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ATHWART IMMERSIONS IN EUCLIDEAN SPACE

S. A. ROBERTSON and F. J. CRAVEIRO de CARVALHO

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Abstract

Let f and g denote immersions of the n-manifolds M and N, respectively, in R^{n+1}. We say that f is athwart to g if $f(M)$ and $g(N)$m have no tangent hyperplane in common. In this paper necessary conditions for athwartness are obtained.

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Introduction

We study the following problem. Let M and N be smooth ($= C^\infty$), closed, connected manifolds and let f and g be smooth immersions of M and N, respectively, in Euclidean $(n+1)$-space R^{n+1}. We say that f is athwart to g, written $f \pitchfork g$, if and only if $f(M)$ and $g(N)$ have no tangent hyperplane in common. In what circumstances is $f \pitchfork g$?

It is easy to think of instances where $f \pitchfork g$ and others where f is not athwart to g. Some examples, for the case $n = 1$, are shown in Figure 1 below. In examples (i) and (ii) $f \pitchfork g$ while f is not athwart to g in (iii) and (iv).

We observe that, for all $n \geqslant 1$, if f is a convex embedding of the n-sphere S^n and if $g(N)$ is inside $f(S^n)$ then $f \pitchfork g$. However convexity of f is obviously not essential for athwartness. It is natural to consider whether any analogue of case (i) in Figure 1 exists for $n > 1$. In fact, we shall prove the following two main theorems which give necessary conditions for athwartness and show that there is an interesting difference between the cases $n = 1$ (Theorem 4.1) and $n > 1$ (Theorem 5.1). Precise definitions are given in Section 2.

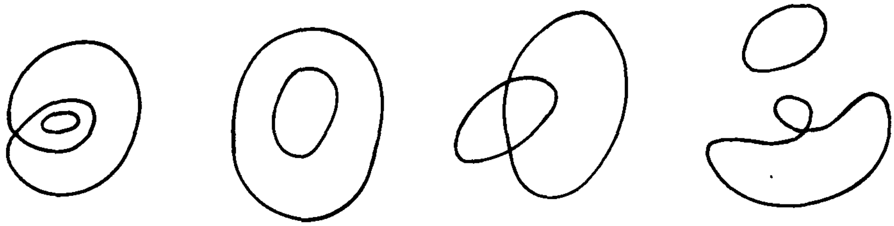

Figure 1

THEOREM 4.1. *Let f and g be two immersions of S^1 into R^2. If $f \mathrel{\bar{\pitchfork}} g$ then the image of one of the immersions is inside all the loops of the other.*

THEOREM 5.1. *Let $f: M \to R^{n+1}$, $g: N \to R^{n+1}$ be immersions such that $f \mathrel{\bar{\pitchfork}} g$. Then one of the manifolds, say M, is diffeomorphic to S^n, f is an embedding with starshaped inside and $g(N)$ is contained in the interior of the kernel of the inside of f.*

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2. Notations and definitions

Throughout the paper we shall be dealing with compact, connected C^∞ n-manifolds without boundary. All the maps are C^∞.

For any immersion $f: M \to R^{n+1}$ the tangent n-plane to $f(M)$ at $f(x)$ will be denoted by T_x and is the affine n-dimensional subspace of R^{n+1} determined by $f(x)$ and $f_{*x}(T_x M)$, where $f_{*x}: T_x M \to T_{f(x)} R^{n+1} \equiv R^{n+1}$. Such an immersion induces a C^∞ map $F: M \to R_n^{n+1}$, where R_n^{n+1} denotes the Grassmannian of affine n-planes in R^{n+1} [5].

Transversality will be denoted by the usual symbol $\pitchfork$. Thus, from Section 1, if $f: M \to R^{n+1}$ and $g: N \to R^{n+1}$ are immersions then $f \mathrel{\bar{\pitchfork}} g$ if and only if $T_x \pitchfork T_y$, for any $x \in M, y \in N$.

A *loop* is a C^∞ map $g: [a, b] \to R^2$ such that $g|[a, b)$ is injective, $g'(t) \neq 0$ for $t \in [a, b]$, and $g(a) = g(b)$. A loop is a Jordan curve and therefore the complement of its image in R^2 consists of two disjoint open connected subsets of R^2

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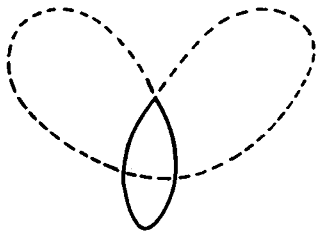

Figure 2

according to the Jordan Curve Theorem [4]. Of these two subsets one is bounded and will be called the *inside* of the loop g (or the inside of $g([a, b])$) while the other is unbounded and is called the *outside*. If a set S is contained in the inside (outside) of a loop g we say that S is *inside* (*outside*) g (or $g([a, b])$).

If we look at an immersion of S^1 into R^2 as a periodic map $f: R \to R^2$ of period one we can speak of loops of an immersion. We shall not distinguish between two loops of f with the same image in R^2. Figure 2 illustrates this idea: the image of the "cloverleaf" immersion of S^1 is indicated by a broken line with the image of one of its six loops shown as a solid line.

If $f: M \to R^{n+1}$ is an embedding then the complement of $f(M)$ in R^{n+1} is the union of two disjoint open connected sets B and U of which B is bounded and U is unbounded [1]. We say that B is the *inside* of f (or $f(M)$) and U is the *outside*. As before we shall speak of being inside (outside) f (or $f(M)$).

If V is a subset of R^{n+1} the *kernel* of V is the set $\{p \in V \mid tp + (1-t)q \in V$ for all $q \in V$ and $0 \leqslant t \leqslant 1\}$.

3. A basic result

THEOREM 3.1. *Let $f: M \to R^{n+1}$ and $g: N \to R^{n+1}$ be immersions. If $f(M)$ has two tangent n-planes such that one meets $g(N)$ and the other does not then f is not athwart to g.*

PROOF. Assume that, under the above hypothesis, $f \pitchfork g$. Then $g \pitchfork T_x$, for any $x \in M$.

Let U denote the set $\{x \in M \mid T_x \cap g(N) \neq \varnothing\}$. This set and its complement are both non-empty. Since $g(N)$ is compact then U is obviously closed. We show next that it is also open.

Let $x \in U$. By transversality there exist y_1, y_2 in N such that $g(y_1)$ and $g(y_2)$ do not lie in the same half-space complementary to T_x. The set of hyperplanes which separate $g(y_1)$ and $g(y_2)$ is open in R_n^{n+1}. Therefore, since the map induced by f is continuous, there exists an open neighbourhood U_x of x such that, for any $y \in U_x$, T_y separates $g(y_1)$ and $g(y_2)$ and so $T_y \cap g(N) \neq \varnothing$.

Having proved that U is simultaneously open and closed we conclude that M is not connected. Consequently athwartness must fail somewhere.

COROLLARY 3.1. *Let $f: M \to R^{n+1}$ and $g: N \to R^{n+1}$ be immersions such that $f(M) \cap g(N) \neq \varnothing$. Then f is not athwart to g.*

PROOF. Suppose that $f = (f_i)$, $g = (g_i)$, $i = 1, \ldots, n$. Looking, for instance, at the absolute maxima of f_1 and g_1 we see that either $f(M)$ and $g(N)$ have a common tangent n-place or one of them has a tangent n-plane which does not meet the other. Since $f(M) \cap g(N) \neq \varnothing$ we are then in the position of the previous theorem.

We remark that, for $n = 1$, Corollary 3.1 is also a corollary of Theorem 4 in [2]. From now on we shall deal with the cases $n = 1$, $n \geqslant 2$ separately.

4. Plane curves

THEOREM 4.1. *Let f and g be two immersions of S^1 into R^2. If $f \pitchfork g$ then the image of one of the immersions is inside all the loops of the other.*

PROOF. Suppose that neither of the images lies inside all the loops of the other. Then either $f(S^1) \cap g(S^1) \neq \varnothing$ or $f(S^1) \cap g(S^1) = \varnothing$. If $f(S^1) \cap g(S^1) \neq \varnothing$, by Corollary 3.1, there is a common tangent. If $f(S^1) \cap g(S^1) = \varnothing$ then we must have two loops, one of each curve, such that neither of them is inside the other. As before, we can assume that there is a tangent to one of the images which does not intersect the other and again we are in the position of Theorem 3.1 in view of the following lemma.

LEMMA 4.1. *Let $f: [a, b] \to R^2$ be a loop and p an outside point. Then at least one tangent to the loop passes through p.*

PROOF. We shall assume that no tangent line to f passes through p. Let $\Psi: [a, b] \to S^1$ be given by $\Psi(t) = (f(t) - p)/\|f(t) - p\|$. Consider the covering map: $g: R \to S^1$, with $g(t) = (\cos 2\Pi t, \sin 2\Pi t)$. Let $x \in R$ be such that $g(x) = \Psi(a) = \Psi(b)$ and $\tilde{\Psi}$ the lift of Ψ which starts at x. The winding number of f relative to p is given by $\tilde{\Psi}(b) - \tilde{\Psi}(a)$. Since we are assuming that no tangent line

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passes through p, Ψ has no critical points and the same happens to $\tilde{\Psi}$. This map is either increasing or decreasing and thus the winding number is non-zero. Therefore, p is an inside point.

5. Hypersurfaces

THEOREM 5.1. *Let $f\colon M \to R^{n+1}$, $g\colon N \to R^{n+1}$ be immersions such that $f \pitchfork g$. Then one of the manifolds, say M, is diffeomorphic to S^n, f is an embedding with starshaped inside and $g(N)$ is contained in the interior of the kernel of the inside of f.*

PROOF. We shall use the following remarkable result due to Halpern [3].

Let $f\colon M \to R^{n+1}$, $\dim M = n$, be an immersion. If $\bigcup_{x\in M} T_x \neq R^{n+1}$ then M is diffeomorphic to S^n, f is an embedding, the inside of $f(M)$ is starshaped and $R^{n+1} \setminus \bigcup_{x\in M} T_x$ is the interior of the kernel of the inside of f.

Since $f \pitchfork g$, there exists a tangent n-plane to one of the images, say $f(M)$, which does not meet the other, as in Section 3. By Theorem 3.1, no tangent to $f(M)$ meets $g(N)$. The theorem now follows from Halpern's result.

Added in Proof. The conditions of Theorem 5.1 are not only necessary but also sufficient [3].

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Faculty of Mathematical Studies
The University
Southampton, S09 5NH
England

Departamento de Matemática
Universidade de Coimbra
3000 Coimbra
Portugal