ON A CONJECTURE OF MELZAK

G. C. Shephard

(received March 7, 1964)

Melzak [2] has shown that there exists a convex pseudo-polyhedron Q (the convex hull of a convergent sequence of points together with its limit point) in E_3 which is s-universal for triangles, that is, all possible triangles occur (up to similarity) as plane sections of Q. He conjectured that no polyhedron P has this property. In this short note we give an elementary proof of this conjecture. (In [1], Klee discusses in detail the problem of the existence of s-universal polyhedra for various classes $\mathcal{K}$ of convex bodies. He asserts on p. 263, but does not prove, that Melzak's conjecture is true.)

The proof is by contradiction. Suppose a bounded convex polyhedron P (the convex hull of a finite number of points) in E_3 is s-universal for triangles. Let $\{\alpha_i\}$ be any monotonic decreasing sequence of real numbers, each less than $\frac{1}{2}\pi$, tending to the limit 0 as $i \to \infty$. Let Δ_i be an isosceles triangle with angles α_i, α_i, $\pi - 2\alpha_i$, and let δ_i be the plane section of P similar to Δ_i. We shall refer to the vertex of δ_i where the angle $\pi - 2\alpha_i$ occurs as the <u>apex</u> of the triangle, the opposite side as the <u>base</u> of the triangle, and the distance between these two as the <u>height</u> of the triangle. Since the sides of δ_i lie on the faces of P there is a correspondence

$$\Delta_i \leftrightarrow (p_1^i,\ p_2^i;\ q^i)$$

between the triangles Δ_i and the semi-ordered triples of faces

Canad. Math. Bull. vol. 7, no. 4, October 1964

of P (semi-ordered because $\triangle_i$ is isosceles so the order of p_1^i, p_2^i is not significant). As P has a finite number of faces there are only a finite number of these semi-ordered triples, and so some subsequence of $\{\triangle_i\}$ consists of triangles corresponding to the same triple $(p_1, p_2; q)$. By change of notation we may write $\{\triangle_i\}$ for this subsequence.

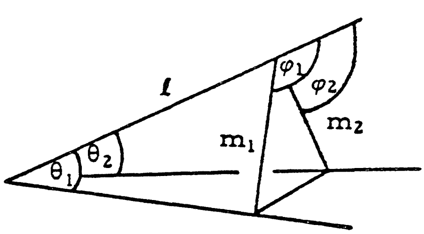
Two cases now arise:

1. The plane q is parallel to the line $\ell = p_1 \cap p_2$. Let d be the distance between q and ℓ; then it is easy to see that any plane section of p_1, p_2, q must be a triangle whose height is at least d. This means that δ_i has height at least d and base at least $2d \cot \alpha_i$. As $i \to \infty$, $\alpha_i \to 0$, $2d \cot \alpha_i \to \infty$ and so the sequence of triangles $\{\delta_i\}$ is unbounded. This is a contradiction since we have assumed that P is bounded.

2. The plane q is not parallel to the line ℓ. Let a variable plane h cut the faces p_1, p_2 in lines m_1, m_2 making angles φ_1, φ_2 with ℓ respectively (see the diagram). If h varies in such a manner that $m_1 \hat{} m_2 \to \pi$, then it is easy to see that either

$$\varphi_1 \to 0 \text{ and } \varphi_2 \to \pi \text{ or } \varphi_1 \to \pi \text{ and } \varphi_2 \to 0 .$$

But if h meets q neither of these possibilities can occur, since if θ_1, θ_2 are the angles between $q \cap p_1$, $q \cap p_2$ and ℓ



https://doi.org/10.4153/CMB-1964-052-5 Published online by Cambridge University Press

respectively, then $\varphi_1 > \theta_1$ and $\varphi_2 > \theta_2$. We conclude that $m_1\widehat{\ }m_2$ is bounded away from π. This is a contradiction since, by our original assumption, the triangles δ_i (whose apex angles are arbitrarily close to π) occur amongst these sections.

Hence each of the two cases leads to a contradiction and the conjecture is proved.

The boundedness of P is an essential condition for the truth of the conjecture. A simple continuity argument establishes that any infinite triangular prism is s-universal for triangles. Further, if P is a tetrahedron at one vertex of which the three face angles are α $(0 < \alpha < \pi/3)$, then P is s-universal for triangles whose angles all exceed α.

REFERENCES

1. V. Klee, Polyhedral sections of convex bodies. Acta Math. 103 (1960), 243-267.

2. Z.A. Melzak, A property of convex pseudopolyhedra. Canadian Bull. Math. 2 (1959), 31-32.

University of Birmingham

https://doi.org/10.4153/CMB-1964-052-5 Published online by Cambridge University Press