SECOND NILPOTENT *BFC* GROUPS

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

The *BFC* number of a group G is defined to be the least upper bound n of the cardinals of the conjugacy classes of G, provided this is finite, and we then say that G is n-*BFC*. It was shown by B. H. Neumann [2] that the derived group G' of such a group is finite, and J. Wiegold [5] proved that

$$|G'| \leqq n^{\frac{1}{2}n^4(\log_2 n)^3}.$$

This bound was sharpened by I. D. Macdonald [1] to

$$|G'| \leqq n^{6n(\log_2 n)^3},$$

and P. M. Neumann has recently communicated the (unpublished) result that $|G'| \leqq n^{q(n)}$ with $q(n)$ a quadratic in $\log_2 n$, an immense improvement on the above. J. A. H. Shepperd and J. Wiegold [4] improved the bound in two special cases, showing that if G is soluble, $|G'| \leqq n^{p(n)}$ with $p(n)$ a quintic in $\log_2 n$, and that if G is nilpotent of class 2,

$$|G'| \leqq n^{(\log_2 n)^2}.$$

It is conjectured that for any n-*BFC* group G,

$$|G'| \leqq n^{\frac{1}{2}(1+\log_2 n)},$$

Wiegold [5] having shown that this bound is attained by certain nilpotent groups of class 2.

The aim of this paper is to prove this conjecture in the case of nilpotent groups of class 2.

We may make two simplifications. Macdonald [1] showed that if G is any n-*BFC* group then there exists a finite n-*BFC* group G_0 with $G_0' \cong G'$, and moreover that if G is nil-c then G_0 may be chosen to be nil-c. Secondly, since we are concerned only with nilpotent groups, which are the direct product of their Sylow subgroups, we can restrict our considerations to p-groups. Thus we shall assume that all groups are finite p-groups.

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The method of proof is by contradiction, using a minimal counter-example. However, attempts to prove the conjecture as it stands were fruitless and we prove in fact a slightly stronger result. Much of the proof involves long and tedious commutator manipulation which has been omitted. We split the work into four cases, each pertaining to a different hypothesis.

This work formed a part of my Ph. D. thesis submitted to the Victoria University of Manchester, and I should like to express my indebtedness to Dr J. A. H. Shepperd for all his assistance and encouragement.

2. Notation

Groups are written multiplicatively and the notation of Scott [3] is followed with the following additions.

$\prod_{i=1}^{k} \times A_i$ direct product $A_1 \times \cdots \times A_k$.

$\mathscr{C}_n^k$ direct product of k copies of the cyclic group $\mathscr{C}_n$.

$\beta_H(a)$ number of distinct conjugates a^h, $h \in H$.

$\beta(G)$ $\max_{a \in G} \beta_G(a)$, i.e. the BFC number of G.

$\Gamma_H(a)$ $\langle [a, h] \mid h \in H \rangle$.

Let G be a nil-2 group. Then we have the following commutator identities:

$$[ab, c] = [a, c][b, c], \quad [a, bc] = [a, b][a, c],$$
$$[a, b]^n = [a^n, b] = [a, b^n]$$

for all $a, b, c \in G$, $n \in \boldsymbol{Z}$. As a consequence of these, for a nil-2 group G,

$$|\Gamma_H(a)| = \beta_H(a)$$

for any $a \in G$, $H \subseteqq G$.

3. Results

THEOREM A. *Let G be a finite p^n-BFC nil-2 p-group. Then*

(1) $|G'| \leqq p^{\frac{1}{2}n(n+1)}$,

(2) *if $|G'| = p^{\frac{1}{2}n(n+1)}$ then there exists an $(n+1)$-generator subgroup H of G such that*

$$G' = H' \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)}.$$

PROOF. Suppose that the theorem is false and let G be a counter-example of minimal order. Then either

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(1) $|G'| > p^{\frac{1}{2}n(n+1)}$,

or (2) $|G'| = p^{\frac{1}{2}n(n+1)}$, but there is no $(n+1)$-generator subgroup H of G with $G' = H' \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)}$.

Using elementary counting methods on the conjugates of suitable elements, we may prove that the theorem holds for $n = 1$ and $n = 2$.[1]

As will be seen, this is necessary since the method employed in general requires $n \geqq 3$, but such a counting method is too unwieldy for larger n. Thus we shall assume

(1) $$n \geqq 3.$$

We divide the proof into several cases, considering each separately.

CASE I. $|G'| > p^{\frac{1}{2}n(n+1)}$.

Suppose that $|G'| > p^{\frac{1}{2}n(n+1)}$. Now G is nil-2, so it is possible to choose elements $a, b \in G$ such that $[a, b] = z$ say, with $z^p = 1$. Now $\langle z \rangle \lhd G$ so we may define a homomorphism $\sigma : G \to G/\langle z \rangle = \Sigma$, say. Then

$$|\Sigma'| = \frac{|G'|}{|G' \cap \langle z \rangle|} \geqq p^{\frac{1}{2}n(n+1)}.$$

But $\beta(\Sigma) \leqq p^n$ so by the minimality of G, $|\Sigma'| \leqq p^{\frac{1}{2}n(n+1)}$. Hence

$$|\Sigma'| = p^{\frac{1}{2}n(n+1)}.$$

Again by the minimality of G, $\beta(\Sigma) = p^n$, and so there exists $\Delta \subseteqq \Sigma$ with $\Delta = \langle \gamma_0, \gamma_1, \cdots, \gamma_n \rangle$ such that

$$\Delta' = \Sigma' \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)}.$$

Hence, since Δ is nil-2, $\beta_\Sigma(\gamma_i) = p^n$ for $0 \leqq i \leqq n$.

Let $g_i \in G$ with $g_i\sigma = \gamma_i$ and let $H = \langle g_0, g_1, \cdots, g_n \rangle$. Then it follows that

(2) $$\beta_H(g_i) = p^n, \qquad 0 \leqq i \leqq n,$$

and that

$$\Gamma_H(g_i) \cong \mathscr{C}_p^n, \qquad H' \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)}.$$

Since $\beta_G(g_i) \leqq p^n$, we must therefore have for all $g \in G$

(3) $$[g_i, g] \in \Gamma_H(g_i), \qquad 0 \leqq i \leqq n.$$

At this stage, it can be shown by using a lengthy but straightforward induction argument on i that we may specify the elements a, b and g_i in a more precise manner. In fact we can prove the following.

[1] That $G' \cong \mathscr{C}_p$ for any p-BFC group G was proved by Wiegold [5].

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Lemma. *There exist* $a, b, g_0, \cdots, g_n \in G$ *such that if* $H = \langle g_0, \cdots, g_n \rangle$, *then* $G = \langle H, a, b \rangle$ *and*

$$G' = H' \times \langle [a, b] \rangle \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)+1},$$

with, moreover,

$$[a, g_0] = [b, g_0] = 1.$$

We now begin again and factor out a different p-cycle, so obtaining another system of generators for G. Let $x = [g_0, g_1] \in H'$. Then using x in place of z, we can prove in a method analogous to the above the following.

Lemma. *There exist* $u_0, u_1, \cdots, u_n \in G$ *such that if* $U = \langle u_0, \cdots, u_n \rangle$, *then* $G = \langle U, g_0, g_1 \rangle$ *and*

$$G' = U' \times \langle [g_0, g_1] \rangle \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)+1}.$$

It is now relatively easy to show that for some $u, v \in U$,

$$U = \langle g_2, \cdots, g_n, u, v \rangle.$$

Certainly $[u, v] \notin H'$, but this apart, we have reasonable freedom of choice for u and v, and from (3), for $0 \leqq i \leqq n$, there exist $h_i \in H$ such that

$$[u, g_i] = [h_i, g_i].$$

It can be shown that we may choose the u, v and g_i in such a way that all the above properties hold and, in addition, either

$$[u, g_2] = [g_1, g_2] \text{ and } [u, g_0] = 1, \quad \text{or} \tag{4}$$

$$[u, g_2] = [g_0, g_2] \text{ and } [u, g_1] = 1. \tag{5}$$

The two cases arise according to the form of the h_i, but since the only difference is that the roles of g_0 and g_1 are interchanged, we may without loss of generality assume (4) holds.

Consider now $\Gamma_G(ug_2)$. Since $ug_2 \in U$, $\beta_U(ug_2) = p^n$, and so $\Gamma_G(ug_2) = \Gamma_U(ug_2)$. Hence in particular there exist λ and α_i, $2 \leqq i \leqq n$, such that

$$[ug_2, g_0] = [ug_2, g_2^{\alpha_2} \cdots g_n^{\alpha_n} v^{\lambda}].$$

Expanding and manipulating these commutators, it ultimately follows from (4) and from the fact that $[u, v] \notin H'$ that

$$[g_2, g_0] = [g_2, g_1^{-\alpha_2} g_3^{\alpha_3} \cdots g_n^{\alpha_n}].$$

But since H is $(n+1)$-generator and since $\Gamma_H(g_2)$ is elementary abelian, this implies that $\beta_H(g_2) \leqq p^{n-1}$, which is contrary to (2).

Hence it is impossible that $|G'| > p^{\frac{1}{2}n(n+1)}$, and so the minimal counter-example must falsify (2) of Theorem A.

Thus we may assume that the minimal counter-example G is such that

$$|G'| = p^{\frac{1}{2}n(n+1)},$$

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but that there is no $(n+1)$-generator subgroup H of G with

$$H' = G' \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)}.$$

Since $\beta(G) = p^n$, there exists $a \in G$ with $\beta_G(a) = p^n$. Let $\Gamma = \Gamma_G(a)$, so that $|\Gamma| = p^n$. Since $\Gamma \lhd G$, we may define a homomorphism

$$\lambda : G \to G/\Gamma = \Lambda$$

say. Then

$$|\Lambda'| = p^{\frac{1}{2}n(n-1)},$$

so that either $\beta(\Lambda) = p^{n-1}$ or $\beta(\Lambda) = p^n$. These two possibilities we consider separately.

CASE II. $\beta(\Lambda) = p^{n-1}$.

Suppose $\beta(\Lambda) = p^{n-1}$. Then by the minimality of G, there exists $\Delta = \langle \alpha_0, \cdots, \alpha_{n-1} \rangle \subseteqq \Lambda$ such that

$$\Delta' = \Lambda' \cong \mathscr{C}_p^{\frac{1}{2}n(n-1)}.$$

Let $a_i \in G$ such that $a_i\lambda = \alpha_i$ and let $M = \langle a_0, \cdots, a_{n-1} \rangle$. Then either $\beta(M) = p^n$ or $\beta(M) = p^{n-1}$, and we deal with each of these in turn.

CASE II (i). $\beta(M) = p^n$.

Suppose $\beta(M) = p^n$. Then without loss of generality we may assume

$$\beta_M(a_0) = p^n, \tag{6}$$

and by considering $\{\Gamma_M(aa_0)\}\lambda$, it can be shown that

$$\beta_M(aa_0) \geqq p^{n-1}. \tag{7}$$

Since M is n-generator, it follows from (6) that $\Gamma_M(a_0)$ is not elementary, so we may assume

$$|[a_0, a_1]| \geqq p^2. \tag{8}$$

Now for any j, $0 \leqq j \leqq n-1$, if $\beta_M(a_j) = p^n$, then $[a_j, a] \in M'$. On the other hand, if $\beta_M(a_j) = p^{n-1}$, then from (8) and the structure of Δ', $\beta_M(a_0 a_j) = p^n$ and so $[a_0 a_j, a] \in M'$. But by (6), $[a_0, a] \in M'$ and hence again $[a_j, a] \in M'$. Thus for all j, $0 \leqq j \leqq n-1$,

$$[a_j, a] \in M'.$$

Hence if we now define $N = \langle M, a \rangle$, we have

$$N' = M',$$

and we now establish a bound on the order of N'.

Suppose that $\beta_M(a_k) = p^n$ for some k, $0 \leqq k \leqq n-1$. Since M is n-generator and since Δ' is elementary,

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$$\Gamma_M(a_k) \cong \mathscr{C}_{p^2} \times \mathscr{C}_p^{n-2},$$

and so for some $\hat{k} \neq k$,

$$|[a_k, a_{\hat{k}}]| = p^2.$$

Hence

$$\Gamma_M(a_k) \cap \operatorname{Ker} \lambda = \Gamma_M(a_{\hat{k}}) \cap \operatorname{Ker} \lambda = \langle [a_k, a_{\hat{k}}]^p \rangle,$$

and so

$$(9) \qquad |\{\Gamma_M(a_k) \cap \operatorname{Ker} \lambda\}\{\Gamma_M(a_{\hat{k}}) \cap \operatorname{Ker} \lambda\}| = p.$$

Let $x \in M' \cap \operatorname{Ker} \lambda$. Then $x = x_0 \cdots x_{n-2}$ say, with

$$x_i \in \langle [a_i, a_j] \mid i < j \leqq n-1 \rangle,$$

and so if $\xi_i = x_i \lambda$, $\xi_0 \cdots \xi_{n-2} = 1$ and $\xi_i \in \langle [\alpha_i, \alpha_j] | \, i < j \leqq n-1 \rangle$. But the $[\alpha_i, \alpha_j]$ generate independent p-cycles and hence $\xi_i = 1$ for all i, $0 \leqq i \leqq n-2$. Thus $x_i \in \operatorname{Ker} \lambda$ and so

$$M' \cap \operatorname{Ker} \lambda = \prod_{i=0}^{n-1} \{\Gamma_M(a_i) \cap \operatorname{Ker} \lambda\}.$$

Hence from (9),

$$|M' \cap \operatorname{Ker} \lambda| \leqq p^{\frac{1}{2}n}.$$

Thus, since $\Delta' = M'/\{M' \cap \operatorname{Ker} \lambda\}$,

$$(10) \qquad |N'| = |M'| \leqq p^{\frac{1}{2}n(n-1)} p^{\frac{1}{2}n} = p^{\frac{1}{2}n^2}.$$

We prove finally that $\Gamma \subseteqq ZN'$ for some central subgroup Z of order p mod N'. Certainly $\Gamma \nsubseteq N'$, and hence there exists $b \in G$ such that, say,

$$[a, b] = z \notin N'.$$

Let $y = [a_0, b]$; then from (6), $\Gamma_G(a_0) = \Gamma_M(a_0)$ and hence $y \in N'$. From (7), $|\Gamma_G(aa_0) \colon \Gamma_M(aa_0)| \leqq p$, whence in particular, $[aa_0, b]^p \in N'$. Thus

$$(11) \qquad z^p \in y^{-p} N' = N'.$$

Let $R = \langle N, b \rangle$. Then from (7), $\beta_R(aa_0) = p^n$. Also since $[aa_0, b] = zy$ with $y \in N'$ and $z \notin N'$,

$$\Gamma_G(aa_0) \subseteqq \langle N', z \rangle.$$

Hence, since $\Gamma_G(a_0) \subseteqq N'$ by (6),

$$\Gamma \subseteqq \langle N', z \rangle.$$

Thus $\Gamma M' \subseteqq \langle N', z \rangle \subseteqq G'$, and

$$|\Gamma M'| = |\Gamma| \cdot |M'/\{\Gamma \cap M'\}| = |\Gamma| \cdot |\Delta'|$$
$$= p^{\frac{1}{2}n(n+1)} = |G'|.$$

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Hence $G' = \langle N', z\rangle$, and so by (10) and (11),

$$p^{\frac{1}{2}n(n+1)} \leqq p \cdot p^{\frac{1}{2}n^2}.$$

Thus $n \leqq 2$. But this is contrary to our assumption (1), and so Case II.i. is impossible.

CASE II (ii). $\beta(M) = p^{n-1}$.

Suppose then that $\beta(M) = p^{n-1}$. We show in this case that in fact G does satisfy Theorem A, by constructing an $(n+1)$-generator subgroup with the desired properties.

By the minimality of G, $|M'| = p^{\frac{1}{2}n(n-1)}$, and so there exists $N = \langle c_0, \cdots, c_{n-1}\rangle \subseteqq M$ such that

$$N' = M' \cong \mathscr{C}_p^{\frac{1}{2}n(n-1)}. \tag{12}$$

Clearly $\Gamma \cap N' = E$, and so

$$G' = N' \times \Gamma. \tag{13}$$

We next prove that Γ is elementary. It is evident from (12) that for all i, $0 \leqq i \leqq n-1$, $\beta_N(c_i) = p^{n-1}$, and since $\Gamma \cap N' = E$, we can show that for each k, $0 \leqq k \leqq n-1$, the $n-1$ commutators $[ac_k, c_i]$, with $i \neq k$, are independent modulo pG. Thus if we define $N_k = \langle c_0, \cdots, c_{k-1}, c_{k+1}, \cdots c_{n-1}\rangle$, we have for $0 \leqq k \leqq n-1$,

$$\beta_{N_k}(ac_k) \geqq p^{n-1}. \tag{14}$$

Since $\beta_N(c_k) = p^{n-1}$, $\Gamma_G(c_k)$ is of order at most p mod N', so for all $g \in G$, $[c_k, g]^p \in N'$. But from (12) and (13), $[c_k, g]^p \in \Gamma$, and hence for all k,

$$[c_k, g]^p = 1. \tag{15}$$

Let $z \in \Gamma$. Since G is nil-2, there exists $b \in G$ such that $[a, b] = z$, and by (14) there exists $w \in N_0$ such that $[ac_0, b]^p = [ac_0, w]$. Hence from (15),

$$[a, b]^p = [a, w][c_0, w], \tag{16}$$

so that $[c_0, w] \in N' \cap \Gamma = E$. Now $w \in N_0$, so for some γ_i, $w = c_1^{\gamma_1} \cdots c_{n-1}^{\gamma_{n-1}} x$ with $x \in N'$. But since $\beta_{N_0}(c_0) = p^{n-1}$, it follows that p divides γ_i, $1 \leqq i \leqq n-1$, and so from (15), $[a, w] = 1$. Thus from (16), $z^p = 1$, whence, since z was arbitrary, Γ is elementary. Hence

$$G' = N' \times \Gamma \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)}.$$

We prove now that $[a, c_i] \neq 1$ for some i, $0 \leqq i \leqq n-1$. For suppose to the contrary that for all i

$$[a, c_i] = 1. \tag{17}$$

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Then $[ac_0, c_i] = [c_0, c_i]$, so that

$$\beta_N(ac_0) = p^{n-1}, \tag{18}$$

and

$$\Gamma_N(ac_0) \subseteqq N'. \tag{19}$$

Suppose first that for some $h \in G$ and for some j, $0 \leqq j \leqq n-1$, $[c_j, h] \notin N'$, so that $[c_j, h] = uv$, say, with $v \in N'$ and $1 \neq u \in \Gamma$. Since $\beta_N(c_j) = p^{n-1}$,

$$\Gamma_G(c_j) = \langle uv \rangle \times \Gamma_N(c_0) \subseteqq \langle u, N' \rangle. \tag{20}$$

Now $n \geqq 3$ by (1) and Γ is elementary, so there exist $b_1, b_2 \in G$ such that $[a, b_1] = x_1 \neq 1$ and $[a, b_2] = x_2 \neq 1$, with

$$\langle x_1 \rangle \cap \langle x_2 \rangle = E \quad \text{and} \quad \langle x_1, x_2 \rangle \cap \langle u \rangle = E. \tag{21}$$

Consider $\Gamma_G(ac_j)$; $[ac_j, b_1] = x_1 y_1$, say, with $y_1 \in \langle u, N' \rangle$ by (20), and hence from (21)

$$\Gamma_G(ac_j) = \langle x_1 y_1 \rangle \times \Gamma_N(ac_j) \subseteqq \langle x_1, u, N' \rangle.$$

But $[ac_j, b_2] = x_2 y_2$ say, with $y_2 \in \langle u, N' \rangle$, whence from (21) and since $\Gamma \cap N' = E$, $\beta_G(ac_j) > p^n$, contrary to G being $p^n - BFC$.

Hence for all i, $\Gamma_G(c_i) \subseteqq N'$, and a similar argument again yields a contradiction.

Thus we have shown that (17) cannot hold, so for some k, $0 \leqq k \leqq n-1$,

$$[a, c_k] \neq 1. \tag{22}$$

Let $T = \langle a, c_0, \cdots, c_{n-1} \rangle$, and consider $\Gamma_T(ac_k)$. We can show that $[ac_k, c_k] \notin \Gamma_{N_k}(ac_k)$, whence from (14) it follows that $\beta_T(ac_k) = p^n$. Thus $\Gamma_G(ac_k) \subseteqq T'$. Also $[a, c_k] \notin N'$ and so, since $\beta_N(c_k) = p^{n-1}$, $\beta_T(c_k) = p^n$. Thus $\Gamma_G(c_k) \subseteqq T'$, and hence $\Gamma \subseteqq T'$. But $N \subseteqq T$, so that $G' = N' \times \Gamma \subseteqq T'$, and hence

$$G' = T' \cong \mathscr{C}_p^{\frac{1}{2}n(n+1)}.$$

Now T is $(n+1)$-generator, and so G satisfies Theorem A, contrary to being a counter-example.

CASE III. $\beta(\Lambda) = p^n$.

Suppose finally that $\beta(\Lambda) = p^n$. Clearly there exists $\eta \in \Lambda$ with $\beta_\Lambda(\eta) = p^n$, and so if $b \in G$ with $b\lambda = \eta$, then

$$\beta_G(b) = p^n.$$

Also, since $\operatorname{Ker} \lambda = \Gamma_G(a)$,

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$$\Gamma_G(b) \cap \Gamma_G(a) = E.$$

Let $c_1, c_2, \cdots, c_k \in G$ be such that

$$\Gamma_G(a) = \prod_{i=1}^{k} \times \langle [a, c_i] \rangle.$$

It is fairly easy to show that $C_G(a) = C_G(b)$, whence

$$\Gamma_G(b) = \prod_{i=1}^{k} \times \langle [b, c_i] \rangle,$$

with $|[b, c_i]| = |[a, c_i]|$ for all i, $1 \leqq i \leqq k$. Let $x_i = [a, c_i]$, $y_i = [b, c_i]$.

We now define $T = \Gamma_G(a)\Gamma_G(c_1)$ and establish a bound on the order of T, proving that $|T| \leqq p^{2n-2}$. Since

$$|T| = \frac{|\Gamma_G(a)|\,|\Gamma_G(c_1)|}{|\Gamma_G(a) \cap \Gamma_G(c_1)|} \leqq \frac{p^{2n}}{|\Gamma_G(a) \cap \Gamma_G(c_1)|},$$

we must show that $|\Gamma_G(a) \cap \Gamma_G(c_1)| \geqq p^2$. Now if $|\Gamma_G(a) \cap \Gamma_G(c_1)| < p^2$, $\Gamma_G(a) \cap \Gamma_G(c_1) = \langle x_1 \rangle$ with $|x_1| = p$. Then, by investigating the independence of $[ac_1, a]$, $[ac_1, b]$ and $[ac_1, c_i]$ for $2 \leqq i \leqq k$, we can deduce that $\beta_G(ac_1) > p^n$, contrary to G being p^n-BFC. Hence we have, as desired,

$$|T| \leqq p^{2n-2}. \tag{23}$$

Since $T \lhd G$ we may define a homomorphism $\sigma : G \to G/T = \Sigma$ say, and our final step is to prove that $\beta(\Sigma) \leqq p^{n-2}$. Let $\gamma \in \Sigma$ and let $g \in G$ with $g\sigma = \gamma$. We must prove

(1) if $\beta_G(g) = p^{n-1}$ then $|\Gamma_G(g) \cap T| \geqq p$,

(2) if $\beta_G(g) = p^n$ then $|\Gamma_G(g) \cap T| \geqq p^2$.

Suppose first that $\beta_G(g) = p^{n-1}$ with $\Gamma_G(g) \cap T = E$. Then $[a, g] = 1$ and so $[b, g] = 1$, whence $[gc_1, a^{-1}] = x_1$ and $[gc_1, b^{-1}] = y_1$. By considering $\Gamma_G(gc_1)$, it ultimately follows that $\beta_G(gc_1) > p^n$, contrary to hypothesis, so that $|\Gamma_G(g) \cap T| \geqq p$ as desired.

Suppose then that $\beta_G(g) = p^n$. If $\Gamma_G(g) \cap T = E$ we can proceed as above and obtain our contradiction. Thus we are left to consider the case when $|\Gamma_G(g) \cap T| = p$. The technique used is similar to, but much lengthier than, the above. If $[a, g] \neq 1$ we achieve a contradiction by considering $\Gamma_G(ag)$, whereas if $[a, g] = 1$ we use $\Gamma_G(ac_1)$. In this way we show that $|\Gamma_G(g) \cap T| \geqq p^2$ as desired.

Combining these results, we have at once that $\beta(\Sigma) \leqq p^{n-2}$, and hence by the minimality of G

$$|\Sigma'| \leqq p^{\frac{1}{2}(n-2)(n-1)}.$$

Hence by (23), since $|G'| = |\Sigma'||T|$,

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$$p^{\frac{1}{2}n(n+1)} \leqq p^{\frac{1}{2}(n-2)(n-1)} p^{2n-2},$$

giving $0 \leqq -1$, the desired contradiction.

This completes an analysis of all the cases, and so we have shown that no counter-example G can exist, and Theorem A follows.

The result for an arbitrary nil-2 n-BFC group G is an immediate consequence. For if $n = p_1^{n_1} \cdots p_k^{n_k}$ with p_i distinct primes, then G is the direct product of $p_i^{n_i}$-BFC p_i-groups and an abelian group of order prime to n. Thus

$$|G'| \leqq \prod_{i=1}^{k} p_i^{\frac{1}{2}n_i(n_i+1)},$$

and since $n_i \leqq \log_2 n$ for all i, we have finally:

THEOREM B. *Let G be a nil-2 n-BFC group. Then* $|G'| \leqq n^{\frac{1}{2}(1+\log_2 n)}$.

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