $f(c) \in f(\overline{P})$, and, since, by (13), 1.1 and 1.2, c is the only point in $f^{-1}\{f(c)\}$, we have $c \in \overline{P}$. Since Q is not empty, it now follows from (17), (18) and (19) that

$$\begin{aligned} d(f_*^{-1}, Q, x) &\neq 0, \quad \text{for } x \in P, \\ &= 0, \quad \text{for } x \in E \sim \overline{P}. \end{aligned}$$

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4.2. Lemma. $\text{Fr}\,\{f(A)\} \subseteq f\{\text{Fr}\,(A)\}$.

Proof. Suppose that the lemma is not true ; i.e., that there exists a point $b \in \text{Fr}\,\{f(A)\}$ with $b \notin f\{\text{Fr}\,(A)\}$. Let Q be the component of $E \sim f\{\text{Fr}\,(A)\}$ containing b and put $P = f^{-1}(Q)$. By Lemma 4.1, $d(f, P, y) \neq 0$ for all $y \in Q$. But, since Q is open and $b \in \text{Fr}\,\{f(A)\}$, Q must contain a point y' of $E \sim f(A)$; hence $y' \notin f(\bar{P})$ and, by 3.2.2, $d(f, P, y') = 0$. This is a contradiction.

4.3. Lemma. $d\{f_*^{-1}, f(A), x\} \neq 0$, *for* $x \in \text{Int}\,(A)$,
$= 0$, *for* $x \in E \sim A$.

Proof. Let $x \in E \sim \text{Fr}\,(A)$. Denote by $\mathcal{Q}$ the collection consisting of all those components of $E \sim f\{\text{Fr}\,(A)\}$ that are contained in $f(A)$. By Lemma 4.2,

$$f(A) \sim f\{\text{Fr}\,(A)\} = \bigcup_{Q \in \mathcal{Q}} Q. \qquad (20)$$

Hence, by 3.2.3,

$$d\{f_*^{-1}, f(A), x\} = \sum_{Q \in \mathcal{Q}} d(f_*^{-1}, Q, x). \qquad (21)$$

(Empty sums are regarded as zero.)

Suppose that $x \in \text{Int}\,(A)$. By 1.1 and (20), there is exactly one $Q \in \mathcal{Q}$, say Q', such that $x \in f^{-1}(Q)$. Therefore, by Lemma 4.1,

$$d(f_*^{-1}, Q, x) \neq 0, \quad \text{when } Q = Q',$$
$$= 0, \quad \text{when } Q \neq Q'.$$

Hence by (21), $d\{f_*^{-1}, f(A), x\} \neq 0$.

If $x \in E \sim A$, there is no $Q \in \mathcal{Q}$ with $x \in f^{-1}(Q)$, so that, by (21), $d\{f_*^{-1}, f(A), x\} = 0$.

4.4. Theorem. $f\{\text{Fr}\,(A)\} = \text{Fr}\,\{f(A)\}$.

Proof. Because of Lemma 4.2, we have only to prove that

$$f\{\text{Fr}\,(A)\} \subseteq \text{Fr}\,\{f(A)\}.$$

Suppose that this inequality is not true ; i.e., that there exists a point $b \in f\{\text{Fr}\,(A)\}$ with $b \notin \text{Fr}\,\{f(A)\}$. Put $a = f_*^{-1}(b)$. Then $a \in \text{Fr}\,(A)$ and $b \in \text{Int}\,\{f(A)\}$.

(i) Let $a \in \overline{\text{Int}\,(A)}$. Let C be the component of $E \sim f_*^{-1}[\text{Fr}\,\{f(A)\}]$ which contains a. Then C is open and therefore contains a point a' of $\text{Int}\,(A)$ and a point a'' of $E \sim A$. By 3.2.4,

$$d\{f_*^{-1}, f(A), a'\} = d\{f_*^{-1}, f(A), a''\},$$

and this contradicts Lemma 4.3.

(ii) Let $a \notin \overline{\text{Int}\,(A)}$. We have $b \notin f\{\overline{\text{Int}\,(A)}\}$. Since $b \in \text{Int}\,\{f(A)\}$, there exists a $U \in \mathcal{U}$ with $\overline{b + U}$ contained in $f(A)$ but not intersecting $f\{\overline{\text{Int}\,(A)}\}$. Then $\overline{b + U} \subseteq f\{\text{Fr}\,(A)\}$ and $f_*^{-1} \mid \overline{b + U}$ is a 1–1 completely continuous movement. Therefore, by Lemma 4.2,

$$\text{Fr}\,\{f_*^{-1}(\overline{b + U})\} \subseteq f_*^{-1}\{\text{Fr}\,(\overline{b + U})\},$$

so that $a \notin \text{Fr}\,\{f_*^{-1}(\overline{b + U})\}$, and hence $a \in \text{Int}\,\{f_*^{-1}(\overline{b + U})\} \subseteq \text{Int}\,(A)$. This is a contradiction.

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ON CERTAIN RELATIONS BETWEEN PRODUCTS OF BILATERAL HYPERGEOMETRIC SERIES

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1. Introduction. Darling [3] in 1932 and Bailey [2] in 1933 gave certain theorems on products of hypergeometric series. Again in 1948 Sears [**4**] used the relation which expresses the ${}_M\Phi_{M-1}(x)$ series in terms of M other series of the same type to derive transformations between products of both basic and ordinary hypergeometric series. In this paper I give certain general theorems on products of bilateral hypergeometric series together with some of their interesting special cases.

The following notation is used throughout the paper :

$$(a\,;\ n)=(1-a)(1-aq)\ldots(1-aq^{n-1}),\quad (a\,;\ 0)=1,$$

$$(a\,;\ -n)=(-1)^n q^{\frac{1}{2}n(n+1)}/a^n(q/a\,;\ n),\quad |\,q\,|<1,$$

$$(a)_n=a(a+1)\ldots(a+n-1),\ (a)_0=1,\quad (a)_{-n}=(-1)^n/(1-a)_n,$$

$${}_r\Psi_r\left[\begin{matrix}a_1,\, a_2,\, \ldots\, ,\, a_r\, ;\, z\\ b_1,\, b_2,\, \ldots\, ,\, b_r\end{matrix}\right]=\sum_{n=-\infty}^{\infty}\frac{(a_1\,;\,n)(a_2\,;\,n)\ldots(a_r\,;\,n)}{(b_1\,;\,n)(b_2\,;\,n)\ldots(b_r\,;\,n)}z^n,$$

$${}_rH_r\left[\begin{matrix}a_1,\, a_2,\, \ldots\, ,\, a_r\, ;\, z\\ b_1,\, b_2,\, \ldots\, ,\, b_r\end{matrix}\right]=\sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_r)_n}z^n,$$

$$\Pi\left[\begin{matrix}a_1,\, a_2,\, \ldots\, ,\, a_r\\ b_1,\, b_2\, \ldots\, ,\, b_r\end{matrix}\right]=\prod_{n=0}^{\infty}\frac{(1-a_1q^n)(1-a_2q^n)\ldots(1-a_rq^n)}{(1-b_1q^n)(1-b_2q^n)\ldots(1-b_rq^n)},$$

$$\Gamma\left[\begin{matrix}a_1,\, a_2,\, \ldots\, ,\, a_r\\ b_1,\, b_2,\, \ldots\, ,\, b_r\end{matrix}\right]=\frac{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_r)}{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_n)},$$

and idem $(a\,;\,b)$ means that the preceding expression is repeated with a and b interchanged.

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