# FINITE GROUPS WITH ALL MAXIMAL SUBGROUPS OF PRIME OR PRIME SQUARE INDEX

JOSEPH KOHLER

**1. Introduction.** In this paper finite groups with the property $M$, that every maximal subgroup has prime or prime square index, are investigated. A short but ingenious argument was given by P. Hall which showed that such groups are solvable.

B. Huppert showed that a finite group with the property $M^1$, that every maximal subgroup has prime index, is supersolvable, i.e. the chief factors are of prime order. We prove here, as a corollary of a more precise result, that if $G$ has property $M$ and is of odd order, then the chief factors of $G$ are of prime or prime square order. The even-order case is different. For every odd prime $p$ and positive integer $m$ we shall construct a group of order $2^a p^b$ with property $M$ which has a chief factor of order larger than $m$.

These results can be stated in another form by using a theorem due to Huppert (**7**, Satz 1). If $G$ is a finite group with property $M^1$, then all subgroups of $G$ have property $M^1$ and if $G$ has property $M$ with $|G|$ odd, then all subgroups of $G$ have property $M$.

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**2. Notations and definitions.** The following is a list of notations which will be used: $|G|$ = the order of $G$; $H < G$ means $H$ is a subgroup of $G$; $Z(G)$ = the centre of $G$; $\Delta(G)$ = the intersection of the non-normal maximal subgroups of $G$; $\phi(G)$ = the Frattini subgroup of $G$; $J_p$ = the field with $p$ elements; $GL(n, p)$ = the group of non-singular $n \times n$ matrices over $J_p$; $\langle A, B\rangle$ = the group generated by the subsets $A$ and $B$ of $G$; $(A, B)$ = commutator subgroup of $A$ and $B$; $G^n = \langle X^n | X \in G\rangle$.

*Definition.* Let $p$ be a prime which divides $|G|$, where $G$ is solvable. If among the chief factors of $G$ which have order a power of $p$ the exponent $s$ is the largest one that occurs, then $s$ is the $p$-rank of $G$. This will be denoted by $r_p(G)$.

**3. The main theorems.** Let $G$ be a group with property $M$. By (**4**,

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Theorem 10.5.7), $G$ is solvable. However, the proof yields more information. This we state as the following theorem.

THEOREM 3.1 (P. Hall). *Let $G$ be a group with property $M$. Then there exists a normal series $G > K > 1$ with $|K|$ prime to 6, $|G/K| = 2^a 3^b$, and $K$ has an ordered Sylow tower.*

*Proof.* See the proof of (**4**, Theorem 10.5.7).

If 2 does not divide $|G|$, then $G$ has an ordered Sylow tower. The same is true if 3 does not divide $|G|$.

LEMMA 3.1. *A subgroup $K$ of $GL(2, p)$ which has odd order prime to $p$ is abelian.*

*Proof.* For $p = 3$ the lemma is trivial. Thus we may assume that $p > 3$. The group $GL(2, p)$ has a normal subgroup $G$ of index 2 consisting of those matrices whose determinant is a square. We observe that $G > K$ and $G > Z$, the centre of $GL(2, p)$ consisting of all scalar multiples of the identity. A list of subgroups of $G/Z$ can be found in (**2**, pp. 447–450). The subgroups of odd order prime to $p$ are cyclic, and this order is a divisor of $p + 1$ or $p - 1$. Thus $KZ/Z$ is cyclic and therefore $KZ$ is abelian. Hence $K$ is abelian.

LEMMA 3.2. *Let $G$ be a finite abelian group and let $\rho$ be an irreducible representation of $G$ over the field $F$. Then $\rho(G)$ is cyclic.*

*Proof.* Let $A$ be an $F$–$G$ module, which yields the representation $\rho$. Since $A$ is irreducible, it follows from Schur's lemma that the ring of operator endomorphisms of $A$ forms a division ring $D$. $D$ is isomorphic to the ring of square matrices whose elements $\alpha$ satisfy $\rho(x)\alpha = \alpha\rho(x)$ for every $x$ in $G$ (**4**, Corollary 16.6.1). But the matrices $\rho(x)$ for $x$ in $G$ are among the choices for $\alpha$ and therefore belong to the centre of $D$. Hence $\rho(G)$ is a finite subgroup of the multiplicative group of a field and therefore cyclic.

LEMMA 3.3. *Let $G$ be an irreducible subgroup of $GL(2, p)$ with $|G|$ odd. Then $G$ is cyclic and $|G|$ divides $p^2 - 1$.*

*Proof.* By Lemma 3.1, $G$ is abelian and by Lemma 3.2, $G$ is cyclic.

Let $A$, $\rho$, and $D$ be the same as in the preceding lemma for the group $G = \langle g \rangle$ and the field $F = J_p$. If $a \neq 0$ with $a$ in $A$, then $A$ is spanned by the vectors $a_i = \rho(g^i)$, $i = 0, 1, \ldots$, since $\rho$ is irreducible. Since $A$ is a cyclic $\rho(g)$ module, any linear transformation on $A$ which commutes with $\rho(g)$ is in the algebra spanned by $\rho(g^i)$, $i = 0, 1, \ldots$. But $D$ is just the set of linear transformations on $A$ which commute with $\rho(g)$. Thus $D$ is the field spanned by the $\rho(g^i)$, $i = 0, 1, \ldots$, from which it follows that $(D{:}F)$ equals the dimension $n$ of $A$ over $J_p$. Every non-zero element $X$ of the field $D$ satisfies $X^{p^n - 1} = 1$.

For the lemma, $n = 2$ and $g = \rho(g)$, giving $g^{p^2-1} = 1$.

LEMMA 3.4. *Let $G$ be a cyclic group whose order is a divisor of $p^2 - 1$. Then every irreducible representation of $G$ over $J_p$ has degree one or two.*

*Proof.* The second paragraph of the above proof applies. Note that $D$ can be regarded as $F$ with $\rho(g)$ adjoined. Thus $X^{p^2-1} = 1$ for all $X$ in $D$. Hence the degree of $D$ over $F$ is 1 or 2 so that $A$ has dimension 1 or 2 over $J_p$, proving the lemma.

LEMMA 3.5. *Let $G$ be an abelian group of exponent dividing $p^2 - 1$. Then every irreducible representation of $G$ over $J_p$ has degree one or two.*

*Proof.* Let $\rho$ be an irreducible representation of $G$ over $J_p$. By Lemma 3.2, $\rho(G)$ is cyclic and by hypothesis $\rho(G)$ has order dividing $p^2 - 1$. By Lemma 3.4, $\rho(G)$ has degree one or two.

The author is grateful to the referee for pointing out the following theorem.

THEOREM 3.2. *If $G$ is a finite solvable group and $p$ any prime, let $S_p(G)$ denote the largest integer such that $G$ has a maximal subgroup of index $p^s$. Then*
(1) *$S_p(G) = 1$ implies $r_p(G) = 1$;*
(2) *$S_p(G) = 2$ and $|G|$ odd imply $r_p(G) = 2$.*

*Proof.* We prove (2) by induction on $|G|$. Let $K = G^{p^2-1}G'$, so that $G/K$ is the largest abelian quotient group of $G$ having exponent $p^2 - 1$. We may assume that $K \neq 1$. Let $M$ be a minimal normal subgroup of $G$ contained in $K$. Then $S_p(G/M) \leqslant 2$ and $|G: M|$ is odd, so that $r_p(G/M) \leqslant 2$ by induction. This gives $r_p(G) \leqslant 2$ unless $|M| = p^s$ with $s > 2$.

Assume $s > 2$. Let $C$ be the centralizer of $M$ in $G$. If $C > K$, then by Lemma 3.5 we have $s \leqslant 2$, so that $D = K \cap C < K$; $D \neq K$ There is a chief factor $E/D$ of $G$ with $E < K$ and $E/D \cong CE/C$, which is a minimal normal subgroup of $G/C$. Now $G/C$ is isomorphic to an irreducible subgroup of $GL(s, p)$ and therefore cannot have a normal $p$-subgroup so that $CE/C$ and hence $E/D$ has order prime to $p$. Therefore, if $Q$ is a Sylow $p$-complement of $K$, then $E < DQ$.

Consider the representation $\rho$ of $G$ on one of its chief $p$-factors in $K/M$. Since $r_p(G/M) \leqslant 2$, it follows from Lemma 3.3 that $G/\ker p$ is cyclic and has order dividing $p^2 - 1$. Thus $\ker \rho > K$ so that $K$ centralizes all chief $p$-factors of $K/M$. Hence $K/M$ has a normal $p$-complement $MQ/M$. Here $MQ/M$ is a characteristic subgroup of $K/M$ so that $MQ$ is a proper normal subgroup of $G$.

Let $N$ be the normalizer of $Q$ in $G$. Then $MN = G$, since $Q^x$ is conjugate to $Q$ in $MQ$ for all $x$ in $G$. Since $M$ is abelian and normal in $G$, $M \cap N$ is a proper normal subgroup of $MN = G$. If $N > M$, then $(M, Q) = 1$ so that $Q < C$. But $Q < C$ implies $DQ < C$ and thus $E < C, E \neq C$, a conflict. Hence $N \not> M$. Owing to the minimality of $M$, the only alternative is that $M \cap N = 1$. Thus $N$ is a maximal subgroup of $G$ with $|G:N| = p^s$, which contradicts $S_p(G) = 2$. This proves (2). The proof of (1) is similar but simpler.

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THEOREM 3.3. *Let G be a group of odd order which has property M. Then $r_p(G) \leqslant 2$ for all primes p which divide $|G|$.*

*Proof.* By Theorem 3.1, $G$ is solvable. The theorem follows at once from Theorem 3.2.

THEOREM 3.4. *Let G be a group of odd order which has property M. Then all subgroups of G also have property M.*

*Proof.* This is an immediate consequence of Theorem 3.3 together with (**7**, Satz 1).

**4. Construction of examples.** In this section examples of groups of order $2^a p^b$, $p$ an odd prime, will be constructed which have property $M$ but contain subgroups which do not have this property. In fact given any positive integer $m$ such a group will be constructed which has a subgroup containing a maximal subgroup of index at least $m$.

LEMMA 4.1. *Let G be a group and let $G_{t+1}$ be the* $(t + 1)$st *term in the descending central series of G. Then*

$$(a_1, \ldots, xy, \ldots, a_t) \equiv (a_1, \ldots, x, \ldots, a_t)(a_1, \ldots, y, \ldots, a_t) \pmod{G_{t+1}}.$$

*Proof.* (**6**, Theorem 2.84).

Let $p$ be an odd prime. Let $D$ be the subgroup of $GL(2, p)$ which is generated by

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$D$ is absolutely irreducible and $|D| = 8$. Let $K = A_1 * \ldots * A_t$ be the free product of the groups $A_1, \ldots, A_t$, where each $A_i$ is elementary abelian and of order $p^2$. Suppose that $A_i = \langle a_i, b_i \rangle$. Define

$$\begin{aligned} a_i^{x_i} &= a_i, & a_i^{y_i} &= b_i^{-1}, \\ b_i^{x_i} &= b_i^{-1}, & b_i^{y_i} &= a_i, \end{aligned}$$

for $i = 1, \ldots, t$. Then $H_i = \langle x_i, y_i \rangle$ is a group of automorphisms of $A_i$ which is isomorphic to $D$. We extend $H_i$ to a group of automorphisms of $K$ by assuming that each element of $H_i$ induces the identity on $A_j$ for $j \neq i$. The group $\langle H_1, \ldots, H_t \rangle = H$ is a group of automorphisms of $K$ and

$$H = H_1 \times \ldots \times H_t.$$

Let $K_{t+1}$ be the $(t + 1)$st term of the descending central series of $K$. Since $K_{t+1}$ is characteristic in $K$, it follows that $H$ induces automorphisms on $K/K_{t+1}$. Also the group $L = K/K_{t+1}$ is a finite $p$-group since it is nilpotent and generated by finitely many elements of order $p$.

A commutator of the form $(\alpha_1, \ldots, \alpha_t)$, where $\alpha_i = a_i$ or $\alpha_i = b_i$ is said to be of type $A$. Let $G$ denote the semi-direct product of $L$ by $H$. Let $W$ denote the subgroup of $G$ which is generated by the commutators of type $A$.

LEMMA 4.2. *W is a normal, elementary abelian p-subgroup of G which is in the centre of L.*

*Proof.* By (**4**, Corollary 10.2.1), $W$ is contained in the centre of $L$, in particular $W$ is abelian. Also a commutator of type $A$ has order $p$ (mod $K_{t+1}$) or else is the identity (mod $K_{t+1}$). This follows from

$$1 \equiv (\alpha_1^p, \alpha_2, \ldots, \alpha_t) \pmod{K_{t+1}}.$$

since $\alpha_i^p = 1$ in $K$. But

$$(\alpha_1^p, \alpha_2, \ldots, \alpha_t) \equiv (\alpha_1, \alpha_2, \ldots, \alpha_t)^p \pmod{K_{t+1}}$$

by Lemma 4.1. Each element in $G$ has a unique representation in the form $kh$ where $k \in L$ and $h \in H$. Then for $(\alpha_1, \ldots, \alpha_t) K_{t+1}$ in $W$ we have

$$[(\alpha_1, \ldots, \alpha_t)K_{t+1}]^{kh} = (\alpha_1, \ldots, \alpha_t)^{kh} K_{t+1}.$$

Now $h$ can be written in the form $h = h_1h_2 \ldots h_t$, where $h_i \in H_i$ for $i = 1, \ldots, t$. Since $h_i$ and $h_j$ commute for $i \neq j$ and $h_i$ centralizes $A_j$ for $j \neq i$, we conclude that

$$(\alpha_1, \ldots, \alpha_t)^h K_{t+1} = (\alpha_1^{h_1}, \ldots, \alpha_t^{h_t})K_{t+1}.$$

But $\alpha_i^{h_i}$ is either $a_i^\epsilon$ or $b_i^\delta$, where $\epsilon = \pm 1$ and $\delta = \pm 1$. So by Lemma 4.1, a commutator of type $A$ (mod $K_{t+1}$) under an element of $G$ is again a commutator of type $A$ or else the inverse of such a commutator. Hence $W$ is a proper normal subgroup of $G$.

LEMMA 4.3. *The commutators of type A* (mod $K_{t+1}$) *are a basis for W.*

*Proof.* We must prove the independence of the commutators of type $A$ (mod $K_{t+1}$). Suppose that

$$\Pi(\alpha_1, \ldots, \alpha_t)^{\theta(\alpha_1,\ldots,\alpha_t)} \equiv 1 \pmod{K_{t+1}}, \tag{1}$$

where the product extends over the commutators of type $A$ and at least one $\theta(\alpha_1, \ldots, \alpha_t)$ is not congruent to zero (mod $p$). Without loss of generality we may assume that $\theta(a_1, \ldots, a_t) \not\equiv 0 \pmod{p}$. Then

$$(a_1, \ldots, a_t)^{\theta(a_1,\ldots,a_t)} \equiv 1 \pmod{\bar{K}_{t+1}},$$

where $\bar{K} = \langle a_1\rangle * \ldots * \langle a_t\rangle$. Here we are using the fact that there is a homomorphism of $K$ onto $\bar{K}$ which maps each $b_i$ into 1 and each $a_i$ into itself. Raising both sides of this relation to a suitable power yields

$$(a_1, \ldots, a_t) \equiv 1 \pmod{\bar{K}_{t+1}}. \tag{2}$$

By (**4**, Theorem 12.1.1), any group $T = \langle c_1, \ldots, c_t\rangle$ with $c_i^p = 1$ for $i = 1, \ldots, t$ is a homomorphic image of $\bar{K}$ under the correspondence $a_i \to c_i$. If $T_{t+1}$ is the identity, then the kernel of this homomorphism contains $\bar{K}_{t+1}$. Hence the mapping $a_i\bar{K}_{t+1} \to c_i$ is a homomorphism from $\bar{K}/\bar{K}_{t+1}$ onto $T$.

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We shall construct a group $T = \langle c_1, \ldots, c_t \rangle$ with $c_i^p = 1$, $T_{t+1} = 1$, and $(c_1, \ldots, c_t) \neq 1$. This will contradict relation (2) and prove that (1) cannot hold. The elements of $T$ are square matrices of size $(t + 1)$ with entries from $J_p$. Let $I$ be the identity matrix and $E_{ij}$ be the matrix with a 1 in position $(i, j)$ and zeros elsewhere. Let $c_i = I + E_{i+1,i}$ for $i = 1, \ldots, t$. The inverse of $c_i$ is $I - E_{i+1,i}$, and $(c_1, \ldots, c_t) = I + \epsilon E_{t+1,1}$, where $\epsilon = \pm 1$ depending upon whether $t$ is even or odd. In either case $(c_1, \ldots, c_t) \neq 1$.

As a consequence of Lemma 4.2, $W$ can be regarded as a vector space on which $G$ operates. By Lemma 4.3, a basis for this vector space is given by the commutators of type $A$. Using this basis it is easy to see that the representation which $G$ induces on $W$ is the Kronecker product of $t$ groups isomorphic with $D$ and is therefore absolutely irreducible. Hence $W$ is a chief factor of $G$.

THEOREM 4.1. *$G$ has property $M$, but its subgroup $HW$ contains $H$ as a maximal subgroup and* $|HW:H| = p^{2t}$.

*Proof.* If $M$ is a maximal subgroup of $G$, then $M$ must contain the commutator subgroup $L'$ of $L = K/K_{t+1}$. For $L'$ is a proper normal subgroup of $G$, and if $L' \not< M$, then $L'M = G$, $L'(M \cap L) = L$ and therefore $M \cap L = L$ by (**4**, Corollary 10.3.3), which is a contradiction. The maximal indices of $G$ are the same those of $G/L'$. But the chief factors of $G/L'$ have orders 2 or $p^2$ so that $S_p(G) = 2$, $S_2(G) = 1$. Thus $G$ has property $M$.

Consider the subgroup $HW$ of $G$. $W$ is operated on absolutely irreducibly by $H$ and therefore $H$ is a maximal subgroup of $HW$. Note that $|HW:H| = |W| = p^{2t}$.

## 5. Two theorems on groups with property $M$.

By a theorem of Gaschutz (**3**, Satz 16) $\Delta(G)$ is a nilpotent subgroup of $G$. Groups with property $M$ can be characterized by the factor group $G/\Delta(G)$.

LEMMA. *If $G$ is a subdirect product of primitive solvable groups on a prime or prime square number of letters, then $r_p(G) \leqslant 2$ for every prime $p$ which divides* $|G|$.

*Proof.* It is sufficient to prove the lemma for primitive solvable groups on a prime or prime square number of letters. A primitive solvable group $T$ contains a unique minimal normal subgroup $B$ and $T/B$ is isomorphic to a group of automorphisms of $B$. Also $|B|$ equals the degree of $T$. If $|B| = q$, a prime, then $T/B$ is cyclic and $r_p(T) = 1$ for all primes $p$ which divide $|T|$. If $|B| = q^2$, then examination of the solvable subgroups of $GL(2, q)$ shows that $r_p(T/B) \leqslant 2$ for all primes $p$ which divide $|T/B|$.

THEOREM 5.1. *A finite group $G$ has property $M$ if and only if $G/\Delta(G)$ is isomorphic to a subdirect product of primitive solvable groups on a prime or prime square number of letters.*

*Proof.* Assume that $G/\Delta(G)$ has the above property. The lemma and **(3**, Satz 16**)** imply that $r_p(G) \leqslant 2$ for all primes $p$ which divide $|G|$. By **(7**, Satz 1**)**, $G$ has property $M$.

Assume that $G$ has property $M$. Decompose the maximal subgroups of $G$ into conjugate classes and represent $G$ by conjugation on these classes. This gives a permutation representation $\pi$, where the sets of transitivity are just the sets of conjugate maximal subgroups. Consider the restriction of $\pi$ to one of these sets. If $M$ is an element of this set, the restricted representation $\pi^*$ is equivalent to that arising from the cosets of $N(M)$, the normalizer of $M$ in in $G$; cf. **(2**, p. 242**)**.

The group $N(M)$ is equal to $G$ or $M$. If $G$, then $\pi^*$ is the identity; and if $M$, then $\pi^*$ is primitive of degree $|G{:}M|$, which is a prime or the square of a prime. Hence $G$ modulo the kernel of $\pi$ is a subdirect product of primitive solvable groups on a prime or prime square number of letters.

We determine the kernel of $\pi$. An element $x$ of $G$ will be in this kernel if and only if it normalizes every maximal subgroup of $G$. A non-normal maximal subgroup of $G$ is its own normalizer so that the kernel of $\pi$ is $\Delta(G)$.

THEOREM 5.2. *Let $G$ be a group whose order is not divisible by* 6. *If every maximal subgroup of $G$ has property $M$, then $G$ is solvable.*

*Proof.* Use induction on $|G|$. The hypothesis is satisfied by all factor groups of $G$. Let $M$ be any maximal subgroup of $G$. By Theorem 3.1, $M$ has an ordered Sylow tower. Let $p$ be the smallest prime which divides $|G|$ and let $P$ be a Sylow $p$-subgroup of $G$. Let $N$ be the normalizer of $P$ in $G$. By induction every proper subgroup of $G$ has an ordered Sylow tower.

Assume $G$ is $p$-normal. If $N = G$, then $P$ is a solvable normal subgroup of $G$ and the proof is completed by induction. If $N$ is a proper subgroup of $G$, then it has an ordered Sylow tower, so that $N$ contains a normal $p$-complement. By Theorem **(4**, 14.4.6**)**, $G$ contains a normal subgroup with a $p$-factor group. This normal subgroup has an ordered Sylow tower and therefore $G$ is an extension of a solvable group by a $p$-group. Thus $G$ is solvable.

Assume that $G$ is not $p$-normal. By **(4**, Lemma 19.3.2**)**, $G$ satisfies the hypothesis of a theorem of Burnside **(4**, Theorem 4.2.5**)**. Hence $G$ contains a $p$-subgroup $H = h_1 h_2 \ldots h_r$, where each $h_i$ is a proper normal subgroup of $H$. The groups $h_1, h_2, \ldots, h_r$ form a complete set of conjugates in $N(H)$, the normalizer of $H$ in $G$, and $r > 1$ is prime to $p$. If $H$ is not normal in $G$, then $N(H)$ is a proper subgroup of $G$ and has an ordered Sylow tower. Hence $N(H)$ contains a normal $p$-complement $K$. Thus $(K, H) = 1$ so that $N(H) = K \times H$. Thus $N(H)/C(H)$ is a $p$-group, where $C(H)$ denotes the centralizer of $H$ in $G$. But $h_1, \ldots, h_r$ form a complete set of conjugates in $N(H)$ so that $N(H)/C(H)$ has order divisible by $r$. This contradiction proves that it is a proper normal subgroup of $G$.

This theorem is not true if 6 divides $|G|$. The linear fractional groups LF$(2, p)$

are simple for $p > 3$. From the discussion given in (**2**, Chapter 20), it follows that the maximal subgroups of LF(2, $p$) all have property $M$, if $p$ is not congruent to $\pm 1$ (mod 5).

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*California Institute of Technology*
*Pasadena, California*

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