SOME INFINITE FIBONACCI GROUPS

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The Fibonacci groups are a special case of the following class of groups first studied by G. A. Miller (7). Given a natural number n, let θ be the automorphism of the free group $F = \langle x_1, \ldots, x_n \mid \rangle$ of rank n which permutes the subscripts of the generators in accordance with the cycle $(1, 2, \ldots, n)$. Given a word w in F, let R be the smallest normal subgroup of F which contains w and is closed under θ. Then define $G_n(w) = F/R$ and write $A_n(w)$ for the derived factor group of $G_n(w)$. Putting, for $r \geqq 2$, $k \geqq 1$,

$$w = x_1 \ldots x_r x_{r+k}^{-1},$$

with subscripts reduced modulo n, we obtain the groups $F(r, n, k)$ studied in (1) and (2), while the $F(r, n, 1)$ are the ordinary Fibonacci groups $F(r, n)$ of (3), (5) and (6). To conform with earlier notation, we write $A(r, n, k)$ and $A(r, n)$ for the derived factor groups of $F(r, n, k)$, and $F(r, n)$ respectively.

Our purpose in the present note is threefold: firstly to explain the connection between the two apparently different formulae for $| A(r, n) |$ given in (5) (numbered (3) and (6)), secondly to derive necessary and sufficient conditions for $A_n(w)$ to be infinite, and finally to apply this to the groups $A(r, n, k)$ by way of example. The second item is the analogue of a result of Dunwoody (4), while the third extends Theorems 6 and 7 of (1) and at the same time complements Theorem 1 of (2).

The symbol $| . |$ will stand indiscriminately for the order of an element of a group, the order of a group or the absolute value of a complex number, it being clear from the context which is intended. $\mathbf{Z}[x]$ denotes the ring of polynomials over the integers $\mathbf{Z}$, while $\mathbf{Z}\langle x \mid x^n \rangle$ is the integral group ring of a cyclic group of order n, so that

$$Z\langle x \mid x^n \rangle \cong Z[x]/(x^n - 1),$$

as rings. Finally, we observe the usual conventions with regard to empty sums (for example, formula (1) below when $s = 0$) and empty products (for example, the definition of $f * g$ when f is a constant polynomial).

The n-generator, n-relation presentation given above for $F(r, n)$ shows that a relation matrix for $A(r, n)$ is the circulant matrix C whose first row is

$$\underbrace{k+1, \ldots, k+1}_{s},\ k-1,\ \underbrace{k, \ldots, k,}_{n-s-1}$$

where,

$$r = kn + s, \quad 0 \leqq s < n.$$

Thus we obtain formula (6) of (5), viz.

$$|A(r,n)| = \pm\det C = \pm \prod_{i=1}^{n}\left(\sum_{j=1}^{s}(k+1)\omega_i^{j-1}+(k-1)\omega_i^{s}+\sum_{j=s+2}^{n}k\omega_i^{j-1}\right), \quad (1)$$

where $\omega_1, \ldots, \omega_n$ are distinct nth roots of unity. We can express this more conveniently as follows: let b_{i-1} be the exponent-sum of x_i in $w = x_1 \ldots x_r x_{r+1}^{-1}$ and put

$$f(x) = b_0 + b_1 x + \ldots + b_{n-1}x^{n-1}.$$

Writing

$$g(x) = x^n - 1,$$

formula (1) reduces to

$$|A(r,n)| = \prod_{g(\xi)=0} |f(\xi)|. \quad (2)$$

The presentation for $F(r, n)$ on r generators $(x_1, \ldots, x_r)$ obtained using Tietze transformations in the obvious way, leads to the relation matrix

$$M^n - I$$

for $A(r, n)$, where M is the companion matrix of the polynomial

$$f'(x) = x^r - x^{r-1} - \ldots - x - 1.$$

Thus we have another expression for $|A(r, n)|$:

$$|A(r,n)| = \prod_{i=1}^{n} |\xi_i^n - 1|,$$

where $\xi_1, \ldots, \xi_r$ are the zeros of $f'(x)$, in other words

$$|A(r,n)| = \prod_{f'(\xi)=0} |g(\xi)|. \quad (3)$$

Now we have

$$-f'(x) = (1+x^n+\ldots+x^{(k-1)n})(1+x+\ldots+x^{n-1})+x^{kn}(1+x+\ldots+x^{s-1}-x^s),$$

so that

$$f+f' \equiv 0 \pmod{(g)}. \quad (4)$$

Definition. Let $f, g \in \mathbf{Z}[x]$ and define $f*g \in \mathbf{Z}$ to be the product of the values of g on the complex zeros of f.

The following easily-proved theorem now shows that (2) and (3) are in fact the same formula.

Theorem 1. *For polynomials $f, g, f' \in \mathbf{Z}[x]$, we have*

(i) $f*g = \pm g*f$, *if f and g are monic,*
(ii) $g*f = g*f'$, *if $f \equiv f' \pmod{g}$,*
(iii) $g*ff' = (g*f)(g*f')$.

Passing now to the $G_n(w)$, let b_{i-1} be the exponent sum of x_i in w and define

$$f(x) = b_0 + b_1 x + \ldots + b_{n-1}x^{n-1},$$

as above, a member of $\mathbf{Z}[x]$. Call $f(x)$ the polynomial associated with w.

Dunwoody's Theorem (**4**) asserts that $G_n(w)$ is perfect if and only if the polynomial associated with w is a unit in $\mathbf{Z}\langle x \mid x^n \rangle$. Our next result is an analogue of this.

Theorem 2. *Let $f(x)$ be the polynomial associated with the word w in the free group of rank n, and let $g(x) = x^n - 1$. The following three assertions are then equivalent:*

(*a*) *$G_n(w)$ has an infinite abelian factor group,*
(*b*) *$g * f = 0$,*
(*c*) *$f(x)$ is a zero-divisor in $\mathbf{Z}\langle x \mid x^n \rangle$.*

Proof. We first prove the equivalence of (*a*) and (*b*). (*a*) is equivalent to the assertion that $A_n(w)$ is infinite. Now a relation matrix for $A_n(w)$ is the circulant matrix C with first row $(b_0, b_1, \ldots, b_{n-1})$, and so $A_n(w)$ is infinite if and only if $\det C = 0$. But $\det C = \pm g * f$, so the result follows. To show that (*b*) and (*c*) are equivalent, first note that $g * f = 0$ if and only if g and f have a common zero, that is $f(\xi) = 0$ for some nth root ξ of unity. Let $\phi_k(x)$ be the cyclotomic polynomial of order k; (*b*) is equivalent to the assertion: $\phi_k(x)$ is a divisor of $f(x)$ for some k dividing n. Since $\phi_k(x)\phi_k'(x) = x^n - 1$ for some $\phi_k'(x) \in \mathbf{Z}[x]$, this condition is clearly equivalent to (*c*).

Thus we see that $A_n(w)$ is infinite if and only if the polynomial associated with w vanishes on some nth root of unity. We apply this to the classification of the infinite members of the set $A(r, n, k)$, thereby generalising Theorems 6 and 7 of (**1**), as well as Corollary 2 of (**5**). Theorem 1 of (**2**) asserts that $F(r, n, k)$ is metacyclic of order $r^n - 1$ provided that

$$r \equiv 1 \pmod{n} \quad \text{and} \quad (n, k) = 1.$$

Our theorem shows that $F(r, n, k)$ is infinite if

$$r \equiv 1 \pmod{n} \quad \text{and} \quad (n, k) \neq 1.$$

Theorem 3. *$A(r, n, k)$ is infinite if and only if at least one of the following two conditions holds:*

(i) $(r-1, k, n) \neq 1$,

(ii) *$v_2(r+1)$ and $v_2(n)$ are each greater than $v_2(k-1)$,*

where v_2 denotes the 2-part of a positive integer and $v_2(0) = \infty$.

Proof. By Theorem 2, $A(r, n, k)$ is infinite if and only if the polynomial

$$1 + x + \ldots + x^{r-1} - x^{r+k-1}$$

vanishes on an nth root of unity. Since $r \geqq 2$, this is equivalent to the vanishing of

$$(x^{r+k} - x^{r+k-1}) - (x^r - 1) \tag{5}$$

on some non-trivial nth root of unity. We first assume that some nth root ξ of unity is a root of (5), so that

$$\xi^{r+k} - \xi^{r+k-1} = \xi^r - 1. \tag{6}$$

It follows that

$$|\xi-1|=|\xi^r-1|,$$

so that either

$$\xi^r=\xi \quad \text{or} \quad \xi^r=\xi^{-1}.$$

In the first case, substitution in (6) yields

$$\xi^k(\xi-1)=(\xi-1),$$

so that

$$\xi\neq 1,\ \xi^k=1,\ \xi^{r-1}=1,\ \xi^n=1,$$

from which (i) follows. In the second case, substitution in (6) yields

$$\xi^{k-1}-\xi^{k-2}=\xi^{-1}-1,$$

and multiplication of this by $\xi/(\xi-1)$ gives

$$\xi^{k-1}=-1.$$

The conditions

$$\xi\neq 1,\ \xi^{k-1}=-1,\ \xi^{r+1}=1,\ \xi^n=1$$

now give condition (ii).

For the converse, let $h \neq 1$ be a common factor of $r-1$, k and n. If ξ is a primitive hth root of unity, then $\xi \neq 1$, $\xi^n = 1$ and (5) vanishes on ξ. If, on the other hand, (ii) holds, let $h = 2.v_2(k-1)$, so that h is a divisor both of $r+1$ and n, and $h \neq 1$. Again let ξ be a primitive hth root of unity, so that again $\xi \neq 1$, $\xi^n = 1$. Since $\xi^{2(k-1)} = 1$ and $\xi^{k-1} \neq 1$, we must have $\xi^{k-1} = -1$ and ξ is again a zero of (5), which completes the proof.

REFERENCES

(1) C. M. Campbell and E. F. Robertson, A note on a class of generalized Fibonacci groups (to appear).

(2) C. M. Campbell and E. F. Robertson, The orders of certain metacyclic groups, *Bull. London Math. Soc.* (to appear).

(3) J. H. Conway, Solution to Advanced Problem 5327 *Amer. Math. Monthly*, **74** (1967), 91-93.

(4) M. J. Dunwoody, Cyclic presentations and 3-manifolds (to appear).

(5) D. L. Johnson, A note on the Fibonacci groups, *Israel J. Math.* (to appear).

(6) D. L. Johnson, J. W. Wamsley and D. Wright, The Fibonacci groups, *Proc. London Math. Soc.* (to appear).

(7) G. A. Miller, Groups generated by *n* operators each of which is the product of the n − 1 remaining ones, *Amer. J. Math.* **30** (1908), 93-98.

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