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Almost Everywhere Convergence of Convolution Measures

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Abstract. Let $(X, \mathcal{B}, m, \tau)$ be a dynamical system with $(X, \mathcal{B}, m)$ a probability space and τ an invertible, measure preserving transformation. This paper deals with the almost everywhere convergence in $\mathrm{L}^1(X)$ of a sequence of operators of weighted averages. Almost everywhere convergence follows once we obtain an appropriate maximal estimate and once we provide a dense class where convergence holds almost everywhere. The weights are given by convolution products of members of a sequence of probability measures $\{\nu_i\}$ defined on $\mathbb{Z}$. We then exhibit cases of such averages where convergence fails.

1 Introduction

1.1 Preliminaries

Let $(X, \mathcal{B}, m)$ be a non-atomic, separable probability space. Let τ be an invertible, measure preserving transformation of $(X, \mathcal{B}, m)$. Given a probability measure μ defined on $\mathbb{Z}$, one can define the operator $\mu f(x) = \sum_{k\in\mathbb{Z}} \mu(k) f(\tau^k x)$ for $x \in X$ and $f \in \mathrm{L}^p(X)$ where $p \geq 1$. Note that this operator is well defined for almost every $x \in X$ and that it is a positive contraction in all $\mathrm{L}^p(X)$ for $p \geq 1$, *i.e.*, $\|\mu f\|_p \leq \|f\|_p$.

Given a sequence of probability measures $\{\mu_n\}$ defined on $\mathbb{Z}$, one can subsequently define a sequence of operators as follows: $\mu_n f(x) = \sum_{k\in\mathbb{Z}} \mu_n(k) f(\tau^k x)$. The case where the weights are induced by the convolution powers of a single probability measure defined on $\mathbb{Z}$ has already been studied. More specifically, given μ a probability measure on $\mathbb{Z}$, let μ^n denote the n-th convolution power of μ, which is defined inductively as $\mu^n = \mu^{n-1} * \mu$, where $\mu^2(k) = (\mu * \mu)(k) = \sum_{j\in\mathbb{Z}} \mu(k-j)\mu(j)$ for all $k \in \mathbb{Z}$. In [2] and [3] the authors studied the sufficient conditions on μ that give L^p, $(p \geq 1)$, convergence of the sequence of operators of the form

$$\mu_n f(x) = \sum_{k\in\mathbb{Z}} \mu^n(k) f(\tau^k x).$$

The type of weighted averages that will be considered in this paper are those whose weights are induced by the convolution product of members of a sequence of probability measures $\{\nu_i\}$ defined on $\mathbb{Z}$. Given this sequence of probability measures $\{\nu_i\}$,

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we define another sequence of probability measures $\{\mu_n\}$ on $\mathbb{Z}$ in the following way:

$$\mu_1 = \nu_1,$$
$$\mu_2 = \nu_1 * \nu_2,$$
$$\vdots$$
$$\mu_n = \nu_1 * \cdots * \nu_n.$$

We then define the sequence of operators

$$\mu_n f(x) = \sum_{k\in\mathbb{Z}} (\nu_1 * \cdots * \nu_n)(k) f(\tau^k x) = \sum_{k\in\mathbb{Z}} \mu_n(k) f(\tau^k x).$$

Note that these operators $\mu_n f(x)$ are well defined for almost every $x \in X$ and that they are positive contractions in all $\mathrm{L}^p(X)$, for $1 \le p \le \infty$.

If one defines $T_m f(x) = \sum_{k\in\mathbb{Z}} \nu_m(k) f(\tau^k x)$, we may view $\mu_n f(x) = \nu_1 * \cdots * \nu_n f(x)$ as the composition of $T_1, T_2, \dots, T_n$ *i.e.*, $\mu_n f(x) = T_n \cdots T_1 f(x)$. Therefore, the almost everywhere convergence of $\mu_n f(x)$ may be viewed as a special case of the almost everywhere convergence of the sequence $S_n f(x) = T_n \cdots T_1 f(x)$, where the T_i's are positive contractions of L^p $\forall p \ge 1$. If one defines

$$S_n f(x) = T_1^* \cdots T_n^* T_n \cdots T_1 f(x),$$

where T_i^* denotes the adjoint of T_i, one encounters a much studied situation. In our case this would correspond to successive convolution of ν_i and $\tilde{\nu}_i$, where $\tilde{\nu}_i$ is defined by $\tilde{\nu}_i(k) = \nu_i(-k)$. When $f \in L^p$ for $1 < p < \infty$ and the T_i's are positive contractions and $T_n 1 = T_n^* 1 = 1$, Rota established the almost everywhere convergence [11]. Akcoglu extended this result to the situation where the T_i's are not necessarily positive [1]. Concerning $p = 1$, Ornstein constructed an example of a self-adjoint operator T satisfying the above for which $T \cdots T f(x) = T^n f(x)$ fails to converge almost everywhere [7].

The above failure when $p = 1$ is in contrast to the almost everywhere convergence of the Cesaro averages $\frac{1}{n}\sum_{k=1}^{n} T^k f(x)$ (see [8]).

1.2 Definitions and Past Results

Before we mention a few of the results regarding weighted averages with convolution powers, some definitions are essential.

Definition 1.1 A probability measure μ defined on a group G is called *strictly aperiodic* if and only if the support of μ cannot be contained in a proper left coset of G.

A key theorem by Foguel that we will use repeatedly is the following.

Theorem 1.2 ([4]) *If G is an abelian group and $\hat{G}$ denotes the character group of the group G, then the following are equivalent for a probability measure μ:*

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(i) *μ is strictly aperiodic;*
(ii) *if $\gamma \neq 1$, $\gamma \in \hat{G}$, then $|\hat{\mu}(\gamma)| < 1$.*

Definition 1.3 If $p > 0$, the *p-th moment* of μ is given by $\sum_{k\in\mathbb{Z}} |k|^p \mu(k)$ and is denoted by $m_p(\mu)$. The *expectation* of μ is $\sum_{k\in\mathbb{Z}} k\mu(k)$ and is denoted by $E(\mu)$.

In [2] Bellow and Calderón proved the following theorem.

Theorem 1.4 *Let μ be a strictly aperiodic probability measure defined on $\mathbb{Z}$ that has expectation 0 and finite second moment. The sequence of operators*

$$\mu_n f(x) = \sum_{k\in\mathbb{Z}} \mu^n(k) f(\tau^k x)$$

converges almost everywhere for $f \in L^1(X)$.

The proof of this theorem involves translating properties of the measure into equivalent conditions on the Fourier transform of the measure.

2 Convolution Measures

In this section we discuss sufficient conditions on the sequence of probability measures $\{\nu_i\}$ so that the operators

$$\mu_n f(x) = \sum_{k\in\mathbb{Z}} \mu_n(k) f(\tau^k x) = \sum_{k\in\mathbb{Z}} (\nu_1 * \cdots * \nu_n)(k) f(\tau^k x)$$

converge a.e. for $f \in \mathrm{L}^1(X)$. We will show that the maximal operator of this sequence is of weak-type $(1,1)$, and then we establish a dense class where a.e. convergence holds. Almost everywhere convergence will follow from Banach's Principle.

2.1 Maximal Inequality

To establish a maximal inequality we will use the following theorems.

Theorem 2.1 ([2]) *Let (μ_n) be a sequence of probability measures on $\mathbb{Z}$, $f\colon X \to \mathbb{R}$ and the operators*

$$(\mu_n f)(x) = \sum_{k\in\mathbb{Z}} \mu_n(k) f(\tau^k x).$$

Let $Mf(x) = \sup_n |\mu_n f(x)|$ denote the maximal operator. Assume that there is $0 < \alpha \leq 1$ and $C > 0$ such that for $n \geq 1$,

$$|\mu_n(x+y) - \mu_n(x)| \leq C \frac{|y|^\alpha}{|x|^{1+\alpha}} \quad \text{for } x, y \in \mathbb{Z}, 2|y| \leq |x|.$$

Then the maximal operator M satisfies a weak-type $(1,1)$ inequality; namely, there exists C such that for any $\lambda > 0$

$$m\{x \in X : (Mf)(x) > \lambda\} \leq \frac{C}{\lambda} \|f\|_1 \quad \text{for all } f \in L^1(X).$$

A sufficient condition to obtain the assumption of Theorem 2.1 is given by the following corollary.

Corollary 2.2 ([2]) *Let μ_n be a sequence of probability measures defined on $\mathbb{Z}$ and let $\hat{\mu}_n(t)$ denote its Fourier transform for $t \in [-1/2, 1/2)$. We assume that*

$$\sup_n \int_{-1/2}^{1/2} |\hat{\mu}_n''(t)||t|\, dt < \infty.$$

Then there exist $0 < \alpha \le 1$ and $C > 0$ such that for $n \ge 1$

$$|\mu_n(x+y) - \mu_n(x)| \le C \frac{|y|^\alpha}{|x|^{1+\alpha}} \quad \text{for } x, y \in \mathbb{Z}, 2|y| \le |x|.$$

Theorem 2.3 *Let (ν_n) be a sequence of strictly aperiodic probability measures on $\mathbb{Z}$ such that*

(i) $E(\nu_n) = 0 \ \forall n$;
(ii) $\phi(n) = \sum_{i=1}^n m_2(\nu_i) = O(n)$;
(iii) *there exist a constant C and an integer $N_0 > 0$, such that $|\hat{\nu}_n(t)| \le e^{-Ct^2}$ for $n > N_0$ and $t \in [-1/2, 1/2)$.*

Then for $\mu_n = \nu_1 * \cdots * \nu_n$ we have that

$$\sup_n \int_{-1/2}^{1/2} |\hat{\mu}_n''(t)||t|\, dt < \infty,$$

and therefore the maximal operator $Mf(x) = \sup_{n\in\mathbb{Z}} |\mu_n f(x)|$ is weak-type $(1,1)$.

Proof Without loss of generality we can assume that $N_0 = 1$. Let $a_n = 4\pi^2 m_2(\nu_n)$. Under our hypothesis one can show that for $\hat{\nu}_n(t) = \sum_k \nu_n(k) e^{2\pi i k t}$ and $t \in [-1/2, 1/2)$,

$$|\hat{\nu}_n'(t)| \le a_n |t|, \quad \text{for } t \in [-1/2, 1/2),$$
$$|\hat{\nu}_n''(t)| \le a_n, \qquad \text{for } t \in [-1/2, 1/2).$$

Observe that since $\mu_n = \nu_1 * \cdots * \nu_n$,

$$\hat{\mu}_n(t) = \prod_{i=1}^n \hat{\nu}_i(t),$$

$$\hat{\mu}_n'(t) = \sum_{j=1}^n \prod_{\substack{i=1\\ i\ne j}}^n \hat{\nu}_i(t) \hat{\nu}_j{}'(t),$$

$$\hat{\mu}_n''(t) = \sum_{j=1}^n \prod_{\substack{i=1\\ i\ne j}}^n \hat{\nu}_i(t)\hat{\nu}_j{}''(t) + \sum_{j=1}^n \sum_{\substack{k=1\\ k\ne j}}^n \prod_{\substack{i=1\\ i\ne j,k}}^n \hat{\nu}_i(t) \hat{\nu}_j{}'(t) \hat{\nu}_k{}'(t).$$

These imply that

$$|\hat{\mu}_n''(t)| \leq \sum_{j=1}^{n} a_j e^{-(n-1)Ct^2} + \sum_{j=1}^{n} a_j \sum_{\substack{k=1\\k\neq j}}^{n} a_k e^{-(n-2)Ct^2}|t|^2$$

$$\leq 4\pi^2\phi(n)e^{-(n-1)Ct^2} + 16\pi^4\phi(n)^2 e^{-(n-2)Ct^2}|t|^2,$$

so that

$$\int_{-1/2}^{1/2} |\hat{\mu}_n''(t)||t|\,dt \leq 4\pi^2\phi(n)\int_{-1/2}^{1/2} e^{-(n-1)Ct^2}|t|\,dt + 16\pi^4\phi(n)^2\int_{-1/2}^{1/2} e^{-(n-2)Ct^2}|t|^3\,dt \leq \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\begin{aligned}\mathrm{I}_1 &= 4\pi^2\phi(n)\int_{-1/2}^{1/2} e^{-(n-1)Ct^2}|t|\,dt = 8\pi^2\phi(n)\int_0^{1/2} e^{-(n-1)Ct^2}t\,dt\\ &= 8\pi^2\phi(n)\Big[\frac{e^{-(n-1)Ct^2}}{-2(n-1)C}\Big]_0^{1/2} = 8\pi^2\phi(n)\Big(\frac{e^{\frac{-(n-1)C}{4}}}{-2(n-1)C} + \frac{1}{2(n-1)C}\Big)\\ &= 4\pi^2\frac{\phi(n)}{C(n-1)}\Big(1 - e^{-\frac{(n-1)C}{4}}\Big),\end{aligned}$$

and

$$\begin{aligned}\mathrm{I}_2 &= 16\pi^4\phi(n)^2\int_{-1/2}^{1/2} e^{-(n-2)Ct^2}|t|^3\,dt = 32\pi^4\phi(n)^2\int_0^{1/2} e^{-(n-2)Ct^2}t^3\,dt\\ &= 16\pi^4\phi(n)^2\int_0^{1/4} e^{-(n-2)Cu}u\,du\\ &= 16\pi^4\phi(n)^2\Big(-\frac{ue^{-(n-2)Cu}}{(n-2)C}\Big|_0^{1/4} + \frac{1}{(n-2)C}\int_0^{1/4} e^{-(n-2)Cu}\,du\Big)\\ &= 16\pi^4\phi(n)^2\Big(-\frac{e^{-\frac{(n-2)C}{4}}}{4(n-2)C} - \frac{1}{(n-2)^2C^2}e^{-(n-2)Cu}\Big|_0^{1/4}\Big)\\ &= 16\pi^4\phi(n)^2\Big(-\frac{e^{-\frac{(n-2)C}{4}}}{4(n-2)C} - \frac{1}{(n-2)^2C^2}(e^{-\frac{(n-2)C}{4}} - 1)\Big)\\ &= 16\pi^4\Big(-\frac{1}{4C}\Big(\frac{\phi(n)}{n-2}\Big)^2 e^{-\frac{(n-2)C}{4}}(n-2) - \frac{1}{C^2}\Big(\frac{\phi(n)}{n-2}\Big)^2(e^{-\frac{(n-2)C}{4}} - 1)\Big).\end{aligned}$$

Both integrals I_1 and I_2 are bounded, given that $\phi(n) = O(n)$. Hence,

$$\sup_n \int_{-1/2}^{1/2} |\hat{\mu}_n''(t)||t|\,dt < \infty. \qquad \blacksquare$$

Lemma 2.4 ([9]) *Let $f(t)$ be a characteristic function of a random variable X. Then for all real numbers t, $1-|f(2t)|^2 \le 4(1-|f(t)|^2)$.*

This lemma helps us prove the following result, which is a modification of a theorem found in [9].

Lemma 2.5 *If $|\hat{\mu}(t)| \le c < 1$ for $\frac{1}{2} > |t| \ge b$ and for some b such that $|b| < \frac{1}{4}$, then $|\hat{\mu}(t)| \le 1 - \frac{1-c^2}{8b^2}t^2$ for $|t| \le b$.*

Proof For $t = 0$ the claim is obvious. Choose t such that $|t| < b$. We can find n such that $2^{-n}b \le |t| < 2^{-n+1}b$. Then $b \le 2^n|t| < 2b$. Hence $|\hat{\mu}(2^n t)| \le c$. Lemma 2.4 implies that by induction $1 - |f(2^n t)|^2 \le 4^n(1 - |f(t)|^2)$ holds for all t and any characteristic function f. Using the fact that $\hat{\mu}(t) = \overline{f(2\pi t)}$ for $-1/2 \le t < 1/2$, we have that

$$1 - |\hat{\mu}(2^n t)|^2 = 1 - |\overline{f(2^n 2\pi t)}|^2 \le 4^n(1 - |f(2\pi t)|^2) = 4^n(1 - |\hat{\mu}(t)|^2),$$

which implies that

$$1 - |\hat{\mu}(t)|^2 \ge \frac{1}{4^n}(1 - |\hat{\mu}(2^n t)|^2) \ge \frac{1}{4^n}(1 - c^2) \ge \frac{1-c^2}{4b^2}t^2.$$

Then $|\hat{\mu}(t)| \le 1 - \frac{1-c^2}{8b^2}t^2$ for $|t| < b$ follows. ∎

Lemma 2.6 *If μ is a strictly aperiodic probability measure on $\mathbb{Z}$ and $\hat{\mu}(t)$ denotes the Fourier transform of μ for $t \in (-1/2, 1/2]$, then there exist positive constants $c < 1$ and d such that*

$$|\hat{\mu}(t)| \le 1 - \frac{1-c^2}{8d^2}t^2 \quad \textit{for } |t| \le d,$$

which implies that there exists $C > 0$ such that $|\hat{\mu}(t)| \le e^{-Ct^2}$ for $t \in [-1/2, 1/2)$.

The third condition of Theorem 2.3 replaces the condition of strict aperiodicity in the case when all of the ν_i's are the same measure, *i.e.*, $\nu_i = \nu$.

Lemma 2.7 *Let $\{\nu_n\}$ be a sequence of probability measures on $\mathbb{Z}$. The following are equivalent.*

(i) $\forall \delta > 0$

$$\overline{\lim_{n\to\infty}} \sup_{|t|>\delta} |\hat{\nu}_n(t)| < 1 \textit{ (asymptotically strictly aperiodic).}$$

(ii) *There exist C and N_0 such that*

$$|\hat{\nu}_n(t)| \le e^{-Ct^2} \textit{ for } n > N_0.$$

Proof (ii) ⇒ (i) is obvious. To show that (i) ⇒ (ii) , since for $\delta > 0$

$$\overline{\lim_{n\to\infty}} \sup_{|t|>\delta} |\hat{\nu}_n(t)| < 1,$$

given $\epsilon > 0$, we can choose $\delta > 0$ and $N \in \mathbb{Z}$ such that $\sup_{|t|>\delta} |\hat{\nu}_n(t)| < 1 - \epsilon$ for $n > N$. By Lemma 2.4, $|\hat{\nu}_n(t)| \le 1 - Kt^2$ for some constant K, $n \ge N$ and $|t| < \delta$. So that there exists a constant C such that $|\hat{\nu}_n(t)| \le e^{-Ct^2}$ for all $t \in [-1/2, 1/2)$ for $n \ge N$. ∎

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2.2 Dense Set and Almost Everywhere Convergence in $\mathbf{L}^1(X)$

Lemma 2.8 *Let μ_n be a sequence of probability measures on $\mathbb{Z}$ such that*

(i) *there is $0 < \alpha \le 1$ and $C > 0$ such that for $n \ge 1$*

$$|\mu_n(x+y) - \mu_n(x)| \le C\frac{|y|^\alpha}{|x|^{1+\alpha}} x, y \in \mathbb{Z} 2|y| \le |x|,$$

(ii) $\hat{\mu}_n(t) \xrightarrow{n\to\infty} 0$ *for a.e. $t \in [-1/2, 1/2)$.*

Then $\|\mu_n - \mu_n * \delta_1\|_1 \xrightarrow{n\to\infty} 0$.

Proof Note that by the first assumption,

$$\begin{aligned}|\mu_n(k) - \mu_n * \delta_1(k)| &= |\mu_n(k-1+1) - \mu_n(k-1)| \\ &\le C\frac{1}{(k-1)^{1+\alpha}}, \quad \text{for } 2 < |k-1|.\end{aligned}$$

This implies that the sequence $|\mu_n(k) - \mu_n * \delta_1(k)|$ is bounded by a summable function. By Lebesgue's dominated convergence theorem the condition $\|\mu_n - \mu_n * \delta_1\|_1 \xrightarrow{n\to\infty} 0$ holds if we show that $|\mu_n(k) - \mu_n(k-1)| \xrightarrow{n\to\infty} 0$ for all k. Indeed, observe that

$$\begin{aligned}|\mu_n(k) - \mu_n(k-1)| &= \Big|\int_{-1/2}^{1/2} \hat{\mu}_n(t)(e^{-2\pi ikt} - e^{-2\pi i(k-1)t})\, dt\Big| \\ &\le \int_{-1/2}^{1/2} |\hat{\mu}_n(t)||e^{-2\pi ikt}||1 - e^{2\pi it}|\, dt \to 0 \text{ as } n \to \infty,\end{aligned}$$

by (ii) and the bounded convergence theorem. ∎

Theorem 2.9 *Let (ν_n) be a sequence of strictly aperiodic probability measures on $\mathbb{Z}$ such that*

(i) *$E(\nu_n) = 0$, $\forall n$;*

(ii) *$\phi(n) = \sum_{i=1}^n m_2(\nu_i) = O(n)$;*

(iii) *there exist a constant C and an integer $N_0 > 0$, such that $|\hat{\nu}_n(t)| \le e^{-Ct^2}$ for $n > N_0$ and $t \in [-1/2, 1/2)$.*

The sequence of operators $\{\mu_n f\}$ converges almost everywhere in $L^1(X)$.

Proof Since the maximal operator has been shown to be of weak-type $(1,1)$ (Theorem 2.3), it is enough to show that convergence holds on the dense class $\{f + g - g\circ\tau : f\circ\tau = f, g \in \mathrm{L}_\infty\}$. Clearly, $\mu_n f$ converges almost everywhere for τ-invariant functions f. Then to show that $(\mu_n g - \mu_n(g\circ\tau))$ converges almost everywhere for $g \in \mathrm{L}_\infty$, it is enough to show that $\|\mu_n g - \mu_n(g\circ\tau)\|_\infty \xrightarrow{n\to\infty} 0$. But

$$\begin{aligned}\|\mu_n g - \mu_n(g\circ\tau)\|_\infty &\le \|\mu_n g - (\mu_n * \delta_1)g\|_\infty \\ &\le \|\mu_n - \mu_n * \delta_1\|_1 \|g\|_\infty,\end{aligned}$$

so that it is enough to show $\|\mu_n - \mu_n * \delta_1\|_1 \xrightarrow{n\to\infty} 0$, which holds according to Lemma 2.8. ∎

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3 Collections with Uniformly Bounded Second Moments

Lemma 3.1 *Let $A \subseteq \mathbb{C}^4$ be the set*

$$A = \{(a_1, a_2, z_1, z_2) : a_1 + a_2 = 1, a_1, a_2 \geq 0, |z_1| = |z_2| = 1\},$$

and let $S(\delta, \eta) \subseteq A$ be the set

$$S(\delta, \eta) = \{(a_1, a_2, z_1, z_2) \ : \ a_1, a_2 \geq \delta \text{ and } |z_1 - z_2| \geq \eta\}, 0 < \delta, 0 < \eta\}.$$

Then there exists $\rho = \rho(\delta, \eta) < 1$ such that for $(a_1, a_2, z_1, z_2) \in S(\delta, \eta)$, $|a_1 z_1 + a_2 z_2| \leq \rho$ holds.

Proof By the triangle inequality for points in A $|a_1 z_1 + a_2 z_2| = 1$ if and only if $a_1 z_1 = \lambda a_2 z_2$ for $\lambda \geq 0$, which implies that $(a_1, a_2, z_1, z_2) \in A$. Therefore $F(a_1, a_2, z_1, z_2) = a_1 z_1 + a_2 z_2$ has modulus 1 on A only on the set $R = \{(a_1, a_2, z_1, z_2), a_1 = a_2, z_1 = z_2\}$. Observe that the points in $S(\delta, \eta)$ are bounded away from R. Since $S(\delta, \eta)$ is a compact subset of A and F is continuous on A, the claim follows. ■

Lemma 3.2 *Let ν be a probability measure on $\mathbb{Z}$ with $m_1(\nu) \leq a$ and*

$$\sup_{\beta, r \in \mathbb{Z}} \nu(\beta\mathbb{Z} + r) \leq \rho < 1.$$

Suppose l/s is a rational number in $(-1/2, 1/2]$ with $|s| \leq M$ and $|l| \leq \lfloor \frac{|s|}{2} \rfloor$. Then there exists $0 \leq \sigma = \sigma(a, \rho) < 1$ such that $|\hat{\nu}(l/s)| \leq \sigma$.

Proof Let $|s| \leq M$. For $|l| \leq \lfloor s/2 \rfloor$, we have $\hat{\nu}(\frac{l}{s}) = \sum_{m \in \mathbb{Z}} \nu(m) e^{2\pi i m (l/s)}$. Write $d = \gcd(l, s)$; then $l = d\alpha$, $s = d\beta$, and $m = \gamma\beta + r$ for some $0 \leq r < \beta$. Then

$$\hat{\nu}\left(\frac{l}{s}\right) = \sum_{r=0}^{\beta-1} \nu(\beta\mathbb{Z} + r) e^{2\pi i r (\alpha/\beta)}.$$

By assumption there exist two cosets $\beta\mathbb{Z} + r_1$, $\beta\mathbb{Z} + r_2$ and a value δ that depends only on M and ρ, such that $\nu(\beta\mathbb{Z} + r_1), \nu(\beta\mathbb{Z} + r_2) \geq \delta$. Therefore,

$$\hat{\nu}\left(\frac{l}{s}\right) = \nu(\beta\mathbb{Z} + r_1) e^{2\pi i r_1 (\alpha/\beta)} + \nu(\beta\mathbb{Z} + r_2) e^{2\pi i r_2 (\alpha/\beta)} + \sum_{m \notin \beta\mathbb{Z} + r_1 \cup \beta(\mathbb{Z} + r_2),} \nu(m) e^{2\pi i m (\alpha/\beta)}.$$

Also since $\gcd(\alpha, \beta) = 1$,

$$|e^{2\pi i r_1 (\alpha/\beta)} - e^{2\pi i r_2 (\alpha/\beta)}| = |1 - e^{2\pi i (r_2 - r_1)(\alpha/\beta)}| \geq \eta > 0,$$

where η depends on M and ρ since $|\beta| \leq |s| \leq M$. Therefore, by Lemma 3.1 there exists a $0 \leq \sigma' = \sigma'(M, \rho) < 1$ such that

$$|\nu(\beta\mathbb{Z} + r_1) e^{2\pi i r_1 (\alpha/\beta)} + \nu(\beta\mathbb{Z} + r_2) e^{2\pi i r_2 (\alpha/\beta)}| \leq \sigma'(\nu(\beta\mathbb{Z} + r_1) + \nu(\beta\mathbb{Z} + r_2)).$$

It follows that there exists $0 \leq \sigma = \sigma(M, \rho) < 1$ such that $|\hat{\nu}(l/s)| \leq \sigma$. ■

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Theorem 3.3 *Let ν be a probability measure on $\mathbb{Z}$ with $m_1(\nu) \le a$ and*

$$\sup_{\beta, r\in\mathbb{Z}} \nu(\beta\mathbb{Z}+r) \le \rho < 1.$$

Then there exists a $c = c(a,\rho)$ such that $|\hat{\nu}(t)| \le e^{-ct^2}$.

Proof By hypothesis and using Chebyshev's inequality there exist $\delta = \delta(\rho, a)$, $M = M(a)$, and integers k, j such that $|k|, |j| \le M$ and $\nu(k), \nu(j) \ge \delta$. Let $s = k - j$, and consider the points $\{\frac{p}{s} : p = 0, \pm 1, \dots, \pm\lfloor\frac{|s|}{2}\rfloor\}$. By Lemma 3.2 and the mean value theorem, for $p = \pm 1, \dots, \pm\frac{|s|}{2}$ there exists an $\epsilon = \epsilon(a)$ such that for all $t \in (\frac{p}{s} - \epsilon, \frac{p}{s} + \epsilon)$ we have $|\hat{\nu}(t)| \le \sigma + \frac{1-\sigma}{2}$, where σ is the value in Lemma 3.2. Let $I_p = (\frac{p}{s} - \epsilon, \frac{p}{s} + \epsilon)$, where $p = 0, \pm 1, \dots, \pm\lfloor\frac{|s|}{2}\rfloor$, and t_0 a point in the complement of $S = \bigcup_p I_p$. We have

$$\hat{\nu}(t_0) = \nu(k)e^{2\pi i k t_0} + \nu(j)e^{2\pi i j t_0} + \sum_{m\neq k,j} \nu(m)e^{2\pi i m t_0}.$$

Now $|e^{2\pi i k t_0} - e^{2\pi i j t_0}| = |1 - e^{2\pi i s t_0}|$ and this is greater than a value $\eta > 0$, which depends only on s and ϵ which depends only on $m_1(\nu)$ which is bounded by a. Thus by Lemma 3.1

$$|\nu(k)e^{2\pi i k t_0} + \nu(j)e^{2\pi i j t_0}| \le \sigma'(\nu(k) + \nu(j))$$

and therefore $|\hat{\nu}(t_0)| \le \sigma'' < 1$ for some value $\sigma'' = \sigma''(\rho, a)$. We therefore have for $|t| \ge \epsilon$ a value $\sigma''' = \max(\sigma, \sigma'') < 1$ dependent on ρ and a only, such that $|\hat{\nu}(t)| \le \sigma''$. By Lemma 2.4 there exists a c' such that $|\hat{\nu}(t)| \le 1 - c't^2 < 1$ for $0 < |t| < \epsilon$. The conclusion follows by choosing a value c small enough so that $|\hat{\nu}(t)| \le e^{-ct^2}$ for $t \in (-1/2, 1/2]$. ■

Combining Theorems 2.9 and 3.3 we get the following theorem.

Theorem 3.4 *If ν_n is a sequence of probability measures on $\mathbb{Z}$ such that for all n,*

(i) $E(\nu_n) = 0$,
(ii) $m_1(\nu_n) \le a$,
(iii) $\sup_n \sup_{\alpha,\beta} \nu_n(\beta\mathbb{Z} + \alpha) \le \rho < 1$,
(iv) $\phi(n) = \sum_{i=1}^n m_2(\nu_i) = O(n)$.

Then $\mu_n f(x)$ converges a.e. for all $f \in L^1(X)$.

Remark 3.5 Let

$$\nu_n(k) = \begin{cases} \dfrac{1-a_n}{2} & k = \pm 1, \\ a_n & k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $1 > a_n > 0$ and $a_n \to 0$ fast enough so that $\prod_{n=1}^\infty a_n > 0$. Then, using an argument similar to that in [3], one may show that the sequence $\mu_n f$ does not converge a.e. for some $f \in \mathrm{L}^\infty$. Of course, the sequence $\nu_n(k)$ does not satisfy the condition $\sup_n \sup_{\alpha,\beta} \nu_n(\beta\mathbb{Z} + \alpha) \le \rho$ while it does satisfy the condition $m_1(\nu_n) \le a$.

4 The Strong Sweeping Out Property

4.1 Introduction

In this section $(X, \mathcal{B}, m, \tau)$ and τ are as previously. Here we discuss cases where the operators $\mu_n f(x) = \sum_{k\in\mathbb{Z}} \mu_n(k) f(\tau^k x)$ fail to converge, whereas before $\mu_n = \nu_1 * \cdots * \nu_n$. The case where μ_n is given by the convolution powers of a single probability measure μ on $\mathbb{Z}$, *i.e.,* $\mu_n = \mu^n$, has been studied. In the event of convolution powers, the probability measure μ given by $\mu = \frac{1}{2}(\delta_0 + \delta_1)$ is the prototype of bad behavior for the resulting sequence of operators $(\mu^n f)(x)$. Using the central limit theorem, it was shown in [3] that the bad behavior of this prototype is typical, at least if μ has $m_2(\mu) < \infty$ and $E(\mu) \neq 0$([3]). In [6], this result was extended to probability measures with $E(\mu) = 0$ and $m_p(\mu) < \infty$ for $p > 1$.

Definition 4.1 The sequence of measures μ_n is said to have the *strong sweeping out* property, if given $\epsilon > 0$, there is a set $B \in \mathcal{B}$ with $m(B) < \epsilon$ such that

$$\limsup_n \mu_n \chi_B(x) = 1 \text{ a.e.}, \quad \liminf_n \mu_n \chi_B(x) = 0 \text{ a.e.}$$

We will use the following in our constructions.

Proposition 4.2 ([10]) *For any sequence of probability measures μ_N on $\mathbb{Z}$ that are dissipative,* i.e., $\lim_{N\to\infty} \mu_N(k) = 0$ *for all* $k \in \mathbb{Z}$*, if there exists* $b > 0$ *and a dense subset* $D \subset \{\gamma : |\gamma| = 1\}$ *with* $\liminf_{N\to\infty} |\hat{\mu}_N(\gamma)| \geq b$ *for all* $\gamma \in D$*, then for any ergodic dynamical system* $(X, \mathcal{B}, m, \tau)$ *the sequence* μ_n *is strong sweeping out.*

4.2 Strong Sweeping out with Convolution Measures

Theorem 4.3 *If* $\nu_n = a_n\delta_{x_n} + (1 - a_n)\gamma_n$*, where* γ_n *is a probability measure,* $\sum x_n$ *either* $\to \infty$ *or* $\to -\infty$ *and* $\sum_n (1 - a_n) < \infty$*, then* $\{\mu_n = \nu_1 * \cdots * \nu_n\}$ *is a dissipative sequence.*

Proof Without loss of generality, assume that $\sum x_n \to \infty$. Suppose

$$\nu_n = a_n\delta_{x_n} + (1 - a_n)\gamma_n$$

as above. Then we have $\sum P(Z_n \neq x_n) \leq \sum(1 - a_n) < \infty$, where Z_n is a sequence of independent random variables having distribution ν_n. By the Borel–Cantelli lemma $P(Z_n \neq x_n \text{ i.o }) = 0$. Let $\omega \in (Z_n \neq x_n \text{ infinitely often})^c$. Then

$$\begin{aligned} S_N(\omega) = \sum_{m=1}^{N} Z_m(\omega) &= \sum_{Z_m(\omega)\neq x_m} z_n + \sum_{Z_m(\omega)=x_m} x_m \\ &\geq -c(\omega) + \sum_{Z_m(\omega)=x_m} x_m \to \infty \text{ as } N \to \infty, \end{aligned}$$

as $c(\omega)$ is a constant depending on ω. Hence $S_N(\omega) \to \infty$ with probability 1. Therefore, when k is fixed, $P(S_N = k) \to 0$. Indeed, since

$$P\Big(\bigcup_{N=1}^{\infty} (S_m(\omega) > k \,\forall m \geq N)\Big) = 1$$

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and the sequence of sets is increasing, we have $P(S_m(\omega) > k\ \forall m \geq N) \to 1$. But $P(S_N > k) \geq P(S_m > k\ \forall m \geq N)$ so $P(S_N(\omega) = k) \leq 1 - P(S_N(\omega) > k) \to 0$. Hence, $\lim_{n\to\infty} \mu_n(k) = \lim_{n\to\infty}(\nu_1 * \cdots * \nu_n)(k) = 0$ and $\{\mu_n\}$ is a dissipative sequence. ■

Corollary 4.4 *Let $\nu_n = a_n\delta_{x_n} + (1-a_n)\gamma_n$, where γ_n is a probability measure on $\mathbb{Z}$, such that $x_n \in \mathbb{Z}$, $\sum(1-a_n) < \infty$, $|x_n| \geq 1$ and $\sum x_n \to \infty$ or $-\infty$. Then for any ergodic dynamical system $(X, \mathcal{B}, m, \tau)$ the sequence $\mu_n = \nu_1 * \cdots * \nu_n$ is strong sweeping out.*

Proof Theorem 4.3 implies that the sequence $\mu_n = \nu_1 * \cdots * \nu_n$ is dissipative. Note that for $t \in [-1/2, 1/2)$ we have

$$\begin{aligned}|\hat{\mu_n}(t)| &= \prod_{l=1}^{n} |\hat{\nu_l}(t)| = \prod_{l=1}^{n} |a_l e^{-2\pi i x_l t} + (1-a_l)\hat{\gamma_l}(t)| \\ &\geq \prod_{l=1}^{n} \big|\,|a_l| - (1-a_l)|\hat{\gamma_l}(t)|\big| \geq \prod_{l=1}^{n} \big|\,|a_l| - (1-a_l)\big| = \prod_{l=1}^{n}(2a_l - 1) \\ &= \prod_{l=1}^{n} a_l\Big(2 - \frac{1}{a_l}\Big) \geq c\prod_{l=1}^{N} a_l \geq cc' > 0\,.\end{aligned}$$

The result follows by Proposition 4.2. Note that we have used the fact that for $a_l > 0$, $\sum(1-a_l) < \infty$ implies that $\prod a_l$ converges to a nonzero value. ■

Lemma 4.5 *Let $\nu_n = a_n\delta_{x_n} + (1-a_n)\gamma_n$, where γ_n is a probability measure, $E(\nu_n) = 0$, $|x_n| \geq c$, and $a_n \geq d$ for some constants c and d. Then $m_2(\nu_n) \geq \frac{\alpha}{1-a_n}$, where $\alpha = dc^2$.*

Proof Since $E(\nu_n) = a_n x_n + (1-a_n)E(\gamma_n) = 0$, $\frac{a_n x_n}{a_n - 1} = E(\gamma_n)$. Therefore

$$\begin{aligned}m_2(\nu_n) &= a_n x_n^2 + (1-a_n)m_2(\gamma_n) \geq a_n x_n^2 + |E(\gamma_n)|^2(1-a_n) \\ &= a_n x_n^2 + \frac{a_n^2 x_n^2}{1-a_n} \\ &\geq \frac{\alpha}{1-a_n}.\end{aligned}$$

This provides a lower bound on the second moment, *i.e.*, $m_2(\nu_n) \geq \frac{\alpha}{1-a_n}$. If in addition $\sum(1-a_n) < \infty$, once we allow $\prod a_n \geq c > 0$, the second moments $m_2(\nu_n)$ cannot grow arbitrarily slowly. ■

Example 4.6 Let a_n be a sequence such that $\sum(1-a_n) < \infty$. Let $b_n = [\frac{1}{1-a_n}]$, where $[x]$ denotes the integer part of the number x. Consider the measures given by

$$\nu_n(k) = \begin{cases} \dfrac{1+2b_n}{3+2b_n}, & k = 1, \\ \dfrac{1}{3+2b_n}, & k = -b_n, \\ \dfrac{1}{3+2b_n}, & k = -b_n - 1. \end{cases}$$

These measures satisfy the assumptions of Theorem 4.4. As such, the sequence $\mu_n f = (\nu_1 * \cdots * \nu_n) f$ is strong sweeping out. It is noteworthy that all the measures in this example additionally satisfy the property

$$m_2(\nu_n) = \frac{2b_n^2 + 4b_n + 2}{3 + 2b_n},$$

which implies that the second moment grows like $\frac{1}{1-a_n}$. One might think of this sequence ν_n as

$$\nu_n = a_n\delta_1 + \frac{(1-a_n)}{2}(\delta_{-b_n} + \delta_{-b_n-1}) = a_n\delta_1 + (1-a_n)\gamma_n,$$

where $\gamma_n = 1/2(\delta_{-b_n} + \delta_{-b_n-1})$.

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